

On the Abstract Computability in Two-sort Structures

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1 Computability in N^*

Basic notions and definitions

Let N be an arbitrary countable set. Take 0^* which is not an element of N and an operation *ordered pair*, denoted by $\langle \cdot, \cdot \rangle$ such that no element of $N \cup \{0^*\}$ is an ordered pair. By induction we define the sets $N_0 = N \cup \{0^*\}$ and $N_{n+1} = N_n \cup N_n^2$, where $N_n^2 = \{\langle a, b \rangle \mid a \in N_n \ \& \ b \in N_n\}$. Finally, define the Moskovakis' extension of N , $N^* = \bigcup_{n=0}^{\infty} N_n$.

For $s_1 \dots s_{n+1} \in N^*$ we use $\langle s_1, \dots, s_{n+1} \rangle$ to denote $\langle s_1, \langle s_2, \dots, s_{n+1} \rangle \rangle$.

Recall that ω is the set of all natural numbers. By induction we define the set $\omega^* \subseteq N^*$ as follows:

$$\begin{cases} 0^* \in \omega^* \\ (n+1)^* = \langle 0^*, n^* \rangle \in \omega^* \end{cases}$$

Define the *left* and *right* functions L and $R : N^* \dashrightarrow N^*$ as follows:

$$\begin{cases} L(0^*) = R(0^*) = 0^* \\ L(n) = R(n) = 1^* & \text{for } n \in N \\ L(\langle s, t \rangle) = s, R(\langle s, t \rangle) = t \end{cases}$$

By F we denote the set of one argument partial functions $\varphi : N^* \dashrightarrow N^*$. Define the following operations on functions in F :

- 1. composition** $(\varphi \circ \psi) \in F$
($\forall s \in N^*$) ($\forall t \in N^*$)
 $(\varphi \circ \psi)(s) = t \Leftrightarrow (\exists p \in N^*)(\psi(s) = p \ \& \ \varphi(p) = t)$
- 2. pairing** $\Pi(\varphi, \psi) \in F$
($\forall s \in N^*$) ($\forall t \in N^*$)

$$\Pi(\varphi, \psi)(s) = t \Leftrightarrow (\exists p, q \in N^*) (\langle p, q \rangle = t \ \& \ \varphi(s) = p \ \& \ \psi(s) = q)$$

3. iteration $[\varphi, \psi] \in F$

$$(\forall s \in N^*) (\forall t \in N^*)$$

$$[\varphi, \psi](s) = t \Leftrightarrow \exists w_0, \dots, w_n \in N^* (w_0 = s \ \& \ w_n = t \ \& \ \psi(w_0) \in N_0 \ \& \ \forall i_{(0 \leq i < n)} (\varphi(w_i) = w_{i+1} \ \& \ \psi(w_i) \notin N_0))$$

We define the constant functions as follows: if $c \in N^*$, by \check{c} we denote the function $\check{c}: N^* \dashrightarrow N^*$, such that $\check{c}(s) = c$ for all $s \in N^*$.

Definition 1.1

The function $\varphi \in F$ is said to be *prime computable using constants*, relatively the set $\{\psi_1, \dots, \psi_k\}$, where $\psi_i \in F$ (write $\varphi \in \mathbf{PC}(\psi_1 \dots \psi_k)$), if it can be expressed with functions from $\{L, R, \psi_1, \dots, \psi_k\} \cup \{\check{c}_1, \dots, \check{c}_n, \dots\}$, where $\check{c}_1, \dots, \check{c}_n, \dots$ is an enumeration of the elements of N , using composition, pairing and iteration. The function φ is said to be *prime computable with constants* (write $\varphi \in \mathbf{PC}$), if it is prime computable relatively the empty set of functions, i.e. $\varphi \in \mathbf{PC}(\emptyset)$.

The function $\varphi \in F$ is said to be *prime computable* relatively the set $\{\psi_1, \dots, \psi_k\}$ (write $\varphi \in PC(\psi_1 \dots \psi_k)$), if it can be expressed with functions from $\{L, R, \psi_1, \dots, \psi_k\}$ using composition, pairing and iteration. The definition of $\varphi \in PC$ is analogous.

Definition 1.2 The function $\varphi \in F$ is said to be *search computable using constants*, relatively $\psi_1, \dots, \psi_k \in F$, write $\varphi \in \mathbf{SC}(\psi_1 \dots \psi_k)$, if there exists a function $\psi \in \mathbf{PC}(\psi_1, \dots, \psi_k)$, such that

$$(\forall s \in N^*) (\forall t \in N^*) (\varphi(s) = t \iff \exists r \in N^* (\psi(\langle r, s \rangle) = t)).$$

In the same way we define $\varphi \in \mathbf{SC}$, $\varphi \in SC(\psi_1 \dots \psi_k)$ and $\varphi \in SC$.

