Genericity in abstract structure degrees

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Abstract

The generalized notion of genericity in the theory of abstract structure degrees is used to obtain a characterization of abstractly generic predicate of natural numbers as the preimage of some predicate of the denumerable set N and generic regular enumeration.

Introduction

In this paper we deal with *search-computability,* defined by Moschovakis in [2], though for the proofs of most of the propositions we have used the Skordev's definition of search-computability, in [3] Skordev proved both are equivalent.

The idea of considering two sort structures was presented by I.N.Soskov during the cycle of lectures of the seminar on Computability Theory at Sofia University, 1998. The abstract structure degrees are defined also by him during the same seminar as well as their regular enumerations.

The first sort of the mentioned two-sort abstract structures is an arbitrary denumerable set and the other is the set of natural numbers. The presence of the equality among the basic predicates of the structure is required.

In these terms we present an analogue of some notions from the theory of the enumeration degrees, namely the *set genericity* and the related results, applying the techniques used by Copestake in [1]. We generalize the characterization obtained in [6], stating that a set of natural numbers is generic relatively a set B if and only if it is the preimage of some set A using a B-generic B-regular enumeration such that both A and its complement are e-reducible to B.

Here we introduce the notion of *genericity* for abstract predicates. Using the enumerations of two-sort abstract structures (in the way they are used in [4]) we obtain a characterization of this type of abstract genericity, which claims that a predicate A of natural numbers is generic relatively the two-sort abstract structure $\mathfrak{B}$ with one predicate if and only if there exists a predicate Σ on the first sort, which is search computable in $\mathfrak{B}$ and a $\mathfrak{B}$-generic regular enumeration f, such that $A = f_N^{-1}(\Sigma)$.

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1 Preliminaries

We use some standard definitions and notations: $\leq_e$ denotes the enumeration reducibility between sets and Ψ_e denotes the e-th enumeration operator, i.e. $\Psi_e(B) = \{x \mid \exists v(\langle x, v\rangle \in W_e \,\&\, D_v \subseteq B)\}$, where W_e is the recursively enumerable set with Gödel code e, B is a set of natural numbers and D_v is the finite set with code v. Recall the *join* operation for sets of naturals: $A \oplus B$ is the set $\{2x | x \in A\} \cup \{2x+1 | x \in B\}$, used to induce the least upper bound of the e-degrees of A and B.

Given a countable set N and $0^* \notin N$, N^* denotes the *Moschovakis' extension* of N, i.e. the smallest extension of $N \cup \{0^*\}$ closed under the operation *ordered pair* $\langle \cdot, \cdot \rangle$ (we will use the same notation for effective coding of pairs of natural numbers); ω denotes the set of the natural numbers and $\omega^* \subseteq N^*$ is the set of elements $0^*, \ldots, (n+1)^* \ldots$, such that $(n+1)^* = \langle 0^*, n^* \rangle \in \omega^*$. By $\mathcal{F}$ we denote the set of one-argument partial functions $\varphi : N^* \multimap N^*$. We write $\varphi \in \mathbf{SC}(\varphi_1, \ldots, \varphi_n)$ to say that φ is *search computable* in the set of functions $\{\varphi_1 \ldots \varphi_n\} \subseteq \mathcal{F}$, (see [3]).

From now on, we consider the abstract partial two-sort structures:

$$\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \ldots, \Sigma_k \rangle,$$

with two fixed basic predicates in N^2: $=_N$ (equality) and $\neq_N$ (inequality), and partial predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, s.t. $a_i, b_i \geq 0$ but not both zero. This kind of structures will be denoted by $\mathfrak{A}(\Sigma_1 \ldots \Sigma_k)$.

The notation $\Sigma_0 \leq_{\mathbf{SC}} \mathfrak{A}$ says that Σ_0 is search computable in the set of $\mathfrak{A}$'s predicates, including the equality and inequality, i.e. $\widehat{\Sigma}_0 \in \mathbf{SC}(\widehat{\Sigma}_1^{\mathfrak{A}}, \ldots, \widehat{\Sigma}_k^{\mathfrak{A}}, \widehat{=}_N, \widehat{\neq}_N)$, (we also write $\widehat{\Sigma}_0 \in \mathbf{SC}(\mathfrak{A})$), where $\widehat{\Sigma} : N^* \multimap N^*$ is the *semi-characteristic* function of the predicate.

Soskov defined $\mathfrak{A} \oplus \mathfrak{B}$ to be the two-sort structure with predicates $=_N$, $\neq_N$, $\Sigma_1^{\mathfrak{A}} \ldots \Sigma_{k_{\mathfrak{A}}}^{\mathfrak{A}}$, $\Sigma_1^{\mathfrak{B}} \ldots \Sigma_{k_{\mathfrak{B}}}^{\mathfrak{B}}$; $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ if and only if $\forall i_{(1 \leq i \leq k_{\mathfrak{A}})}$: $\Sigma_i^{\mathfrak{A}} \leq_{\mathbf{SC}} \mathfrak{B}$ and $\mathfrak{A} \equiv_{\mathbf{SC}} \mathfrak{B}$ if and only if $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$.

Definition 1.1 (Soskov)
The *abstract structure degrees* are the equivalence classes, induced by the relation $\equiv_{\mathbf{SC}}$ between structures, and we denote them by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$... and for every $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{D}$, $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{D}_s(\mathfrak{A} \oplus \mathfrak{B})$ for some $\mathfrak{A} \in \mathfrak{a}$ and $\mathfrak{B} \in \mathfrak{b}$.

