Софийски Университет „Св.Климент Охридски”

Факултет по Математика и Информатика

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Dimitar Vandev

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Dimitar Vandev

University of Sofia, Faculty of Mathematics and Informatics,
Sofia 1164, J.Bourchier 5,

E-mail: vandev@fmi.uni-sofia.bg

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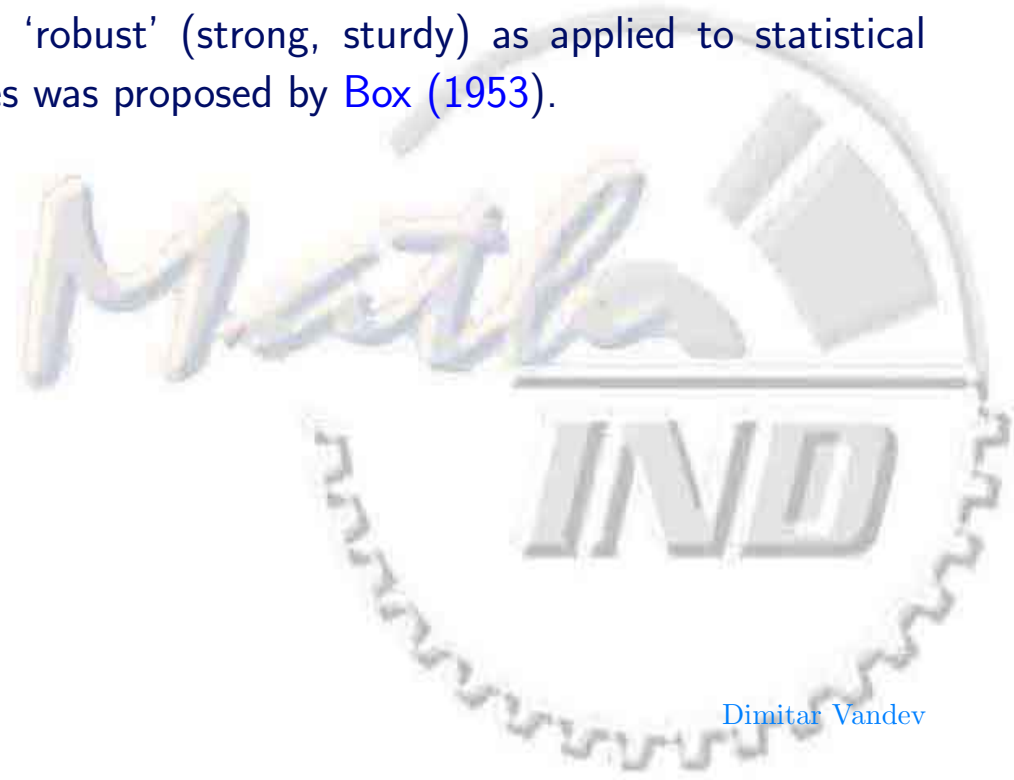
1. General remarks

The field of mathematical statistics called robust statistics appeared due to the pioneer works of Tukey (1960), Huber (1964), and Hampel (1968); it has been intensively developed since the sixties and is rather definitely formed by present. The term 'robust' (strong, sturdy) as applied to statistical procedures was proposed by Box (1953).



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A large part of this talk is based on the recent book of Shevlyakov and Vilchevski (2002).

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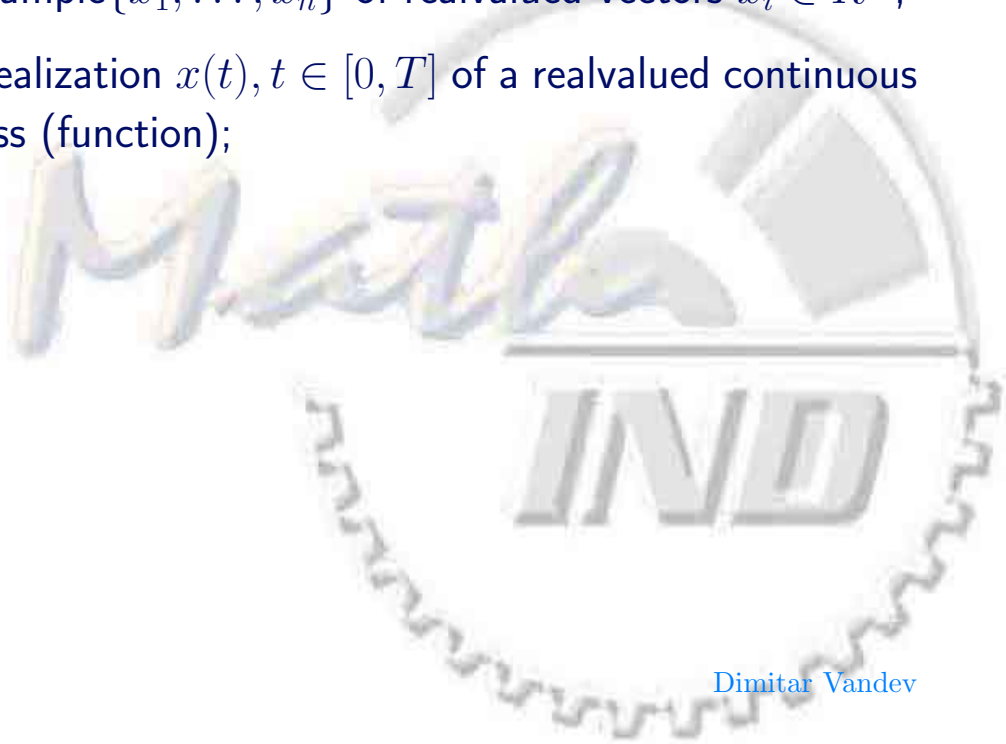

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(iv) as a sample of 'nonnumerical nature' data representing qualitative variables;

(v) as semantic type data (statements, texts, pictures, etc.).