Examples The following functions are *PC* :

1. $0(s) = 0^*$ for $s \in N^*$ $\bar{0} = L^3 \circ [L, L]$
 $\bar{n}(s) = n^*$ $\overline{(n+1)} = \Pi(\bar{0}, \bar{n})$
2. $I(s) = s$ for $s \in N^*$ $I = [L, \bar{0}]$
3. The nowhere defined function \emptyset^* : $\emptyset^* = [I, I] \circ \Pi(I, I)$
4. $\chi_{N_0}(s) = \begin{cases} 0^* & , s \in N_0 \\ 1^* & , s \in N^* \setminus N_0 \end{cases}$ $\chi_{N_0} = L \circ [\Pi(\bar{1}, \bar{0}), R] \circ \Pi(\bar{0}, I)$

$$\begin{aligned}
5. \quad Sg(s) &= \begin{cases} 0^* & , s = 0^* \\ 1^* & , s \in N^* \setminus \{0^*\} \end{cases} & Sg = L \circ [\Pi(\bar{1}, \bar{0}), R] \circ \Pi(L, I) \\
6. \quad \overline{Sg}(s) &= \begin{cases} 0^* & , Sg(s) = 1^* \\ 1^* & , Sg(s) = 0^* \end{cases} & \overline{Sg} = L \circ [\Pi(\bar{0}, \bar{0}), R] \circ \Pi(\bar{1}, Sg) \\
7. \quad \chi_{\omega^*}(s) &= \begin{cases} 0^* & , s \in \omega^* \\ 1^* & , s \in N^* \setminus \omega^* \end{cases} & \chi_{\omega^*} = L \circ [\varphi, R] \circ \Pi(L, I), \text{ where} \\
\varphi(\langle s, \langle s, t \rangle \rangle) &= \begin{cases} \langle 1^*, 0^* \rangle & , s \neq 0^* \\ \langle L(t), t \rangle & , s = 0^* \end{cases} & \varphi = L \circ [\psi, \overline{Sg} \circ L \circ R] \circ \Pi(\Pi(\bar{1}, \bar{0}), I) \\
& & & \text{and } \psi = \Pi(\Pi(L \circ R^3, R^3), L) \\
8. \quad And(s) &= \begin{cases} 0^* & L(s) = R(s) = 0^* \\ 1^* & \text{otherwise} \end{cases} & And = Sg \circ L \circ [\Pi(\bar{1}, \bar{0}), Sg \circ R] \\
Or(s) &= \begin{cases} 0^* & L(s) = 0^* \text{ or } R(s) = 0^* \\ 1^* & \text{otherwise} \end{cases} & Or = \overline{Sg} \circ And \circ \Pi(\overline{Sg} \circ L, \overline{Sg} \circ R) \\
9. \quad N^k &= \{\langle s_1 \dots s_k \rangle \mid s_1, \dots, s_k \in N\} \\
\quad N^1 &= N \\
\chi_{N^k}(s) &= \begin{cases} 0^* & , s \in N \\ 1^* & , s \notin N \end{cases} & \chi_N = \overline{Sg} \circ [\bar{0}, I] \\
& & & \chi_{N^{k+1}} = And \circ \Pi(\chi_N \circ L, \chi_{N^k} \circ R)
\end{aligned}$$

10. If $\varphi \in PC(\psi_1, \dots, \psi_n)$, $\psi \in PC(\psi_1, \dots, \psi_n)$ and $\xi \in PC(\psi_1, \dots, \psi_n)$, then the function $If(\xi, \varphi, \psi)$ is $PC(\psi_1, \dots, \psi_n)$, where

$$If(\xi, \varphi, \psi)(s) = \begin{cases} \varphi(s) & , \text{ if } \xi(s) \in N_0 \\ \psi(s) & , \text{ if } \xi(s) \in N^* \setminus N_0 \end{cases}$$

$$If(\xi, \varphi, \psi) = L \circ [\Pi(\varphi \circ L, \Pi(\bar{1}, \bar{0})), \overline{Sg} \circ L \circ R] \circ [\Pi(\psi \circ L, \Pi(\bar{1}, \bar{0})), R^2] \circ \Pi(I, \Pi(I, I) \circ \chi_{N_0} \circ \xi)$$

11. If $\varphi \in PC(\psi_1, \dots, \psi_n)$ and $\psi \in PC(\psi_1, \dots, \psi_n)$, then there exists a function $\chi \in PC(\psi_1, \dots, \psi_n)$, such that:

$$\left| \begin{array}{l} \chi(\langle 0^*, s \rangle) \simeq \varphi(s) \\ \chi(\langle (x+1)^*, s \rangle) \simeq \psi(\langle x^*, s, \chi(\langle x^*, s \rangle) \rangle) \end{array} \right. ,$$

for all $s \in N^*$ and $x \in \omega^*$; in the other cases χ is not defined.

$\chi = If(\zeta, \xi, \varnothing^*)$, where $\zeta = And(\overline{Sg} \circ \chi_{N_0}, \chi_{\omega^*})$ and $\xi = R^3 \circ \xi_2 \circ \xi_1$, where $\xi_1 = \Pi(L, \Pi(\bar{0}, \Pi(I, \varphi) \circ R))$, $\xi_2 = [\Pi(R \circ L, \Pi(\Pi(\bar{0}, L), \Pi(L \circ R, \psi)) \circ R), L]$.

12. If $\psi \in PC(\psi_1, \dots, \psi_n)$, then there exists a function $\varphi \in PC(\psi_1, \dots, \psi_n)$, such that:

$$\varphi(s) \simeq \mu^* x_{x^* \in \omega^*} [\psi(\langle x^*, s \rangle) \in N_0],$$

i.e. $Rng(\varphi) \subseteq \omega^*$ and $\varphi(s) = x^* \Leftrightarrow \psi(\langle x^*, s \rangle) \in N_0 \ \& \ \forall y < x (\psi(\langle y^*, s \rangle) \downarrow \ \& \ (\psi(\langle y^*, s \rangle) \notin N_0))$, for all $s \in N^*$. Indeed $\varphi = L \circ [\Pi(\Pi(\bar{0}, L), I), \psi] \circ \Pi(\bar{0}, I)$.

The proof of 10, 11 and 12 for **PC** is similar.