We write $\mathfrak{D}$ for the set of all abstract structure degrees with the partial ordering induced by $\leq_{\mathbf{SC}}$. Thus the structure $\langle \mathfrak{D}, \leq_{\mathbf{SC}}, \cup, \mathfrak{O} \rangle$ is an upper semi-lattice with least element the empty structure $\mathfrak{O} = \langle N; \omega; =_N, \neq_N \rangle$.

On the seminar on Computability Theory in 1998, I.Soskov introduced the following definition of search computability and proved its equivalence with the standard ones (see [2] and [3]):

$$\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B} \text{ iff } \forall \alpha (\mathfrak{B} \leq \alpha \Rightarrow \mathfrak{A} \leq \alpha),$$

where $\alpha = (f, R)$ is enumeration and $\mathfrak{A} \leq \alpha$ if and only if $f^{-1}(\mathfrak{A}) \leq_e R$. Here we shall use it for a single predicate $\Sigma \subseteq N^{a_i} \times \omega^{b_i}$ in this particular form:

$$\Sigma \leq_{\mathbf{SC}} \mathfrak{A} \text{ iff } \big(f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A}), \text{ for every } N\text{-enumeration } f_N\big) \qquad (1)$$

where $f_N : \omega \to N$ is total and surjective function, that we shall call N-*enumeration*, $f_N^{-1}(\Sigma) = \{\langle x_1 \ldots x_a, y_1 \ldots y_b\rangle \in \omega \mid (f_N(x_1) \ldots f_N(x_a), y_1 \ldots y_b) \in \Sigma\}$ and for the structure $\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \ldots, \Sigma_k\rangle$, the preimage $f_N^{-1}(\mathfrak{A})$ is defined in such a way that is e-equivalent to $f_N^{-1}(\Sigma_1) \oplus \ldots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(=_N) \oplus f_N^{-1}(\neq_N)$.

2 Enumerations

Many of the definitions and the proofs from [4] concerning the enumeration approach and the normal form theorem are applicable in our case. We recall them in order to use them later in §3 and for the characterization in §4.

Definition 2.1
1) N-string τ_N is a finite function $\tau_N : [0 \ldots n-1] \to N$, with domain an initial segment of ω with length $lh(\tau_N) = n$.

We shall call the strings used in [6] ω-*strings*, i.e. an ω-string is a finite sequence of naturals meant to be an initial segment of ω.
2) $\tau_N \subseteq \sigma_N$ iff $\forall x(x < lh(\tau_N) \Rightarrow \tau_N(x) = \sigma_N(x))$.
3) Code of the N-string τ_N is defined to be $\ulcorner \tau_N \urcorner = \langle n^*, \tau_N(0), \ldots, \tau_N(n-1)\rangle$.

Definition 2.2 ([4])
For a structure $\mathfrak{A}(\Sigma_1 \ldots \Sigma_k)$ with $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, an N-string τ_N and a formula $F_e(z)$ with $e, z \in \omega$, define the *forcing relation* $\tau_n \Vdash_{\mathfrak{A}} F_e(z)$ as follows:

(1) $\tau_N \Vdash_{\mathfrak{A}} F_e(z)$ iff $\exists v(\langle v, z\rangle \in W_e \;\&\; \tau_n \Vdash_{\mathfrak{A}} D_v)$

(2) $\tau_N \Vdash_{\mathfrak{A}} D_v$ iff $\forall u \in D_v(u = \langle i, \langle x_1 \ldots x_{a_i}, y_1 \ldots y_{b_i}\rangle\rangle \;\&$ $1 \leq i \leq k \;\&\; x_1 \ldots x_{a_i} \in Dom(\tau_N) \;\&(\tau_N(x_1) \ldots \tau_N(x_{a_i}), y_1 \ldots y_{b_i}) \in \Sigma_i \vee\; u = \langle 0, 2\langle x, y\rangle\rangle \;\&\; x, y \in Dom(\tau_N) \;\&\; \tau_N(x) = \tau_N(y) \;\&\; u = \langle 0, 2\langle x, y\rangle + 1\rangle \;\&\; x, y \in Dom(\tau_N) \;\&\; \tau_N(x) \neq_N \tau_N(y))$.

Definition 2.3 ([4])
For an N-enumeration $f_N : \omega \to N$ and a structure $\mathfrak{A}(\Sigma_1 \ldots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, define

$$f_N \models_{\mathfrak{A}} F_e(z) \text{ if and only if } z \in \Psi_e(f_N^{-1}(\mathfrak{A})).$$

Definition 2.4 ([4])
We say that the predicate $\Sigma \subseteq N^a \times \omega^b$ has *normal form* in the structure $\mathfrak{A}(\Sigma_1 \ldots \Sigma_k)$, if there exist $e \in \omega$, an N-string δ_N and $x_1 \ldots x_a \notin Dom(\delta_N)$, such that for all $s_1 \ldots s_a \in N$, and for all $y_1 \ldots y_b \in \omega$, $(s_1 \ldots s_a, y_1 \ldots y_b) \in \Sigma$ iff $\exists \tau_N \supseteq \delta_N$ s.t. $\forall_{1 \leq i \leq a}\, (\tau_N(x_i) = s_i) \;\&\; \tau_N \Vdash_{\mathfrak{A}} F_e(\langle x_1 \ldots x_a, y_1 \ldots y_b\rangle)$.