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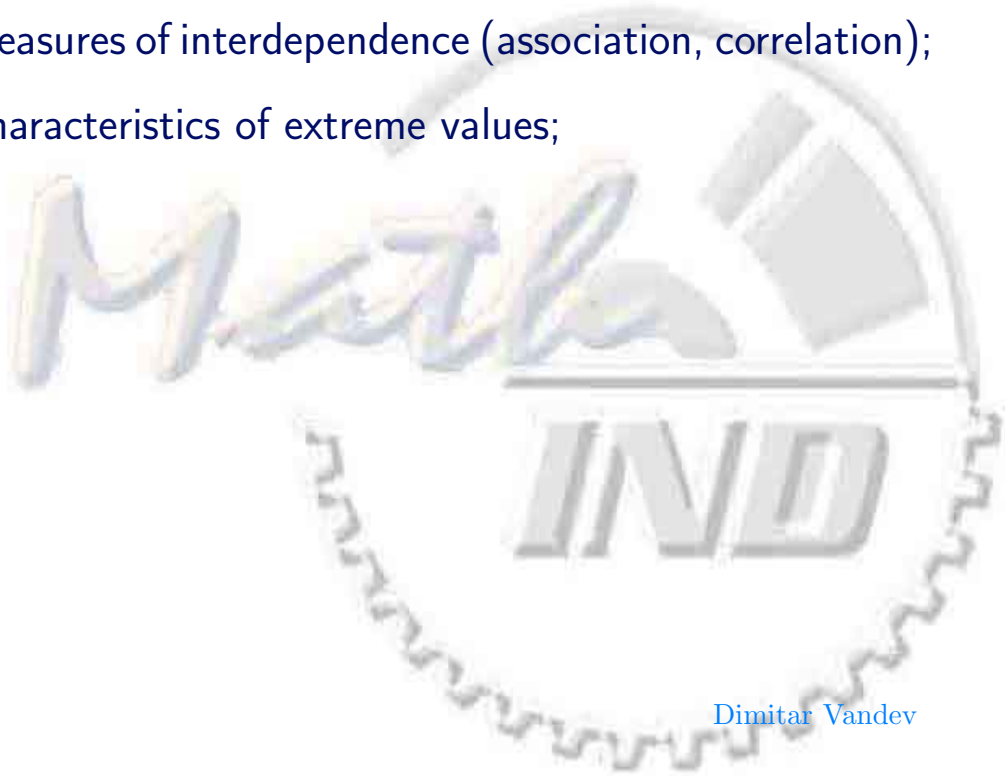

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- the characteristics of data distributions or the measures of shape.

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(A2) Estimation of model parameters describing mass phenomena,

The next step (A2) is to suggest an explanatory underlying model for the observed data and phenomena. It may be a regression model, or a distribution model, or any other, desirably a simple one: an essentially multiparametric model is usually a bad model. Parametric models refer to the first to be considered and examined.

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Finally, all previous aims are only the steps to the last (A3): here we have to state that this aim remains a main challenge to industrial statistics and to science as a whole.

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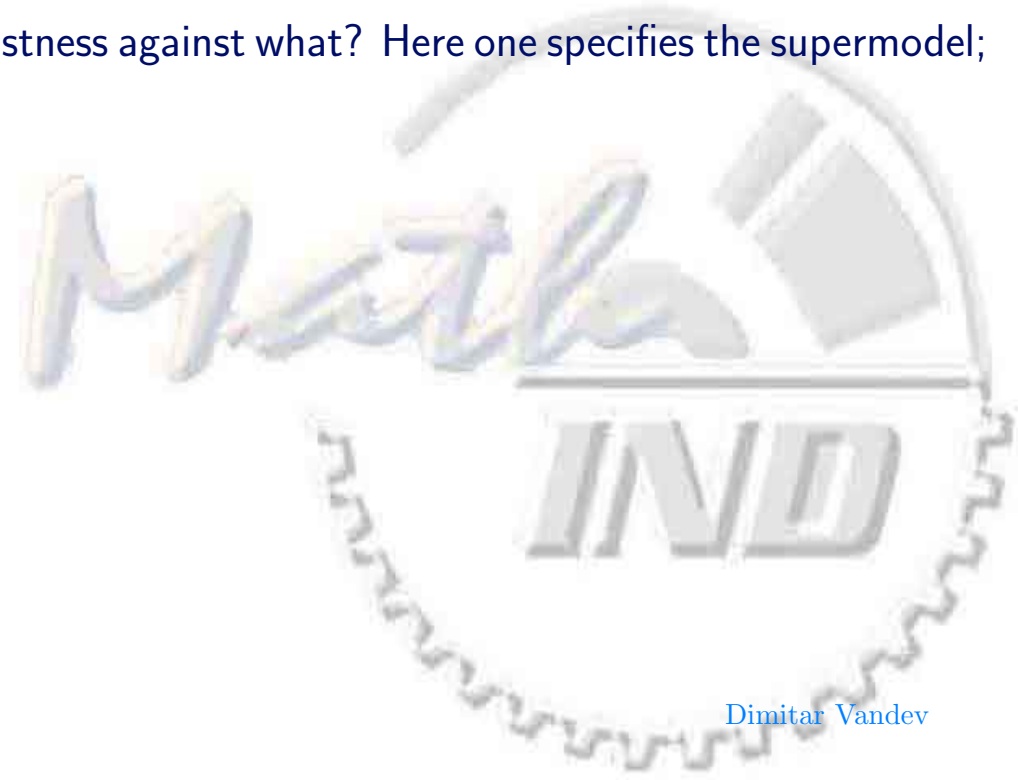

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The wide spectrum of the problems observed in robust statistics can be explained by the fact that there exists a variety of answers to each of the above questions.