Definition 1.3 Having the partial function $\rho : \omega^n \dashrightarrow \omega$, we can define its corresponding function $\rho^* \in F$ as follows:

$$\begin{aligned} \text{for } n \geq 2 \quad : \quad \rho^*(s) = t \quad \Leftrightarrow \quad & \exists x_1 \dots x_n, y \in \omega (s = \langle x_1^* \dots x_n^* \rangle \ \& \\ & y^* = t \ \& \ \rho(x_1, \dots, x_n) = y); \\ \text{for } n = 1 \quad : \quad \rho^*(s) = t \quad \Leftrightarrow \quad & \exists x, y \in \omega (s = x^* \ \& \ y^* = t \ \& \ \rho(x) = y), \end{aligned}$$

for all s and t in N^* .

Proposition 1.4

If the function $\rho : \omega^n \dashrightarrow \omega$ is partial recursive relatively ψ_1, \dots, ψ_k , where each $\psi_i : \omega^{n_i} \dashrightarrow \omega$, then $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$.

PROOF:

Since ρ is partial recursive relatively ψ_1, \dots, ψ_k , it can be expressed with functions from $\{O, S, I_1^m, \dots, I_m^m\} \cup \{\psi_1, \dots, \psi_k\}$, using *superposition*, *prime recursion* and *minimization*. We will prove that $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$ using induction of the structure of ρ :

Let $\rho = O$. Define $\rho^{*'} = \bar{0}$. Then $\rho^* = If(\chi_{\omega^*}, \rho^{*'}, \emptyset^*) \in PC$;

Let $\rho = S$ and define $\rho^{*'} = \Pi(\bar{0}, I)$. Then $\rho^* = If(\chi_{\omega^*}, \rho^{*'}, \emptyset^*) \in PC$;

Let $\rho = I_i^m$ and define $\rho^* = L \circ R^{i-1} \in PC$ for $m > 1$, and for $m = 1$ define $\rho^* = I \in PC$.

Superposition: Let $\rho(x_1 \dots x_n) \simeq f(g_1(x_1 \dots x_n) \dots g_s(x_1 \dots x_n))$ and f, g_1, \dots, g_s are partial recursive relatively ψ_1, \dots, ψ_k . By the induction hypothesis it follows that $f^*, g_1^*, \dots, g_s^* \in PC(\psi_1^*, \dots, \psi_k^*)$. But $\rho^* = f^* \circ \Pi_s(g_1^* \dots g_s^*)$, where $\Pi_1(g^*) = g^*$, and $\Pi_{s+1}(g_1^* \dots g_{s+1}^*) = \Pi(g_1^*, \Pi_s(g_2^* \dots g_{s+1}^*))$ are $PC(\psi_1^*, \dots, \psi_k^*)$ and therefore so is ρ^* .

Prime recursion: Let

$$\left| \begin{array}{l} \rho(0, \bar{y}) \quad \simeq \quad \varphi(\bar{y}) \\ \rho(x+1, \bar{y}) \quad \simeq \quad \psi(x, \bar{y}, \rho(x, \bar{y})) \end{array} \right.$$

where φ and ψ are partial recursive relatively ψ_1, \dots, ψ_k and then $\varphi^*, \psi^* \in PC(\psi_1^*, \dots, \psi_k^*)$. But (see ex.11) there exists $\chi \in PC(\psi_1^*, \dots, \psi_k^*)$, such that $\chi = Pr(\varphi^*, \psi^*)$, i.e.

$$\begin{cases} \chi(\langle 0^*, s \rangle) & \simeq \varphi^*(s) \\ \chi(\langle x+1, s \rangle) & \simeq \psi^*(\langle x^*, s, \chi(\langle x^*, s \rangle) \rangle) \end{cases}$$

But $\rho^* = \chi = Pr(\varphi^*, \psi^*)$ and therefore $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$.

Minimization: Let $\rho(x_1 \dots x_n) \simeq \mu y[\psi(y, x_1 \dots x_n) = 0]$ and ψ is partial recursive relatively ψ_1, \dots, ψ_k . Then $\psi^* \in PC(\psi_1^*, \dots, \psi_k^*)$, but (see ex.12) there exists $\varphi \in PC(\psi_1^*, \dots, \psi_k^*)$ and s.t. $\varphi(s) \simeq \mu^* x_{x^* \in \omega^*}[\psi^*(\langle x^*, s \rangle) \in N_0]$, i.e. $\varphi(s) = x^* \Leftrightarrow \psi^*(\langle x^*, s \rangle) \in N_0$ & $\forall y < x$ ($\psi^*(\langle y^*, s \rangle) \downarrow \notin N_0$), for all $s \in N^*$. Then $\varphi(s) = t \Leftrightarrow \exists z, x_1 \dots x_n \in \omega$ ($s = \langle x_1 \dots x_n \rangle$ & $t = z^*$ & $\psi^*(\langle z^*, s \rangle) = 0^*$ & $\forall y < z$ ($\psi^*(\langle y^*, s \rangle) \downarrow \neq 0^*$)), i.e. $\varphi(s) = t$ if and only if $\exists z, x_1 \dots x_n \in \omega$ ($s = \langle x_1 \dots x_n \rangle$ & $t = z^*$ & $\psi(z, x_1 \dots x_n) = 0$ & $\forall y < z$ ($\psi(y, x_1 \dots x_n) \downarrow \neq 0$)), i.e. $\rho^* = \varphi = \mu^* x^*[\psi^*(\langle x^*, \cdot \rangle) \in N_0] \in PC(\psi_1^* \dots \psi_k^*)$. \square

We have the following: $PC \subseteq SC$, $\mathbf{PC} \subseteq \mathbf{SC}$, $PC \subseteq \mathbf{PC}$ and $SC \subseteq \mathbf{SC}$.