The following is a corrollary from the Normal Form Theorem from [4] for the case of two-sort structures.

Theorem 2.5 (Normal Form Theorem)
Let $\mathfrak{A}(\Sigma_1 \ldots \Sigma_k)$ be a structure, with predicates $\Sigma_i \leq N^{a_i} \times \omega^{b_i}$. Then every predicate $\Sigma \subseteq N^a \times \omega^b$ that Σ is search computable in $\mathfrak{A}$, has a normal form in $\mathfrak{A}$.

3 Generic predicates

Definition 3.1

1) Let $\Sigma \subseteq N^a \times \omega^b$ be a predicate. We define the *characteristic function* of Σ to be the function $\chi_\Sigma : N^* \multimap\!\!\rightarrow N^*$, defined as follows:

$$\chi_\Sigma(s) = \begin{cases} 0^* & \text{if } s = \langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle \ \& \ (s_1 \dots s_a, x_1 \dots x_b) \in \Sigma \\ 1^* & \text{if } s = \langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle \ \& \ (s_1 \dots s_a, x_1 \dots x_b) \notin \Sigma \ . \\ \uparrow & \text{otherwise} \end{cases}$$

2) Let $\mathcal{F}_{a,b}$, where $a + b \geq 1$, be the set of all partial functions $\varphi \in \mathcal{F}$, such that $Dom(\varphi) \subseteq \{\langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle \mid (s_1 \dots s_a, x_1 \dots x_b) \in N^a \times \omega^b\}$ and $Range(\varphi) \subseteq \omega^*$.

3) Define (a,b)-string to be a finite function $\alpha \in \mathcal{F}_{a,b}$ with $Range(\alpha) \subseteq \{0^*, 1^*\}$. We may define the code of the (a,b)-string α (denote $\ulcorner \alpha \urcorner$), to be $\langle k^*, \langle s_1, \alpha(s_1) \rangle, \dots, \langle s_k, \alpha(s_k) \rangle \rangle \in N^*$, if $Dom(\alpha) = \{s_1, \dots, s_k\}$; and $\ulcorner \varnothing \urcorner = 0^*$, for the empty function.

Remark: Since an (a,b)-string may have more than one (but only finitely many) different codes, by $\alpha \in S^* \subseteq N$, we mean that there exists a code of α, which belongs to the set S^*; respectively $\alpha \notin S^*$ means there is no code of α that belongs to the set. We say that S^* is a set of codes of (a,b)-strings when each element is a code of some (a,b)-string, it is not necessary that S^* contains all the codes of an (a,b)-string.

4) *Semi-characteristic function* of the set $S^* \subseteq N$ we call the function $C_{S^*} : N^* \multimap\!\!\rightarrow N^*$, defined as follows:

$$C_{S^*}(s) \cong \begin{cases} 0^* & \text{if } s \in S^* \\ \uparrow & \text{otherwise} \end{cases}$$

For a given set $S^* \subseteq N^*$ and structure $\mathfrak{B}(\Sigma_1 \dots \Sigma_k)$, we write $S^* \in \mathbf{SC}(\mathfrak{B})$, when $C_{S^*} \in \mathbf{SC}(\hat{=}_N, \hat{\neq}_N, \hat{\Sigma}_1, \dots, \hat{\Sigma}_k)$.

5) For every a and b, which are not both zero, and every function $\varphi \in \mathcal{F}_{a,b}$, we define the *graph-predicate* of φ, to be the predicate $\Sigma_\varphi \subseteq N^a \times \omega^{b+1}$, such that for all $s_1, \dots, s_a \in N$ and $x_1, \dots, x_b, y \in \omega$, $(s_1 \dots s_a, x_1 \dots x_b, y) \in \Sigma_\varphi$ iff $\varphi(\langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle) = y^*$.

Definition 3.2

Given a structure $\mathfrak{B}(\Sigma_1 \dots \Sigma_k)$, we say that the predicate $\Sigma \subseteq N^a \times \omega^b$ is $\mathfrak{B}$-*generic* if for every set $S^* \subseteq N^*$ of codes of (a,b)-strings such that $S^* \in \mathbf{SC}(\mathfrak{B})$, the following holds:

$$\exists \alpha \subseteq \chi_\Sigma \big(\alpha \in S^* \vee \forall \beta \supseteq \alpha (\beta \notin S^*)\big).$$

Note: If we consider a structure $\mathfrak{B}(B)$ with one predicate of naturals and a predicate $\Sigma \subseteq \omega$, then Σ is $\mathfrak{B}$-generic in the sense of Definition 3.2 if and only if the set Σ is B-generic set of natural numbers in the classical sense. The proof uses the definition of $\mathbf{SC}$ via enumerations (1).

Proposition 3.3

For every structure $\mathfrak{B} = \mathfrak{B}(\Sigma_1 \dots \Sigma_k)$ and $a, b \in \omega$, such that $a + b \geq 1$, there exists a $\mathfrak{B}$-generic predicate $\Sigma \subseteq N^a \times \omega^b$.

PROOF:

In this proof and from now on $\overline{x}$ will denote a finite sequence of elements for brevity (an appropriate number of them).

We can find such Σ by building its characteristic function as a union of (a, b)-strings, that we build at stages, such that at even stages we satisfy the requirements $Dom(\chi_\Sigma)$ to be a domain of a predicate's characteristic function and at odd stages - the genericity.