2. Huber minimax approach

The convincing arguments for robust statistics are given in Tukey (1960); Huber (1981); Hampel et al. (1986). Here we only recall that the classical examples of robust and non-robust estimators of location are given by the sample median and sample mean, respectively.

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For description of these violations, the concrete forms of neighborhoods of the underlying model are formed with the use of an appropriately chosen metric, for example, the Kolmogorov, Prokhorov, or Lévy Hampel et al. (1986); Huber (1981)). Hence the initial model (basic or ideal) is enlarged up to the socalled supermodel that describes both the ideal model and the deviations from it.

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For example, Hodges and Lehmann (1963) consider the supermodel in the form of all absolutely continuous symmetric distributions.

Birnbaum and Laska (1967) propose the supermodel as a finite collection of distribution functions: $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$.

Various supermodels are used to study deviations from normality: the family of powerexponential distributions with the normal, Laplace, and uniform distributions as particular cases; the family of the Student tdistributions with the normal and Cauchy distributions; also the influence of nonnormality can be studied with the use of the measures of asymmetry and kurtosis, the positive values of the latter indicate gross errors and heavy tails.

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For describing gross errors and outliers, the most popular is the Tukey (1960) supermodel based on the Gaussean low:

$$\mathcal{F} = \left\{ F : F(x) = (1-\varepsilon)\Phi(x) + \varepsilon\Phi(\frac{x-\theta}{k}) \right\}.$$

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Huber (1964) considered more general model

$$\mathcal{F} = \left\{ F : F(x) = (1-\varepsilon)F_0(x) + \varepsilon H(x) \right\}, \qquad (2)$$

where F_0 is some given distribution (the ideal model) and $H(x)$ is an arbitrary continuous distribution.

M-estimators of location

The first general approach to robust estimation is based on the minimax principle (Huber, 1964; Huber, 1972; Huber, 1981). The minimax approach aims at the least favorable situation for which it suggests the best solution.

Let $x_1, \ldots, x_n$ be a random sample from a distribution F with density $f(x - \theta)$ in a convex class $\mathcal{F}$, where θ is the location parameter.

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Assume that F is a symmetric unimodal distribution, hence θ is the center of symmetry to be estimated. Then the M-estimator $\widehat{\theta}_n$ of the location parameter is defined as some solution of the following minimization problem

$$\widehat{\theta}_n = \mathbf{argmin}_{\theta} \sum_{i=1}^{n} \rho(x_i - \theta), \tag{3}$$

where $\rho(u)$ is an even non-negative function called the contrast function; $\rho(x_i - \theta)$ is the measure of discrepancy between the observation x_i and the center θ.

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- for $\rho(u) = |u|$, we have the least absolute values (LAV) method with the sample median as the estimator;
- most important, for a given density $f(x)$, the choice $\rho(u) = -\log f(u)$ yields the maximum likelihood estimator (MLE).

It is convenient to formulate the properties of M-estimators in terms of the derivative of the contrast function $\psi(u) = \rho'(u)$ called the score function. In this case, the M-estimator $\widehat{\theta}_n$ is defined as a solution of the following implicit equation

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Under rather general regularity conditions imposed on the class of score functions Ψ and on the related class of densities $\mathcal{F}$, M - estimators are consistent, asymptotically normal with the asymptotic variance

$$V_M(\psi, f) \stackrel{def}{=} \mathbf{D}\left(n^{1/2}\widehat{\theta}_n\right) = \frac{\mathbf{E}_F \psi^2}{(\mathbf{E}_F \psi')^2} = \frac{\int \psi^2 dF}{(\int \psi' dF)^2}. \tag{5}$$

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Let f^* be the least favorable density in $\mathcal{F}$:

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Then the optimal contrast function and score function are calculated by maximum likelihood method for the least favorable density:

$$\rho^* = -\log f^*, \quad \psi^* = f^{*\prime}/f^*$$

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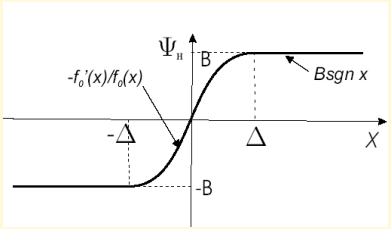

$$\psi^*(x) = \begin{cases} -f_0'(x)/f_0(x), & |x| \le \Delta \\ B\,\mathbf{sgn}(x), & \Delta < |x|. \end{cases}$$

$$f^*(x) = \begin{cases} (1-\varepsilon) f_0(x), & |x| \le \Delta \\ A\exp(-Bx), & \Delta < |x|. \end{cases}$$

Figure 1: Score Function

L-estimators of location

L-estimators were proposed by Daniel (1920) and forgotten for 30 years. The linear combinations of order statistics (L-estimators) are defined as

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and the Winsorized mean:

$$\overline{x}_W(k) = \frac{1}{n}(kx_{(k)} + \sum_{i=k+1}^{n-k} x_{(i)} + kx_{(n-k+1)}) \tag{10}$$

are typical representatives of this class.