In the general case the constant functions are prime computable using constants: $\check{c} \in \mathbf{PC}$, for all $c \in N^*$. And $\check{c} \in PC$, for all $c \in \omega^*$.

2 Abstract structures

Definition 2.1

Let ω be the set of all the natural numbers and N be a countable set. A *structure* we will call an abstract partial two-sort structure

$$\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \dots, \Sigma_k \rangle,$$

with two fixed (basic) predicates in N^2 - equality in N ($=_N$) and inequality in N (\neq_N). $\Sigma_1, \dots, \Sigma_k$ are partial predicates, $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $a_i, b_i \geq 0$ and are not both zero, for all $i, 1 \leq i \leq k$. This kind of structure will be denoted as $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$.

Definition 2.2

For any predicate $\Sigma \subseteq N^a \times \omega^b$ we define its semicharacteristic function $\widehat{\Sigma} : N^* \dashrightarrow N^*$, $\widehat{\Sigma} \in F$, as follows: $Rng(\widehat{\Sigma}) \subseteq \{0^*\}$ and for all $s \in N^*$

$\widehat{\Sigma}(s) = 0^*$ if and only if

$\exists s_1 \dots s_a \in N \exists x_1, \dots, x_b \in \omega$

$(s = \langle s_1 \dots s_a, x_1^* \dots x_b^* \rangle \text{ \& } (s_1 \dots s_a, x_1 \dots x_b) \in \Sigma)$.

Examples:

$=_N = \{(n, n) \mid n \in N\}$,

$\widehat{=}_N(s) = 0^* \Leftrightarrow \exists n \in N (s = \langle n, n \rangle)$

$\neq_N = \overline{=}_N = \{(n, m) \mid n, m \in N \text{ \& } n \neq m\}$

$\widehat{\neq}_N(s) = 0^* \Leftrightarrow \exists n \in N \exists m \in N (s = \langle n, m \rangle \text{ \& } n \neq m)$.

Definition 2.3

Let $\mathfrak{A} = \langle \Sigma_1^{\mathfrak{A}}, \dots, \Sigma_m^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \Sigma_1^{\mathfrak{B}}, \dots, \Sigma_n^{\mathfrak{B}} \rangle$ be structures and $\Sigma_0, \Sigma_1, \dots, \Sigma_k$ be predicates.

1. $\Sigma_0 \leq_{\mathbf{SC}} \{\Sigma_1, \dots, \Sigma_k\}$ if and only if $\widehat{\Sigma}_0 \in \mathbf{SC}(\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_k)$;
2. $\Sigma_0 \leq_{\mathbf{SC}} \mathfrak{A}$ if and only if $\Sigma_0 \leq_{\mathbf{SC}} \{=_N, \neq_N, \Sigma_1^{\mathfrak{A}}, \dots, \Sigma_m^{\mathfrak{A}}\}$;
3. $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ if and only if $\forall i_{1 \leq i \leq m} \Sigma_i^{\mathfrak{A}} \leq_{\mathbf{SC}} \mathfrak{B}$;
4. $\mathfrak{A} \oplus \mathfrak{B} = \langle N; \omega; =_N, \neq_N; \Sigma_1^{\mathfrak{A}} \dots \Sigma_m^{\mathfrak{A}}, \Sigma_1^{\mathfrak{B}} \dots \Sigma_n^{\mathfrak{B}} \rangle = \mathfrak{C}(\Sigma_1^{\mathfrak{A}} \dots \Sigma_m^{\mathfrak{A}}, \Sigma_1^{\mathfrak{B}} \dots \Sigma_n^{\mathfrak{B}})$.
5. $\mathfrak{A} \equiv_{\mathbf{SC}} \mathfrak{B}$ if and only if $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$.

In the same way we define the relation \leq_{SC} (without constants) but henceforth we consider the relation $\leq_{\mathbf{SC}}$ only.

The relation $\leq_{\mathbf{SC}}$ is a preorder (reflexive and transitive relation), and $\equiv_{\mathbf{SC}}$ is an equivalence relation.

S-degrees or *structure degrees* we call the equivalence classes, induced by the relation $\equiv_{\mathbf{SC}}$ between structures: given a structure \mathfrak{A} , an **S**-degree of \mathfrak{A} we denote its structure degree by $\mathfrak{D}_S(\mathfrak{A})$ and mean the set $\{\mathfrak{B} \mid \mathfrak{B} \equiv_{\mathbf{SC}} \mathfrak{A}\}$.

The set $\{\mathfrak{D}_S(\mathfrak{A}) \mid \mathfrak{A} \text{ is an abstract structure}\}$ we denote by \mathfrak{D} . For \mathfrak{a} and \mathfrak{b} in \mathfrak{D} , $\mathfrak{a} \leq_{\mathbf{sc}} \mathfrak{b}$ if and only if there exist structures \mathfrak{A} in \mathfrak{a} and \mathfrak{B} in \mathfrak{b} , such that $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$. For \mathfrak{a} and \mathfrak{b} in \mathfrak{D} and for $\mathfrak{A} \in \mathfrak{a}$ and $\mathfrak{B} \in \mathfrak{b}$ we define $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{D}_s(\mathfrak{A} \oplus \mathfrak{B})$. So the structure $S = \langle \mathfrak{D}, \leq_{\mathbf{SC}}, \cup, \mathfrak{D} \rangle$ is an upper semilattice with least element the empty structure $\mathfrak{D} = \langle N; \omega; =_N, \neq_N \rangle$.