Let us have some enumeration $S_0^* \dots S_n^* \dots$ of the domains the partial functions from $\mathbf{SC}(\mathfrak{B})$, i.e. $S_n^* = Dom(\varphi)$, for $\varphi \in \mathbf{SC}(\mathfrak{B})$.

Stage 0 Define $\alpha_0 = \varnothing$.

Stage 2n+1 We have defined α_q for $q = 2n$. Let $\langle \overline{s}, \overline{x}^* \rangle \in N^*$ be such that $(\overline{s}, \overline{x})$ is *the least* according to some order in $N^a \times \omega^b$ element for which $\langle \overline{s}, \overline{x}^* \rangle \notin Dom(\alpha)$. Define α_{q+1} to extend α_q with one new argument, i.e. such that $\alpha_{q+1}(\langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle) = 0^*$.

Stage 2n+2 We have defined α_q for $q = 2n + 1$.

Case 1. If there exists in S_n^* an (a, b)-string β, extending α_q, define α_{q+1} to be the first such β.

Case 2. Otherwise define $\alpha_{q+1} = \alpha_q$.

Finally we can define $\chi_\Sigma = \bigcup_{q=0}^{\infty} \alpha_q$, that is the characteristic function of some $\mathfrak{B}$-generic predicate. $\square$

Proposition 3.4

Let $\mathfrak{B}$ be an abstract structure and $\Sigma \subseteq N^a \times \omega^b$ a $\mathfrak{B}$-generic predicate. Then the following hold:

P1) The predicate $\overline{\Sigma} \subseteq N^a \times \omega^b$ is $\mathfrak{B}$-generic.

P2) There is no infinite predicate $C \subseteq N^a \times \omega^b$, such that $C \leq_{\mathbf{SC}} \mathfrak{B}$ and $C \subseteq \Sigma$.

P3) Σ is infinite.

P4) $\Sigma \not\leq_{\mathbf{SC}} \mathfrak{B}$.

PROOF:

Each of (P3) and (P4) follows directly from the previous properties. To prove (P1) we may assume it's false. Therefore there is a set of codes of (a, b)-strings, namely $P^* \in \mathbf{SC}(\mathfrak{B})$, such that:

(a) $\forall \alpha \subseteq \chi_{\overline{\Sigma}} \ (\alpha \notin P^* \ \& \ \exists \beta \supseteq \alpha (\beta \in P^*))$.

There is a recursive function translating (codes of) (a, b)-strings into their reverse, e.g. the reverse of α being the (a, b)-string $\overline{\alpha}$, such that $\forall s \in Dom(\alpha)$, $\alpha(x) = 0^*$ iff $\overline{\alpha}(x) = 1^*$. Thus the set $S^* = \{\alpha \mid \overline{\alpha} \in P^*\} \in \mathbf{SC}(\mathfrak{B})$ and therefore there exists $\alpha \subseteq \chi_\Sigma$ (and therefore $\overline{\alpha} \subseteq \chi_{\overline{\Sigma}}$) such that the next (1) or (2) holds:

(1) $\alpha \in S^*$. Then $\overline{\alpha} \in P^*$ and $\overline{\alpha} \subseteq \chi_{\overline{\Sigma}}$, which is a contradiction with (a).

(2) $\forall \beta \supseteq \alpha$ $(\beta \notin S^*)$. But from (a) for $\overline{\alpha}$ follows there exists an (a,b)-string $\beta \in P^*$ extending $\overline{\alpha}$. Since $\overline{\overline{\beta}} = \beta$, we have that $\beta \in S^*$ and $\beta \supseteq \alpha$, which is a contradiction.

In both cases we found a contradiction, therefore Σ is $\mathfrak{B}$-generic.

To prove (P2) we may assume there exists such $C \subseteq N^a \times \omega^b$ and define a set $S^* = \{\alpha \mid \exists s_1 \dots s_a \in N, \exists y_1 \dots y_b \in \omega\, ((s_1 \dots s_a, y_1 \dots y_b) \in C \;\&\; \alpha(\langle s_1 \dots s_a, y_1^* \dots y_b^* \rangle) = 1^*)\}$, that will lead to contradiction.

$\square$

Definition 3.5

Let us define the structure $\mathfrak{A}(\Sigma_1^{\mathfrak{A}} \dots \Sigma_n^{\mathfrak{A}})$ to be total iff $\overline{\Sigma_i^{\mathfrak{A}}} \leq_{\mathbf{SC}} \mathfrak{A}$ for $1 \leq i \leq n$. The generalization the *quasi-minimal* and the *minimal-like* (see [1]) will have the following form:

1. $\mathfrak{A}$ is *quasi-minimal* over $\mathfrak{B}$ if the following two conditions hold:
   - $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$ and $\mathfrak{A} \not\leq_{\mathbf{SC}} \mathfrak{B}$;
   - For every total structure $\mathfrak{C}$, if $\mathfrak{C} \leq_{\mathbf{SC}} \mathfrak{A}$ then $\mathfrak{C} \leq_{\mathbf{SC}} \mathfrak{B}$.
2. $\mathfrak{A}$ is *minimal-like* over $\mathfrak{B}$, if the following two conditions hold:
   - $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$ and $\mathfrak{A} \not\leq_{\mathbf{SC}} \mathfrak{B}$;
   - For every function $\varphi \in \mathcal{F}_{a,b}$, if $\varphi \in \mathbf{SC}(\mathfrak{A})$, there exists a function $\psi \in \mathcal{F}_{a,b}$ such that $\varphi \subseteq \psi$ and $\psi \in \mathbf{SC}(\mathfrak{B})$.