The L-estimators may be easily represented in the form:

$$\widehat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} h(\frac{i}{n+1})x_{(i)}, \qquad (11)$$

where the function h is a function of bounded variation on $[0,1]$, $h(t) = h(1-t)$ and $\int_0^1 h(t)dt = 1$.

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where the function h is a function of bounded variation on $[0,1]$, $h(t) = h(1-t)$ and $\int_0^1 h(t)dt = 1$. These conditions on h along with the regularity conditions $(\mathcal{F}1)$ and $(\mathcal{F}2)$ on the distribution provide consistency and asymptotic normality of L-estimators (8) with asymptotic variance

$$V_L(h,f) \stackrel{def}{=} \mathbf{D}\left(n^{1/2}\widehat{\theta}_n\right) = \int_0^1 K^2(t)dt, \tag{12}$$

where

$$K(t) = (h(t)F^{-1}(t) - \theta), \quad \theta = \int_0^1 h(t)F^{-1}(t)dt$$

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R-estimators proposed in Hodges and Lehmann (1963) are based on rank tests. There are several methods of their construction. Let $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ be independent samples from the distributions $F(x)$ and $F(x-\theta)$ respectively. For testing the hypothesis $\theta = 0$ against the alternative $\theta > 0$ the following statistic is used:

$$W_n(y_1, \ldots, y_n, z_1, \ldots, z_n) = \sum_{i=1}^{n} J(\frac{s_i}{2n+1}) \qquad (13)$$

where s_i is the rank of $y_i, i = 1, \ldots n$, in the united sample of size $2n$.

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1. $J(t)$ is increasing;
2. $J(t) + J(1-t) = 0$ for all $t \in [0, 1]$;

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Under these conditions, Hájek and Šidák (1967) proved that the test with the critical region $W_n > c$ has certain optimal in power properties.

The R-estimator $\widehat{\theta}_n$ based on this test is defined as a solution of the equation:

$$W_n(x_1-\theta,\ldots,x_n-\theta,-(x_1-\theta),\ldots,-(x_n-\theta)) = 0 \quad (14)$$

Under the above conditions, $\widehat{\theta}_n$ is consistent and asymptotically normal with asymptotic variance

$$V_R(J,F) \stackrel{def}{=} \mathbf{D}\left(n^{1/2}\widehat{\theta}_n\right) = \frac{\int_0^1 J^2(t)dt}{(\int J'(F(x))f^2(x)dx)^2}.$$

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For any fixed function $F(x)$, it is possible to find the function $J(t)$ minimizing asymptotic variance $V_R(J,F)$. The test based on such function $J(t)$ also has optimal properties for this F. In particular, the logistic distribution $F(x) = (1+e^{-x})^{-1}$ produces the well known Wilcoxon test. The corresponding estimator of location is the Hodges-Lehmann median:

$$\widehat{\theta}_n = \mathbf{med}\left\{\frac{x_{(i)}+x_{(k)}}{2},\ 1 \le i < k \le n\right\}. \qquad (16)$$

Applications of measures of location

Probably the most important place is the statistical quality control. Stromberg et al. (1998) developed Control Charts for the Median and Interquartile Range.

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Römisch et al. (2001) tested all kinds of estimators for Determination of the Geographical Origin of Wines from East European Countries.

3. Hampel approach

The main advantage of robust methods is their lower sensitivity to possible variations of data statistical characteristics. Thus it is necessary to have specific mathematical tools allowing to analyze the sensitivity of estimators to outliers, roundingoff errors, etc. On the other hand, such tools make it possible to solve the inverse problem: to design estimators with the required sensitivity. Now we introduce the above-mentioned apparatus, namely the sensitivity curves and the influence functions.

The sensitivity curve

Let $\{T_n\}$ be a sequence of statistics. Let $T_n(X)$ denote the statistic from $\{T_n\}$ on the sample $X = (x_1, \ldots, x_n)$, and let $T_{n+1}(x, X)$ denote the same statistic on the sample

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$$SC_n(x; T_n, X) = (n+1)[T_{n+1}(x, X) - T_n(X)] \qquad (17)$$

is called the sensitivity curve for this statistic Tukey (1977).

In particular,

$\overline{x}_n$	$SC_n(x;\overline{x}_n,X)=x-\overline{x}_n$
$\mathbf{med}(X)$ $(n=2k+1)$	$SC_n(x;\mathbf{med}(X),X)=\begin{cases}(n+1)(x_{(k)}-x_{(k+1)})/2, & x\le x_{(k)}\\ (n+1)(x-x_{(k+1)})/2, & x_{(k)}\le x\le x_{(k+2)}\\ (n+1)(x_{(k+2)}-x_{(k+1)})/2, & x_{(k+2)}\le x\end{cases}$
$\overline{x}_{tr}(1)$	$SC_n(x;\overline{x}_{tr}(1),X)=\begin{cases}x_{(1)}, & x\le x_{(1)}\\ x, & x_{(1)}\le x\le x_{(n)}\\ x_{(n)}, & x_{(n)}\le x\end{cases}$

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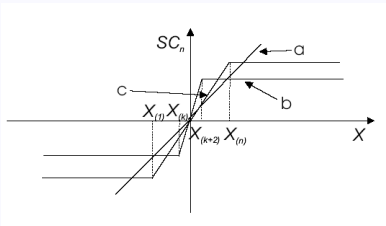

Figure 2: Sensitivity curves

We can see that the sensitivity curve (a.) of the sample mean is unbounded, hence only one extreme observation can completely destroy the estimator. In addition, the maximal error of the trimmed mean (curve c.) is of order $(x_{(n)} - x_{(1)})/n$, while this of median – of $(x_{(k+2)} - x_{(k)})$.