Lemma 2.4

The following functions are prime computable (*PC*):

1. The functions $\varphi_{n,m}$, defined as follows:

$$\varphi_{0,1}(s) = \begin{cases} 0^* & , \text{ if } s \in \omega^* \\ 1^* & , \text{ otherwise} \end{cases} \text{ and } \varphi_{1,0}(s) = \begin{cases} 0^* & , \text{ if } s \in N \\ 1^* & , \text{ otherwise} \end{cases} \text{ and}$$

$$\text{for } n + m \geq 2, \varphi_{n,m}(s) = \begin{cases} 0^* & , \text{ for } s = \langle s_1 \dots s_n, t_1 \dots t_m \rangle \text{ where} \\ & s_1, \dots, s_n \in N \text{ and } t_1, \dots, t_m \in \omega^* . \\ 1^* & , \text{ otherwise} \end{cases}$$

Indeed $\varphi_{0,1} = \chi_{\omega^*}$, $\varphi_{0,m+1} = \text{And} \circ \Pi(\overline{Sg} \circ \chi_{N_0}, \text{And} \circ \Pi(\chi_{\omega^*} \circ L, \varphi_{0,m} \circ R))$ for $m \geq 1$, $\varphi_{1,0} = \chi_N$, $\varphi_{n+1,m} = \text{And} \circ \Pi(\chi_N \circ L, \varphi_{n,m} \circ R)$, for $m \geq 0$ and $n \geq 0$ and $n + m \geq 1$.

2. The functions $\psi_{n,m}$, where for all u and s_j in N^* , $\psi_{n,m}(\langle u, \langle s_1 \dots s_n, \dots \rangle \rangle = \langle s_1 \dots s_n, \underbrace{u \dots u}_{m \text{ times}}, \dots \rangle$, $n \geq 0$, $m \geq 0$.

Indeed $\psi_{0,m+1} = \psi_{0,m} \circ \Pi(L, I)$, for $m \geq 0$; and $\psi_{n+1,m} = \Pi(L \circ R, \psi_{n,m} \circ \Pi(L, R^2))$, for $n, m \geq 0$. Therefore $\psi_{n,m} \in PC$.

3. The functions γ_p^i , such that for all $s_1, \dots, s_p \in N^*$ and $p \geq 1, i \in \omega$, $\gamma_p^i(\langle s_1 \dots s_p \rangle) = \langle s_1 \dots s_p, i^* \rangle = \langle s_1, \langle s_2 \dots \langle s_p, i^* \rangle \dots \rangle \rangle$.

Indeed $\gamma_1^i = \Pi(I, \bar{i}^*)$, $\gamma_{p+1}^i = \Pi(L, \gamma_p^i \circ R)$, i.e. $\gamma_p^i = \psi_{p,i} \circ \Pi(\bar{i}^*, I)$. Therefore $\gamma_p^i \in PC$.

4. The functions Lt_n , $n \geq 1$, such that for u, s_j in N^* , $Lt_n(\langle s_1 \dots s_n \rangle) = s_n$.

Indeed $Lt_1 = I$ and $Lt_{n+1} = Lt_n \circ R$, for $n \geq 1$.

5. The functions Fst_n , $n \geq 1$, such that for u, s_j in N^* , $Fst_n(\langle s_1 \dots s_n \dots \rangle) = \langle s_1 \dots s_n \rangle$.

Indeed $Fst_1 = L$ and $Fst_{n+1} = \Pi(L, Fst_n \circ R)$, for $n \geq 1$.

□

Corollary 2.5

For every structure $\mathfrak{B}^k(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $1 \leq i \leq k$, there exists a structure \mathfrak{B} with one predicate $\mathfrak{B}(\Sigma)$, $\Sigma \subseteq N^a \times \omega^{b+1}$, where $a = \max\{a_i \mid 1 \leq i \leq k\}$ and $b = \max\{b_i \mid 1 \leq i \leq k\}$, such that $\mathfrak{B}^k \equiv_{\text{SC}} \mathfrak{B}$.

PROOF:

Let $a = \max\{a_i \mid 1 \leq i \leq k\}$ and $b = \max\{b_i \mid 1 \leq i \leq k\}$ and define a predicate $\Sigma \subseteq N^a \times \omega^{b+1}$, as follows:

$$\Sigma = \{(s_1 \dots s_a, x_1 \dots x_b, i) \mid 1 \leq i \leq k \ \& \ s_1 \dots s_a \in N \ \& \ x_1 \dots x_b \in \omega \ \& \ (s_1 \dots s_a, x_1 \dots x_b) \in \Sigma_i\}.$$

Here we prove that fact, but we do not give such explicit proof in the rest of this paper.