For the (a,b)-string α we define a predicate α^+ to be the set $\{(s_1 \dots s_a, x_1 \dots x_b) | \alpha(\langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle) = 0^*\}$.

If Σ_0 is a predicate and $\mathfrak{B}(\Sigma_1 \dots \Sigma_k)$ a structure, we denote by $\Sigma_0 \oplus \mathfrak{B}$ the two-sort structure with predicates $\Sigma_0, \Sigma_1 \dots \Sigma_k$.

Proposition 3.6

For given $\mathfrak{B}(\Sigma_1 \dots \Sigma_k)$ and $\mathfrak{B}$-generic predicate Σ_0, the structure $\Sigma_0 \oplus \mathfrak{B}$ is minimal-like over $\mathfrak{B}$.

PROOF:

Since Σ_0 is $\mathfrak{B}$-generic, $\Sigma_0 \not\leq_{\mathbf{SC}} \mathfrak{B}$ and therefore $\mathfrak{B} \lneqq_{\mathbf{SC}} \Sigma_0 \oplus \mathfrak{B}$. Let (a_i, b_i) be the arity of the predicate $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$.

For $\varphi \in \mathcal{F}_{a,b}$, such that $\varphi \in \mathbf{SC}(\Sigma_0 \oplus \mathfrak{B})$, we define its graph-predicate Σ_φ for which $\widehat{\Sigma}_\varphi \in \mathbf{SC}(\Sigma_0 \oplus \mathfrak{B})$, i.e. $\Sigma_\varphi \leq_{\mathbf{SC}} \Sigma_0 \oplus \mathfrak{B}$ and from the *Normal form theorem*(2.5) it follows that Σ_φ has normal form in $\Sigma_0 \oplus \mathfrak{B}$. i.e. there are $e \in \omega$, N-string δ_N and $z_1 \dots z_a \notin Dom(\delta_N)$, such that for all $s_1, \dots, s_a \in N$ and $x_1 \dots x_b, y \in \omega$, $(s_1 \dots s_a, x_1 \dots x_b, y) \in \Sigma_\varphi$ iff $\exists \tau_N \supseteq \delta_N$ s.t. $(\tau_N(z_i) = s_i \;\&\; \tau_N \Vdash_{\Sigma_0 \oplus \mathfrak{B}} F_e(\langle z_1, \dots, z_a, x_1 \dots x_b, y \rangle)$. If by $P_{a,b}$ we denote the set of codes of all (a,b)-strings and by P_N the set of all codes of N-strings, we may define the set S^* to be the set of all $\beta_0 \in P_{a_0,b_0}$, for which there exist $\beta_i \in P_{a_i,b_i}$ for $\forall_{1 \leq i \leq k}$, s.t. $\beta_i^+ \subseteq \Sigma_i$, and there exist $\tau_N^1, \tau_N^2 \in P_N$, both extending δ_N and such that $z_1 \dots z_a \in$

$Dom(\tau_N^1) \cap Dom(\tau_N^2)$, and there exist natural numbers $x_1 \dots x_b \in \omega, y_1 \neq y_2 \in \omega$, such that $\tau_N^\varepsilon \Vdash_{\mathfrak{A}(\beta_0^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1 \dots z_a, x_1 \dots x_b, y_\varepsilon \rangle)$ for each $\varepsilon \in \{1, 2\}$, where $\mathfrak{A}(\beta_0^+, \beta_1^+, \dots, \beta_k^+)$ denotes the structure with finite predicates $\beta_i^+ \subseteq N^{a_i} \times \omega^{b_i}$. Therefore $S^* \in \mathfrak{B}$ and there is an (a_0, b_0)-string $\alpha \subseteq \chi_{\Sigma_0}$, such that $\alpha \in S^*$ or $\forall \beta \supseteq \alpha \ (\beta \notin S^*)$.

In the first case, since $\alpha \subseteq \chi_{\Sigma_0}$, then $\alpha^+ \subseteq \Sigma_0$, and from $\tau_N^\varepsilon \Vdash_{\mathfrak{A}(\alpha^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1 \dots z_a, x_1 \dots x_b, y_\varepsilon \rangle)$ follows that $\tau_N^\varepsilon \Vdash_{\Sigma_0 \oplus \mathfrak{B}} F_e(\langle z_1 \dots z_a, x_1 \dots x_b, y_\varepsilon \rangle)$, and using the normal form of Σ_φ we obtain a contradiction. So it remains the second case $\forall \beta \supseteq \alpha (\beta \notin S^*)$ and now we can define a predicate Σ_ψ as follows:
$\Sigma_\psi = \{(s_1 \dots s_a, x_1 \dots x_b, y) \mid \ \big(\exists \beta_0 \in P_{a_0, b_0} \dots \exists \beta_k \in P_{a_k, b_k}, \ \exists \tau_N \in P_N\big)$ s.t. $\big(\beta_0 \supseteq \alpha \ \& \ \forall_{1 \leq i \leq k} \ \beta_i^+ \subseteq \Sigma_i \ \& \ \tau_N \supseteq \delta_N \ \& \ \forall_{1 \leq j \leq a} \ \tau_N(z_j) = s_j \ \& \ \tau_N \Vdash_{\mathfrak{A}(\beta_0^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1 \dots z_a, x_1 \dots x_b, y \rangle)$, which is the graph-predicate of some function ψ and is search computable in $\mathfrak{B}$, therefore $\psi \in \mathbf{SC}(\mathfrak{B})$.