The influence function

Let F be a fixed distribution and $T(F)$ be a functional defined on some set $\mathcal{F}$ of distributions satisfying conditions $(\mathcal{F}1)$ and $(\mathcal{F}2)$. Let the estimator T_n be constructed in the form $T_n = T(F_n)$. Then we define the influence function as:

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$\mathbf{med}(X)$	$T(F) = F^{-1}(1/2), \quad IF(x,T,F) = \mathbf{sgn}(x)/(2f(0))$
$\overline{x}_{tr}(k)$ $\alpha = k/n$	$IF(x,T,F) = \begin{cases} F^{-1}(\alpha)/(1-2\alpha), x \le F^{-1}(\alpha) \\ x/(1-2\alpha), \quad F^{-1}(\alpha) \le x \le F^{-1}(1-\alpha) \\ F^{-1}(\alpha)/(1-2\alpha), \quad F^{-1}(1-\alpha) \le x \end{cases}$

Table 2: Example influence functions

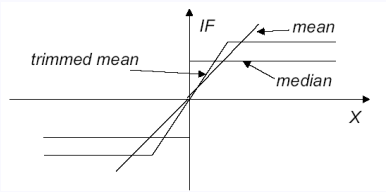
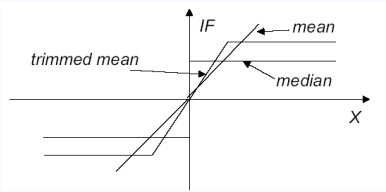


Figure 3: Influence functions

Comparing Fig.2 and Fig.3, we see that the forms of influence and sensitivity curves are similar. In fact $SC_n(x;T,F) \longrightarrow IF(x;T,F)$ as $n \longrightarrow \infty$.



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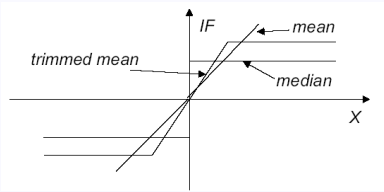
The influence function for the M-estimator with the score function ψ is of the form Hampel et al. (1986)

$$IF(x;\psi,F) = \frac{\psi}{\int \psi dF(x)}$$



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Fernholz (1983) showed that T_n is asymptotically normal with asymptotic variance

$$V(T,F) = \int IF^2(x;T,F)dF(x). \qquad (19)$$

4. Measures of robustness

Local measures of sensitivity

From the influence function, the following robustness measures can be defined (Hampel (1968); Hampel (1974)).

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Gross-error sensitivity

$$\gamma^*(T,F) = \sup_x |IF(x;T,F)|$$

is an upper bound to the asymptotic bias of the estimator and measures the worst influence of an infinitesimal contamination. The estimators T having finite $\gamma^*(T,F)$ are called B-robust.

Local-shift sensitivity

$$\lambda^*(T,F) = \sup_{x \neq y} \frac{|IF(y;T,F) - IF(x;T,F)|}{|y-x|}$$

accounts the effects of rounding-off and grouping of the observations.

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Rejection point

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Change-of-variance function

$$CVF(x;T,F) = \lim_{t \to 0} \frac{V(T,(1-t)F + t\delta_x) - V(T,F)}{t}$$

was introduced by Hampel et al. (1986) by analogy with the influence function IF. Here $V(T,F)$ is the asymptotic variance.

Change-of-variance sensitivity

$$k^*(T,F) = \sup_x \frac{CVF(x;F,T)}{V(T,F)}$$

The estimator $T_n = T(F_n)$ of the functional $T(F)$ is called V-robust if $k^*(T,F) < \infty$.

Hence it is desirable to have a measure of the global robustness of the estimator over the chosen class of distributions, in other words, in the chosen supermodel $\mathcal{F}$.

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All the above-introduced measures of robustness based on the influence function and its derivatives are of a local character being evaluated at the model distribution F. Hence it is desirable to have a measure of the global robustness of the estimator over the chosen class of distributions, in other words, in the chosen supermodel $\mathcal{F}$.

Break-down point

Let d be a distance in the space of all distributions. The break-down point of the estimator $T_n = T(F_n)$ for the functional $T(F)$ at $\mathcal{F}$ is defined by

$$\varepsilon^*(T, \mathcal{F}) = \sup_{\varepsilon<1}\{\varepsilon : \sup_{F:d(F,F_0)<\varepsilon} |T(F) - T(F_0)| < \infty\}.$$

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Breakdown point as applied to the Huber supermodel

$$\varepsilon^*(T, \mathcal{F}) = \sup_{\varepsilon<1}\{\varepsilon : \sup_{F:F=(1-\varepsilon)F_0+\varepsilon H} |T(F) - T(F_0)| < \infty\}. \tag{20}$$

Here is the replacement variant of the finite sample breakdown point given by Hampel et al. (1986).