1) $\mathfrak{B}^k \leq_{\text{SC}} \mathfrak{B}$, proof: For every predicate Σ_i , $1 \leq i \leq k$, for all $s \in N^*$, $\hat{\Sigma}_i(s) = 0^*$ iff $s = \langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^* \rangle \ \& \ (s_1 \dots s_{a_i}, x_1 \dots x_{b_i}) \in \Sigma_i$, i.e. iff $s = \langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^* \rangle \ \& \ (\exists r \in N^*) (r = \langle r_1 \dots r_a, p_1 \dots p_b, c \rangle \ \& \ r_1 =_N s_1 \ \& \ \dots \ \& \ r_{a_i} =_N s_{a_i} \ \& \ p_1 =_N t_1 \ \& \ \dots \ \& \ p_{b_i} =_N t_{b_i} \ \& \ c =_N i^* \ \& \ \hat{\Sigma}(r) = 0^*)$, that is $\hat{\Sigma}_i = If(\varphi_{a_i, b_i}, \hat{\Sigma} \circ \psi_i \circ \gamma_{a_i+b_i}^i, \emptyset^*)$, where φ_{a_i, b_i} and $\gamma_{a_i+b_i}^i$ defined in Lemma2.4 are prime computable, and the function ψ_i is such that for all $s_1 \dots, s_{a_i}, t_1 \dots t_{b_i}, p \in N^*$,

$\psi_i(\langle s_1 \dots s_{a_i}, t_1 \dots t_{b_i}, p \rangle) = \langle s_1 \dots s_{a_i}, \underbrace{c \dots c}_{a-a_i \text{ times}}, t_1 \dots t_{b_i}, \underbrace{d \dots d}_{b-b_i \text{ times}}, p \rangle$, where

$c \in N$ and $d \in \omega^*$ are constants, $\psi_i \in \mathbf{PC}$ is prime computable with constants c , since $\psi_i = \psi_{a_i, a-a_i} \circ \Pi(\hat{c}, I) \circ \psi_{a_i+b_i, b-b_i} \circ \Pi(\hat{d}, I)$, where the functions $\psi_{a_i, a-a_i}$ and $\psi_{a_i+b_i, b-b_i}$ defined in Lemma2.4(2) are prime computable (PC). Then $\hat{\Sigma}_i \in \mathbf{SC}(\hat{=}N, \hat{\neq}N, \hat{\Sigma})$, i.e. $\Sigma_i \leq_{\mathbf{SC}} \mathfrak{B}$, i.e. $\mathfrak{B}^k = (\Sigma_1 \dots \Sigma_k) \leq_{\mathbf{SC}} \mathfrak{B}$.

2) $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{B}^k$, proof:

$\hat{\Sigma}(r) = 0^*$ iff $s = \langle s_1 \dots s_a, x_1^* \dots x_b^*, i^* \rangle$ & $(s_1 \dots s_a, x_1 \dots x_b, i) \in \Sigma$ iff $s = \langle s_1 \dots s_a, x_1^* \dots x_b^*, i^* \rangle$ & $1 \leq i \leq k$ & $s_1 \dots s_a \in N$ & $x_1 \dots x_b \in \omega$ & $(s_1 \dots s_{a_i}, x_1 \dots x_{b_i}) \in \Sigma$. $\hat{\Sigma} = If(And(\varphi_{a_i, b+1}, \varphi_k \circ Lt_{a+b+1}), \psi_k, \emptyset^*)$, where $\varphi_{a, b+1}$ and Lt_{a+b+1} are defined in Lemma2.4 and φ_k is such that for $Dom(\varphi_k) = \omega^*$ and $\varphi_k(x^*) = 0^*$ iff $1 \leq x$ & $x \leq k$, i.e. $\varphi_k = If(\chi_{\omega^*}, And(\delta^*, \rho_k^*), \emptyset^*)$, where δ and ρ_k are functions from ω into ω , $\delta(x) = 0 \Leftrightarrow 1 \leq x$, and $\rho_k(x) = 0$ iff $x \leq k$, and δ^* and ρ^* are prime computable, see Proposition 1.4. The functions ψ_k is such that $\psi_k(\langle s_1 \dots s_a, x_1^* \dots x_b^*, i^* \rangle) = \hat{\Sigma}_i(\langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^* \rangle)$, for $1 \leq i \leq k$. Indeed $\psi_1 = If(\rho_1^*, \hat{\Sigma}_1 \circ Fst_{a_1+b_1}, \emptyset^*)$, that is $\psi_1 \in PC(\hat{\Sigma}_1)$; and $\psi_k = If(\rho_k^*, \hat{\Sigma}_k \circ Fst_{a_k+b_k}, \psi_{k-1}) \in PC(\hat{\Sigma}_1, \dots, \hat{\Sigma}_k)$. Therefore $\hat{\Sigma} \in PC(\hat{\Sigma}_1 \dots \hat{\Sigma}_k)$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{B}^k$. □

3 N-enumerations

Definition 3.1

An N -enumeration is a total and surjective function $f_N : \omega \rightarrow N$. Define the following sets:

1. $f_N^{-1}(\Sigma) := \{\langle x_1 \dots x_a, y_1 \dots y_b \rangle \in \omega \mid (f_N(x_1) \dots f_N(x_a), y_1 \dots y_b) \in \Sigma\}$.
2. $E_f := \{\langle x, y \rangle \in \omega \mid f_N(x) =_N f_N(y)\} = f_N^{-1}(=_N)$;
 $\overline{E}_f := \{\langle x, y \rangle \in \omega \mid f_N(x) \neq_N f_N(y)\} = f_N^{-1}(\neq_N)$; and $E_f^+ := E_f \oplus \overline{E}_f$.

Definition 3.2

Let $f^* : \omega \rightarrow N^*$. For the functions $\varphi \in \mathcal{F}$ and the predicate $\Sigma \subseteq N^a \times \omega^b$, define:

1. $E^* := \{\langle x, y \rangle \in \omega \mid f^*(x) = f^*(y)\}$.
2. $f^{*-1}(\varphi) := \{\langle x, y \rangle \in \omega \mid \varphi(f^*(x))\} = f^{*-1}(Gr_\varphi)$.
3. $f^{*-1}(\Sigma) := \{x \in \omega \mid \hat{\Sigma}(f^*(x)) = 0^*\} = \{x \mid f^*(x) = \langle s_1 \dots s_a, y_1^* \dots y_b^* \rangle \text{ \& } (s_1 \dots s_a, y_1 \dots y_b) \in \Sigma\}$.