Using the above definition and the normal form of Σ_φ it is not difficult to verify that $\Sigma_\varphi \subseteq \Sigma_\psi$, from which follows that $\varphi \subseteq \psi$, and this proves our proposition.

$\square$

Given a structure $\mathfrak{C}(\Sigma_1 \dots \Sigma_k)$ and a predicate $\Sigma \subseteq N^a \times \omega^b$, if $\Sigma \leq_{\mathbf{SC}} \mathfrak{C}$ and $\overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{C}$, then its characteristic function $\chi_\Sigma \in \mathbf{SC}(\mathfrak{C})$. This fact can be used to prove the following:

Proposition 3.7
Given a structure $\mathfrak{B}(\Sigma_1 \dots \Sigma_k)$ and a $\mathfrak{B}$-generic predicate Σ, the structure $\Sigma \oplus \mathfrak{B}$ with predicates $\Sigma, \Sigma_1, \dots, \Sigma_k$ is quasi-minimal over $\mathfrak{B}$.

The above is true for a single predicate, but not in the general case with multiple $\mathfrak{B}$-generic predicates. For example, for any total structure $\mathfrak{A}(\Sigma, \overline{\Sigma})$ with $\mathfrak{B}$-generic predicates Σ and $\overline{\Sigma} \subseteq N^a \times \omega^b$, the structure $\mathfrak{A} \oplus \mathfrak{B}$ is not quasi-minimal over $\mathfrak{B}$.

4 Generic regular enumerations

The regular enumerations are introduced by I.Soskov in [5] and here we shall use their modification for two-sort structures. An enumeration for two-sort structures is the pair $f = (f_N, f_\omega)$, where $f_N : \omega \to N$ and $f_\omega : \omega \to \omega$ are total surjective functions.

$Gr(f_N) = \{(s, x) \mid f_N(x) = s\} \subseteq N \times \omega$ is the graph of f_N.

$Gr(f_\omega) = \{\langle x, y \rangle \mid f_\omega(x) = y\} \subseteq \omega$ is the graph of f_ω.

The enumerations $f = (f_N, f_\omega)$, define a unique structure $\mathfrak{A}(Gr(f_N), Gr(f_\omega))$, denoted by $\mathfrak{A}_f$.

Since every two-sort structure (with finite number of predicates) is equivalent, in terms of search computability, to a structure with one predicate, in this section we consider only structures with one predicate.

Definition 4.1

Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that the enumeration $f = (f_N, f_\omega)$ is $\mathfrak{B}$-regular, if the function f_ω is $f_N^{-1}(\Sigma^{\mathfrak{B}})$-regular enumeration of ω in the sense of [5] and [6], i.e. f_ω is total surjective mapping of ω onto ω, such that $f_\omega(2\omega) = f_N^{-1}(\Sigma^{\mathfrak{B}})$.

Definition 4.2

1) A *pair of strings* $\tau = (\tau_N, \tau_\omega)$ is the pair of an N-string $\tau_N : \omega \multimap\!\!\to N$ and an ω-string $\tau_\omega : \omega \multimap\!\!\to \omega$ (see *Definition 2.1*). The pair $\varnothing = (\varnothing_N, \varnothing_\omega)$ is referred as the empty pair of strings.

2) Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that the pair of strings $\tau = (\tau_N, \tau_\omega)$ is $\mathfrak{B}$-*regular* if $\tau_\omega(2\omega) \subseteq \tau_N^{-1}(\Sigma^{\mathfrak{B}})$, where $\tau_N^{-1}(\Sigma^{\mathfrak{B}})$ $= \{\langle x_1 \dots x_a, y_1 \dots y_b\rangle \in Dom(\tau_N)^a \times \omega^b \mid \& \ (\tau_N(x_1) \dots \tau_N(x_a), y_1 \dots y_b) \in \Sigma^{\mathfrak{B}}\}$ and $\tau_\omega(2\omega) = \{y \mid \exists x \ (\tau_\omega(2x) = y)\}$.

3) The N^*-code of $\tau = (\tau_N, \tau_\omega)$ is denoted by $\ulcorner \tau \urcorner^*$ and defined to be the pair of codes $\ulcorner \tau \urcorner^* = \langle \ulcorner \tau_N \urcorner^*, \ulcorner \tau_\omega \urcorner^* \rangle$, where $\ulcorner \tau_N \urcorner^* = \langle n^*, \langle 1^*, \tau_N(1)\rangle, \dots, \langle n^*, \tau_N(n)\rangle\rangle$ and $\ulcorner \tau_\omega \urcorner^* = \langle m^*, \langle 1^*, (\tau_\omega(1))^*\rangle, \dots, \langle m^*, (\tau_\omega(m))^*\rangle\rangle$, $n = lh(\tau_N)$ and $m = lh(\tau_\omega)$; define $\ulcorner \varnothing_N \urcorner^* = 0^*$ and $\ulcorner \varnothing_\omega \urcorner^* = 0^*$.

4) We say that τ extends σ, write $\sigma \subseteq \tau$, if both $\sigma_N \subseteq \tau_N$ and $\sigma_\omega \subseteq \tau_\omega$. For an enumeration $f = (f_N, f_\omega)$ and a pair of strings $\tau = (\tau_N, \tau_\omega)$, we say that $\tau \subseteq f$ when both $\tau_N \subseteq f_N$ and $\tau_\omega \subseteq f_\omega$.