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Let $\Omega = \{\omega_i \in R^p$, for $i = 1, \ldots, n\}$ be a sample of size n. The breakdown point of an estimator $T(\Omega) \in R^q$ is given by

$$\varepsilon_n^*(T) = \frac{1}{n} \max\{m : \sup_{\widetilde{\Omega}_m} \|T(\widetilde{\Omega}_m)\| < \infty\}, \qquad (21)$$

where $\widetilde{\Omega}_m$ is any sample obtained from Ω by replacing any m of the points in Ω by arbitrary values.

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In other words there should exist a compact set such that the estimator T remains in it even if we replace any m elements of the sample Ω by arbitrary ones. The largest m/n for which this property holds is the breakdown point.

5. Multidimensional Statistics

All the definitions and methods can be easily extended to multivariate and multiparametric case when one estimates location.

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LTS and LMS

The multiple regression is probably most used statistical procedure in the industrial statistics. Consider the model

$$y_i = x_i^T \beta + \varepsilon_i.$$

where y_i is an observed response, x_i is a $p \times 1$–dimensional vector of explanatory variables and β is a $p \times 1$ vector of unknown parameters. Classically ε_i, $i = 1, \ldots, n$ are assumed to be i.i.d. $N(0, \sigma^2)$, for some $\sigma^2 > 0$.

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$$\mathrm{LMS}(r_1, \ldots, r_n) = \operatorname*{argmin}_\theta \operatorname{med}\{r_i^2, i = 1, ..., n\}, \quad (22)$$

$$\mathrm{LTS}(k)(r_1, \ldots, r_n) = \operatorname*{argmin}_\theta \sum_{i=1}^{k} r^2_{\nu(i,\theta)}. \quad (23)$$

Here $\nu(i, \theta)$ is a permutation of the indices, such that $r^2_{\nu(i,\theta)} \leq r^2_{\nu(i+1,\theta)}$. Thus the idea was to minimize the sum of squares using "smallest residuals" only.

Covariance

The estimation of unknown covariance matrix of observed data or estimated parameters turned out to be not so easy. In fact, only two methods are used in practice and very little is known about their properties.

Applications

We should mention here the works of Mili et al. (1991)and Mili et al. (1994) on Power Systems.

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- MVE - minimum volume ellipsoid;
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In both cases one have to choose fixed percentage (e.g. 90%) of observed data having corresponding optimal property. Then the estimator is build using only these data.

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6. Robustified Maximum likelihood

Neykov and Neytchev (1990) proposed to replace in the LMS and LTS estimators the squared residuals with - log likelihood´s of the individual observations and thus to obtain robustified likelihood.

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Both estimators may be easily combined into one. However it took some time (5 years) to understand.

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$$\mathsf{WTL(k)}(x_1, \ldots, x_n) = \operatorname*{argmin}_{\theta} \sum_{i=1}^{k} \{-w_i \log f(x_{\nu(i,\theta)}, \theta)\} \tag{26}$$

where $f(x_{\nu(i,\theta)}, \theta) \geq f(x_{\nu(i+1,\theta)}, \theta)$ are the ordered density values. ν is a permutation of the indices $1, \ldots, n$, which may depend on θ.

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Definition of d-fullness

In order to study the breakdown properties of general estimators of the type (24) and (25) Vandev (1993) developed a d–fullness technique. He proved that their breakdown point is not less than $(n-k)/n$ if k is within the range of values $(n+d)/2 \leq k \leq (n-d)$ for some constant d which depends upon the density considered.

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We remind that a real valued function $g(\theta)$ defined on a topological space Θ is called subcompact, if its Lebesque sets $\{\theta : g(\theta) \leq C\}$ are compact (or empty) for any constant C.

Let the finite set $F = \{f_i(\theta) \geq 0, i = 1, \ldots, n, \text{ for } \theta \in \Theta\}$ be d–full and Θ is a topological space. Consider the estimator of θ

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Theorem. *Under these conditions if $n \geq 3d$ and $(n + d)/2 \leq k \leq n - d$, then the breakdown point of the estimator $R(k)$ is not less than $(n - k)/n$.*

Let the finite set $F = \{f_i(\theta) \geq 0, i = 1, \ldots, n, \text{ for } \theta \in \Theta\}$ be d–full and Θ is a topological space. Consider the estimator of θ

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The value d may be interpreted as
number of observations necessary to make unique guess for the estimated parameter.

Multivariate normal distribution

Vandev and Neykov (1993) determined the value of d for the set of log-density functions for the multivariate normal case. When estimating only the mean $d = 1$. When one need to estimate the covariance matrix $d = p + 1$. Let $x_i \in R^p,\ i = 1, \ldots, n$ have density

$$\varphi(x, \mu, S) = (2\pi)^{-p/2}(\det(S))^{-1/2} \exp(-(x-\mu)'S^{-1}(x-\mu)/2).$$

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Later Marincheva and Vandev (1995) considered a general elliptic family. Atanasov and Neykov (2001) calculated the fullness parameters for the Lognormal, Poisson, Gamma, Geometric and Logarithmic series distributions and thus determined the BPs of the WTL estimators for the corresponding Generalized Linear Models.

7. Linear regression

Theory

Consider the class of regression estimators defined as

$$\widehat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{k} w_i \rho(|r|_{\nu(i,\theta)}), \tag{27}$$

where ρ is strictly increasing continuous function such that $\rho(0) = 0$.