Lemma 3.3

The function $f^* : \omega \rightarrow N^*$ is total and surjective; there exist a recursive function $J : \omega^2 \rightarrow \omega$, such that for all $x, y \in \omega$, $f^*(J(x, y)) = \langle f^*(x), f^*(y) \rangle$, and there exists a recursively enumerable set X , such that $f^*[X] = N$. Then for all $\psi, \varphi_1 \dots \varphi_k \in \mathcal{F}$, if $\psi \in \mathbf{SC}(\varphi_1 \dots \varphi_k)$, then $f^{*-1}(\psi) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

PROOF:

Since $\psi \in \mathbf{SC}(\varphi_1 \dots \varphi_k)$, there exists $\varphi \in \mathbf{PC}(\varphi_1 \dots \varphi_n)$, such that $\psi(s) = t \Leftrightarrow (\exists u \in N^*) (\varphi\langle u, s \rangle) = t$. Therefore $\langle x, y \rangle \in f^{*-1}(\psi) \Leftrightarrow \psi(f^*(x)) = f^*(y) \Leftrightarrow (\exists u \in N^*) (\varphi(\langle u, f^*(x) \rangle) = f^*(y))$ /since f is surjective/ $\Leftrightarrow (\exists t \in \omega) \varphi(\langle f^*(t), f^*(x) \rangle) = f^*(y) \Leftrightarrow (\exists t \in \omega) (\varphi(f^*(J(t, x))) = f^*(y)) \Leftrightarrow (\exists t, z \in \omega) (J(t, x) = z \ \& \ \varphi(f^*(z)) = f^*(y))$, therefore $f^{*-1} \leq_e f^{*-1}(\varphi)$.

By induction on the definition of \mathbf{PC} , we are going to prove that for $\varphi \in \mathbf{PC}(\varphi_1 \dots \varphi_k)$, $f^*(\varphi) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$, from which follows that $f^*(\psi) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

Let c_0 and $c_1 \in \omega$ be such that

$f^*(c_0) = 0^*$ and $f^*(c_1) = 1^*$. Such c_0 and c_1 exist, since f^* is a surjective function.

- If $\varphi = L$
 $\langle z, y \rangle f^{*-1}(L)$ iff $L(f^*(z)) = f^*(y)$ iff $f^*(z) = f^*(y) = 0^* \vee f^*(z) \in N$
 $\& f^*(y) = 1^* \vee f^*(z) \in N^* \setminus N - 0$ & $\exists x \in \omega \langle f^*(y), f^*(x) \rangle f^*(z)$ iff
 $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in X(\langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x \in \omega (f^*(J(y, x)) =$
 $f^*(z))$ iff $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in X(\langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x, t \in$
 $\omega (J(y, x) = t \ \& \ (f^*(t) = f^*(z)))$ iff $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in$
 $X \langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x, t \in \omega (J(y, x) = t \ \& \ \langle t, z \rangle \in E^*)$, i.e.
 $f^{*-1}(L) \leq_e E^*$, and it follows that $f^{*-1}(L) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus$
 $f^{*-1}(\varphi_k) \oplus E^*$.
- For $\varphi = L$ the proof is similar.
- For $\varphi = \varphi_i$, $1 \leq i \leq k$.
 $f^{*-1}(\varphi) = f^{*-1}(\varphi_i) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.
- For $\varphi = \hat{c}$ the constant function, for $c \in N$.
Therefore $f^{*-1}(\varphi) = f^{*-1}(\hat{c}) = f^{*-1}(Gr_{\hat{c}}) = \{\langle x, y \rangle \in \omega \mid \hat{c}f^*(x) =$

$f^*(y)\} = \{\langle x, y \rangle \in \omega \mid c = f^*(y)\}$. Since f^* is surjective, there exists $d \in \omega$, such that $f^*(d) = c$. Then $\langle x, y \rangle \in f^{*-1}(\hat{c})$ iff $\langle x, y \rangle \in E^*$.

The induction hypothesis for ψ_1 and $\psi_2 \in \mathbf{PC}(\varphi_1 \dots \varphi_k)$,
 $f^{*-1}(\psi_j) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

- Composition $\varphi = \psi_1 \circ \psi_2$.

We have the following equivalences $\langle z, y \rangle \in f^{*-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow \exists s \in N^* (\psi_2(f^*(z)) = s \ \& \ \psi_1(s) = f^*(y))$ and since f^* is surjective, this is equivalent to $\exists x \in \omega (\psi(f^*(z)) = f^*(x) \ \& \ \psi_1(f^*(x)) = f^*(y))$, i.e. $\exists x (\langle z, x \rangle \in f^{*-1}(\psi_2) \ \& \ \langle x, y \rangle \in f^{*-1}(\psi_1))$, therefore $f^{*-1}(\varphi) \leq_e f^{*-1}(\psi_1) \oplus f^{*-1}(\psi_2)$, i.e. $f^{*-1}(\varphi) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

- Pairing $\varphi = \Pi(\psi_1, \psi_2)$.