Remark: Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ let $Reg_{\mathfrak{B}}$ denote the set of codes of all $\mathfrak{B}$-regular pairs of strings. Thus $\tau \in Reg_{\mathfrak{B}} \Leftrightarrow \tau_\omega(2\omega) \subseteq \tau_N^{-1}(\Sigma^{\mathfrak{B}})$, and therefore $Reg_{\mathfrak{B}} \in \mathbf{SC}(\mathfrak{B})$.

Definition 4.3

Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that $f = (f_N, f_\omega)$ is $\mathfrak{B}$-generic regular enumeration, if it is $\mathfrak{B}$-regular enumeration and for every set $S^* \subseteq N^*$ of codes of $\mathfrak{B}(\Sigma^{\mathfrak{B}})$-regular pairs of strings, for which $S^* \in \mathbf{SC}(\mathfrak{B})$, there exists a pair of strings $\tau \subseteq f$, such that $\tau \in S^*$ or $\forall \sigma \supseteq \tau \ (\sigma \notin S^*)$.

Proposition 4.4

For every structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, there exists a $\mathfrak{B}$-generic regular enumeration $f = (f_N, f_\omega)$.

Proof:

Let $S_0^*, \dots, S_n^*, \dots$ be a sequence of all the sets $S^* \in \mathbf{SC}(\mathfrak{B})$ and $s_0, \dots, s_n, \dots$ be all the elements of N. We can build a $\mathfrak{B}$-generic regular enumeration in the standard way starting from the empty pair of strings and building an increasing sequence of $\mathfrak{B}$-regular pair of strings, such that, at even stages we will monitor the n-th set S_n^* and take care to satisfy the requirements for genericity. At odd stages we will satisfy $\tau_\omega^q(2\omega) \subseteq (\tau_\omega^q)^{-1}(\Sigma^{\mathfrak{B}})$ and in the same time $f_N^{-1} \subseteq f_\omega(2\omega)$, as follows:

Suppose at *Stage 2n+1* we have defined $\tau_q = (\tau_N^q, \tau_\omega^q)$, for $q = 2n$. We may define τ_N^{q+1} to extend τ_N^q, so that for $x = lh(\tau_N^q)$, $\tau_N^{q+1}(x) = s_n$. For the set

$(\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}})$ we have two possibilities: if it is empty, define $\tau_\omega^{q+1} = \tau_\omega^q$. Otherwise $(\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}}) \neq \emptyset$. In this case we consider the set $A_q = (\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}}) \setminus \tau_\omega^q(2\omega)$ and define τ_ω^{q+1} to extend τ_ω^q such that in the first odd number $x_1 \notin Dom(\tau_\omega^q)$, define $\tau_\omega^{q+1}(x_1) = n$, and in the first even number $x_0 \notin Dom(\tau_\omega^q)$, define $\tau_\omega^{q+1}(x_0)$ to be the first $y \in A_q$, if $A_q \neq \emptyset$, or the first $y \in (\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}})$, if $A_q = \emptyset$.

In this way we obtain the desired enumeration.

$\square$

To prove the following proposition and the lemma, it may be convenient to define two notations for a $(0,1)$-string α and N-string τ_N :

$cmp(\alpha, \tau_N)$ if and only if $\forall x \in \omega\ (x^* \in Dom(\alpha) \Leftrightarrow x \in Dom(\tau_N))$,

$\alpha \sim_\Sigma \tau_N$ if and only if $\forall x^* \in Dom(\alpha)\ \big(\alpha(x^*) = 0^* \Leftrightarrow \tau_N(x) \in \Sigma\big)$.

Proposition 4.5

For a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$ and a $\mathfrak{B}$-generic regular enumeration $f = (f_N, f_\omega)$ the following properties hold:

1) $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}_f$

2) $\mathfrak{A}_f \not\leq_{\mathbf{SC}} \mathfrak{B}$

3) For every predicate $\Sigma \subseteq N^a \times \omega^b$, if $\Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{A}_f$, then $\overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$.

4) For every predicate $\Sigma \subseteq N$, if $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$, then $f_N^{-1}(\Sigma)$ is $\mathfrak{B}$-generic predicate.

5) For every predicate $\Sigma \subseteq N$, if $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$, the structure $\mathfrak{A}(f_N^{-1}(\Sigma), \Sigma^{\mathfrak{B}})$ is quasi-minimal over $\mathfrak{B}$.

PROOF:

These properties follow easily from the definitions and the properties of the enumerations. For example, for the proof of (4) we may assume that $f_N^{-1}(\Sigma)$ is not $\mathfrak{B}$-generic predicate. Then there exists a set of $(0,1)$-strings S, that fails the genericity, and consider the set of $\mathfrak{B}$-regular pairs of strings:

$$P^* = \{\tau \in Reg_{\mathfrak{B}} \mid \exists \alpha \in S(cmp(\alpha, \tau_N)\ \&\ \alpha \sim_\Sigma \tau_N)\}.$$

Since for each τ there is a unique α, such that $cmp(\alpha, \tau_N)$ and $\alpha \sim_\Sigma \tau_N$, and for each α there is such τ_N, we can obtain a contradiction with the genericity of f.