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This class of estimators is regression, scale and affine equivariant following the reasoning of Rousseeuw and Leroy (1987).

Theorem. *The breakdown point of the regression estimators (27) is equal to* $(n-k)/n$ *if the index* k *is within the bounds* $(n+p+1)/2 \leq k \leq n-p-1,\ n \geq 3(p+1)$ *and the data points* $x_i \in R^p$ *for* $i = 1, \ldots, n$ *are in general position.*

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- LMS and LTS estimators of Rousseeuw (1984);
- h-trimmed weighted L_q estimators of Müller (1995) if $\rho(|r|_{(i)}) = |r|^q_{(i)}$,
- rank-based linear regression estimators proposed by Hössjer (1994), where the weights w_i are generated by a function of the residual's ranks.

Regression depth

Rousseeuw and Hubert (1999) introduced a notion of depth in the regression setting. It provides the "rank" of any line (plane), rather than ranks of observations or residuals. In simple regression they can compute the depth of any line by a fast algorithm. For any bivariate dataset Z_n of size n there exists a line with depth at least $n = 3$. The largest depth in Z_n can be used as a measure of linearity versus convexity.

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In both simple and multiple regression they introduce the deepest regression method, which generalizes the univariate median and is equivariant for monotone transformations of the response. Throughout, the errors may be skewed and heteroscedastic.

They explore the analogies between depth in regression and in location, where Tukey's halfspace depth is a special case of this general definition.

Algorithms

Peña and Yohai (1999) propose a procedure for computing a fast approximation to regression estimates based on the minimization of a robust scale. The procedure can be applied with a large number of independent variables where the usual algorithms require an unfeasible or extremely costly computer time. Also, it can be incorporated in any high-breakdown estimation method and may improve it with just little additional computer time. The good performance of the procedure allows identification of multiple outliers, avoiding masking effects.

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References

Andrews, D. F., Bickel, P. J., et al. (1972). *Robust Estimates of Location*. Princeton Univ. Press, Princeton.

Bickel, P. J. (1976). Another look at robustness: a review of reviews and some new developments. *Scand. J. Statist. Theory and Appl.*, 3(4):pp. 145 – 168.

Birnbaum, A. and Laska, E. (1967). Optimal robustness: a general method with application to linear estimators of location. *J. Amer. Statist. Assoc.*, 62:pp. 1230 – 1240.

Box, G. E. P. (1953). Nonnormality and test on variances. *Biometrika*, 40:pp. 318 – 335.

Daniel, C. (1920). Observations weighted according to order. *Amer. J. Math.*, 42:pp. 222 – 236.

Fernholz, L. T. (1983). *Von Mises Calculus for Sta-*

References

Andrews, D. F., Bickel, P. J., et al. (1972). *Robust Estimates of Location*. Princeton Univ. Press, Princeton.

Bickel, P. J. (1976). Another look at robustness: a review of reviews and some new developments. *Scand. J. Statist. Theory and Appl.*, 3(4):pp. 145 – 168.

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Box, G. E. P. (1953). Nonnormality and test on variances. *Biometrika*, 40:pp. 318 – 335.

Daniel, C. (1920). Observations weighted according to order. *Amer. J. Math.*, 42:pp. 222 – 236.

Fernholz, L. T. (1983). *Von Mises Calculus for Sta-*

tistical Functionals. Springer, New York.

HADI, A. and LUCEÑO, A. (1997). Maximum Trimmed Likelihood Estimators: A Unified Approach, Examples and Algorithms. *Comput. Statist. and Data Analysis*, 25:pp. 251 – 272.

HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

HAMPEL, F. R., RONCHETTI, E., ET AL. (1986). *Robust Statistics. The Approach Based on Influence Functions*. Wiley, New York.

HAMPEL, F. R. (1968). *Contributions to the Theory of Robust Estimation*. Ph.D. thesis, University of California, Berkeley.

HAMPEL, F. R. (1974). The influence curve and its role in robust estimation. *J. Amer. Statist. Assoc.*, 69:pp. 383 – 393.

tistical Functionals. Springer, New York.

HADI, A. and LUCEÑO, A. (1997). Maximum Trimmed Likelihood Estimators: A Unified Approach, Examples and Algorithms. *Comput. Statist. and Data Analysis*, 25:pp. 251 – 272.

HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

HAMPEL, F. R., RONCHETTI, E., ET AL. (1986). *Robust Statistics. The Approach Based on Influence Functions*. Wiley, New York.

HAMPEL, F. R. (1968). *Contributions to the Theory of Robust Estimation*. Ph.D. thesis, University of California, Berkeley.

HAMPEL, F. R. (1974). The influence curve and its role in robust estimation. *J. Amer. Statist. Assoc.*, 69:pp. 383 – 393.

HODGES, Z. L. and LEHMANN, E. L. (1963). Estimates of location based on rank tests. *Ann. Math. Statist.*, 34:pp. 598 – 611.

HÖSSJER, O. (1994). Rank-based estimates in the linear model with high breakdown point. *J. Amer. Statist. Assoc.*, 89(425):pp. 149–158.

HUBER, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.*, 35:pp. 73 – 101.

HUBER, P. J. (1981). *Robust Statistics*. John Wiley, New York.

MARINCHEVA, M. Z. and VANDEV, D. (1995). On high breakdown point estimators of location and scale. In: *Statistical Data Analysis, Proceedings of SDA-95, SDA-96*, pp. 51–57. Sofia.