$\square$

Lemma 4.6

Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, given a pair of strings δ, a $\mathfrak{B}$-generic predicate $A \subseteq \omega$ and a predicate $\Sigma \subseteq N$, such that $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$, for which the following two conditions hold:

(1) δ is $\mathfrak{B}$-regular;

(2) $\forall x < lh(\delta_N)\ (x \in A \Leftrightarrow \delta_N(x) \in \Sigma)$,

If $S^* \subseteq N^*$ is a set of codes of $\mathfrak{B}$-regular pairs of strings and $S \leq_{\mathbf{SC}} \mathfrak{B}$, then there exists a pair of strings σ with the following properties:

(a) $\sigma \supseteq \delta$;

(b) σ is $\mathfrak{B}$-regular;

(c) $\forall x < lh(\sigma_N)\ (x \in A \Leftrightarrow \sigma_N(x) \in \Sigma)$, (this is the property (2) for σ);
(d) $\sigma \in S \vee \forall \tau(\tau \supseteq \sigma \Rightarrow \tau \notin S)$.

PROOF:

The proof is very similar to the one of the corresponding lemma in the classical case (Lemma 2.4. in [6]).

□

Proposition 4.7

Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with $\varnothing \neq \Sigma^{\mathfrak{B}}$, a $\mathfrak{B}$-generic predicate $A \subseteq \omega$, and a predicate $\Sigma \subseteq N$, such that $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$, there exists a $\mathfrak{B}$-generic regular enumeration f, such that $A = f_N^{-1}(\Sigma)$.

PROOF:

We can build f by the standard construction of increasing sequence of pairs of strings σ_q, (starting from the empty pair of strings), with the properties (1) and (2) from the above lemma. Moreover, we want them to satisfy four additional properties:
(3) $\exists n \forall e \geq n\ \big(lh(\sigma_N^{2e+1}) \geq lh(\sigma_N^{2e})$ and $lh(\sigma_\omega^{2e+1}) \geq lh(\sigma_\omega^{2e})\big)$.
(4) $\forall s \in N\ \exists e\ (s \in Range(\sigma_N^{2e+1}))$ and $\forall y \in \omega\ \exists e\ (y \in Range(\sigma_\omega^{2e+1}))$.
(5) $\forall p\ \forall x \in (\sigma_N^p)^{-1}(\Sigma^{\mathfrak{B}})\ \exists e\ (x \in \sigma_\omega^{2e+1}(2\omega))$.
(6) $\forall e\ \big($ if $S_e \subseteq Reg_{\mathfrak{B}}$ then $(\sigma_{2e+2} \in S_e\ \vee\ \forall \tau \supseteq \sigma_{2e+2}\ (\tau \notin S_e))\big)$, where S_e is the e-th search computable in $\mathfrak{B}$ set in some given enumeration of all the sets from $\mathbf{SC}(\mathfrak{B})$, and $Reg_{\mathfrak{B}}$ is the set of the $\mathfrak{B}$-regular pair of strings.

These requirements guarantee that $f = \bigcup_{q=0}^{\infty} \sigma_q$ will be a $\mathfrak{B}$-generic regular enumeration and $A = f_N^{-1}(\Sigma)$.

Stage 2e+1 Suppose σ_q is defined, for $q = 2e$. Define σ_N^{q+1} to extend σ_N^q with new elements and to have the property (2) defined in the previous lemma. If $(\sigma_N^q)^{-1}(\Sigma^{\mathfrak{B}})$ is empty, we define $\sigma_\omega^{q+1} = \sigma_\omega^q$, otherwise define σ_ω^{q+1} to extend σ_ω^q with the first two elements $x_0 \in 2\omega \setminus Dom(\sigma_\omega^q)$ and $x_1 \in (2\omega + 1) \setminus Dom(\sigma_\omega^q)$, for which:

- $\sigma_\omega^{q+1}(x_1) =$ the first y s.t. $y \notin Range(\sigma_\omega^q)$;
- $\sigma_\omega^{q+1}(x_0) =$ the first y s.t. $y \in (\sigma_\omega^q)^{-1}(\Sigma^{\mathfrak{B}}) \setminus \sigma_\omega^q(2\omega)$ if not empty, or the first $y \in (\sigma_\omega^q)^{-1}(\Sigma^{\mathfrak{B}})$ otherwise.

Stage 2e+2 Suppose σ_q is defined, for $q = 2e + 1$. Let G be the set of all pairs of strings having the properties (1) and (2) from the previous lemma. We have two possibilities:

Case 1. $\exists \sigma \supseteq \sigma_q\ \big(\sigma \in G\ \&\ (\sigma \in S_e\ \vee\ \forall \tau \supseteq \sigma\ (\tau \notin S_e))\big)$. Define σ_{q+1} to be the first such σ.

Case 2. Otherwise define $\sigma_{q+1} = \sigma_q$.

Now it can be verified that this construction meets the requirements (3) to (6), defined earlier in the current proof. For example to verify (6) we can use the previous lemma to show that Case 2 never happens if S_e is a set of $\mathfrak{B}$-regular pair of strings.

$\square$

Theorem 4.8

Let a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, be given.

Then for any predicate $A \subseteq \omega$, A is $\mathfrak{B}$-generic if and only if there exist a predicate $\Sigma \subseteq N$, such that $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$, and there exists a $\mathfrak{B}$-generic regular enumeration f, such that $A = f_N^{-1}(\Sigma)$.

PROOF:

($\Leftarrow$) The *Proposition 4.5-4*.

($\Rightarrow$) Consider the predicate $\Sigma = \{s\}$ for which it is clear that $\varnothing \neq \Sigma \leq_{\mathbf{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{B}$. From previous proposition it follows that there exists a $\mathfrak{B}$-generic regular enumeration f, such that $A = f_N^{-1}(\Sigma)$.

$\square$

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