MILI, L., CHENIAE, M., ET AL. (1994). Robust State Estimation of Electric Power System. *IEEE Transac-*

HODGES, Z. L. and LEHMANN, E. L. (1963). Estimates of location based on rank tests. *Ann. Math. Statist.*, 34:pp. 598 – 611.

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HUBER, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.*, 35:pp. 73 – 101.

HUBER, P. J. (1981). *Robust Statistics*. John Wiley, New York.

MARINCHEVA, M. Z. and VANDEV, D. (1995). On high breakdown point estimators of location and scale. In: *Statistical Data Analysis, Proceedings of SDA-95, SDA-96*, pp. 51–57. Sofia.

MILI, L., CHENIAE, M., ET AL. (1994). Robust State Estimation of Electric Power System. *IEEE Transac-*

tions on circuit and systems-I: Fundamental Theory and Applications, 41(5):pp. 349 – 358.

Mili, L., Phaniraj, V., et al. (1991). Least Median of Squares Estimation in Power System. *IEEE Transactions on Power Systems*, 6(2):pp. 511 – 523.

Müller, C. H. (1995). Breakdown points for designed experiments. *J. Statist. Plann. Inference*, 45:pp. 413 – 427.

Neykov, N. and Neytchev, P. (1990). A robust alternative of the maximum likelihood estimators. In: *Short communications of COMPSTAT'90*, pp. 99 – 100. Dubrovnik, Yugoslavia.

Peña, D. and Yohai, V. (1999). A fast procedure for outlier diagnostics in large regression problems. *J. Amer. Statist. Assoc.*, 94(446):pp. 434–445. ISSN 0162-1459.

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Peña, D. and Yohai, V. (1999). A fast procedure for outlier diagnostics in large regression problems. *J. Amer. Statist. Assoc.*, 94(446):pp. 434–445. ISSN 0162-1459.

URL mailto:dpena@est-econ.uc3m.es

Römisch, U., Vandev, D., et al. (2001). Determination of the geographical origin of wines from east european countries by methods of multivariate data analysis. In: *Seminar of the International Biometric Society, Region Oesterreich- Schweiz (ROeS)*. 24 - 27 September 2001, Mayrhofen.

Rousseeuw, P. J. and Hubert, M. (1999). Regression depth. *J. Amer. Statist. Assoc.*, 94(446):pp. 388–402.
URL http://www.amstat.org/publications/jasa/rousseeuw.pdf

Rousseeuw, P. J. and Leroy, A. (1987). *Robust Regression and Outlier Detection*. John Wiley and Sons, New York.

Rousseeuw, P. J. (1984). Least median of squares

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URL http://www.amstat.org/publications/jasa/rousseeuw.pdf

Rousseeuw, P. J. and Leroy, A. (1987). *Robust Regression and Outlier Detection*. John Wiley and Sons, New York.

Rousseeuw, P. J. (1984). Least median of squares

regression. *J. Amer. Statist. Assoc.*, 79(388):pp. 871–880. ISSN 0162-1459.

Shevlyakov, G. L. and Vilchevski, N. O. (2002). *Robustness in data analysis: criteria and methods*, volume 6 of *Modern Probability and Statistics'*. VSP, Utrecht. V. Yu. Korolev, V. M. Zolotarev, Editors-in-Chief.

Stromberg, A. J., Griffith, W., et al. (1998). Control Charts for the Median and Iinterquartile Range. Department in Statistics, University of Kentucky, USA.

Tukey, J. W. (1960). A survey of sampling from contaminated distributions. In: Olkin, I. (ed.), *Contributions to Probability and Statistics*, pp. 448 – 485. Stanford Univ. Press, Stanford.

Tukey, J. W. (1977). *Exploratory Data Analysis*.

regression. *J. Amer. Statist. Assoc.*, 79(388):pp. 871–880. ISSN 0162-1459.

Shevlyakov, G. L. and Vilchevski, N. O. (2002). *Robustness in data analysis: criteria and methods*, volume 6 of *Modern Probability and Statistics'*. VSP, Utrecht. V. Yu. Korolev, V. M. Zolotarev, Editors-in-Chief.

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Addison-Wesley, Reading, MA.

VANDEV, D. L. and NEYKOV, N. M. (1993). Robust maximum likelihood in the gaussian case. In: MORGENTHALER, S., RONCHETTI, E., ET AL. (eds.), *New Directions in Statistical Data Analysis and Robustness*, pp. 257 – 264. Birkhauser Verlag, Basel.

VANDEV, D. L. and NEYKOV, N. M. (1998). About regression estimators with high breakdown point. *Statistics*, 32:pp. 111 – 129.

VANDEV, D. (1993). A note on breakdown point of the least median squares and least trimmed squares. *Statist. Probab. Lett.*, 16:pp. 117 – 119.

Addison-Wesley, Reading, MA.

VANDEV, D. L. and NEYKOV, N. M. (1993). Robust maximum likelihood in the gaussian case. In: MORGENTHALER, S., RONCHETTI, E., ET AL. (eds.), *New Directions in Statistical Data Analysis and Robustness*, pp. 257 – 264. Birkhauser Verlag, Basel.

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VANDEV, D. (1993). A note on breakdown point of the least median squares and least trimmed squares. *Statist. Probab. Lett.*, 16:pp. 117 – 119.