Using that f^* is surjective, we obtain the following equivalences $\langle z, y \rangle \in f^{*-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow \exists s_1, s_2 \in N^* (\langle s_1, s_2 \rangle = f^*(y) \ \& \ \psi(f^*(z)) = s_1 \ \& \ \psi_2(f^*(z)) = s_2) \Leftrightarrow \exists x_1, x_2 \in \omega (\langle f^*(x_1), f^*(x_2) \rangle = f^*(y) \ \& \ \psi_1(f^*(z)) = f^*(x_1) \ \& \ \psi_2(f^*(z)) = f^*(x_2)) \Leftrightarrow \exists x_1, x_2 \in \omega (f^*(J(x_1, x_2)) = f^*(y) \ \& \ \psi_2(f^*(z)) = f^{*-1}(x_1) \ \& \ \psi_2(f^*(z)) = f^*(x_2)) \Leftrightarrow \exists x_1, x_2, t (J(x_1, x_2) = t \ \& \ \langle t, y \rangle \in E^* \ \& \ \langle z, x_1 \rangle \in f^{*-1}(\psi_1) \ \& \ \langle z, x_2 \rangle \in f^{*-1}(\psi_2))$, i.e. $f^{*-1}(\varphi) \leq_e f^{*-1}(\psi_1) \oplus f^{*-1}(\psi_2) \oplus E^*$, therefore $f^{*-1}(\varphi) \leq_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

- Iteration $\varphi = [\psi_1, \psi_2]$.

We have that $\langle z, y \rangle \in f^{*-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow [\psi_1, \psi_2](f^*(z)) = f^*(y) \Leftrightarrow \psi_2(f^*(y)) \downarrow \in N_0 \ \& \ \exists w_0, \dots, w_n \in N^* (w_0 = f^*(z) \ \& \ w_n = f^*(y) \ \& \ \forall i (0 \leq i < n) (\psi_1(w_i) = w_{i+1} \ \& \ \psi_2(w_i) \notin N_0))$, and since f^* is surjective we obtain that the last part is equivalent to $\exists x_0, \dots, x_n \in \omega (f^*(x_0) = f^*(z) \ \& \ f^*(x_n) = f^*(y) \ \& \ \forall i (0 \leq i < n) (\psi_1(f^*(x_i)) = f^*(x_{i+1}) \ \& \ \psi_2(f^*(x_i)) \notin N_0))$. Therefore $\langle z, y \rangle \in f^{*-1}(\varphi) \Leftrightarrow (\psi_2(f^*(y)) = f^*(c_0) = 0^* \vee \psi_2(f^*(y)) \in N) \ \& \ \exists x_0 \dots x_n \in \omega (\langle x_0, z \rangle, \langle x_n, y \rangle \in E^* \ \& \ \forall i_{(0 \leq i < n)} (\langle x_i, x_{i+1} \rangle \in f^{*-1}(\psi_1) \ \& \ \psi_2(f^*(x_i)) \downarrow \in N^* \setminus N_0)) \Leftrightarrow (\langle y, c_0 \rangle \in f^{*-1}(\psi_2) \vee \exists t \in X (\psi_2(f^*(y)) = f^*(t))) \ \& \ \exists x_0 \dots x_n \in \omega (\langle x_0, z \rangle, \langle x_n, y \rangle \in E^* \ \& \ \forall i_{(0 \leq i < n)} (\langle x_i, x_{i+1} \rangle \in f^{*-1}(\psi_1) \ \& \ \exists t_{i_1}, t_{i_2}, t_i \in \omega (J(t_{i_1}, t_{i_2}) = t_i \ \& \ \psi_2(f^*(x_i)) = f^*(t_i))) \Leftrightarrow \exists t \in X \cup \{c_0\} (\langle y, t \rangle \in f^{*-1}(\psi_2)) \ \& \ \exists x_0, \dots, x_n \in \omega (\langle x_0, z \rangle \in E^* \ \& \ \langle x_n, y \rangle \in E^* \ \& \ \forall i_{(0 \leq i < n)})$

$(\langle x_i, x_{i+1} \rangle \in f^{*-1}(\psi_1) \ \& \ \exists t_i \in \text{Range}(J) (\langle x_i, t_i \rangle \in f^{*-1}(\psi_2)))$, i.e.
 $f^{*-1}(\varphi) \leq_e f^{*-1}(\psi_1) \oplus f^{*-1}(\psi_2) \oplus E^*$, therefore
 $f^{*-1}(\varphi) = \text{leq}_e f^{*-1}(\varphi_1) \oplus \dots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

□

Corollary 3.4

Let $f^* : \omega \rightarrow N^*$ be a total surjective function and $J : \omega^2 \rightarrow \omega$, such that for all $x, y \in \omega$, $f^*(J(x, y)) = \langle f^*(x), f^*(y) \rangle$ and there is a recursively enumerable set X , such that $f^*(X) = N$, $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $0 \leq i \leq k$, are predicates, such that $\Sigma_0 \leq (\Sigma_1 \dots \Sigma_k)$. Therefore

$$f^{*-1}(\Sigma) \leq_e f^{*-1}(\Sigma_1) \oplus \dots \oplus f^{*-1}(\Sigma_k) \oplus E^*.$$

Proposition 3.5

Let $f_N : \omega \rightarrow N$ be an N -enumeration and $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $0 \leq i \leq k$, such that $\Sigma_0 \leq_{\mathbf{SC}} (\Sigma_1 \dots \Sigma_k)$.

Therefore $f^{*-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_1) \oplus \dots \oplus f^{*-1}(\Sigma_k) \oplus E_f$.

PROOF:

Let $J(x, y) := 2\langle x, y \rangle + 2$, where the pairing is recursive injective functions, for example $\langle x, y \rangle := 2^x \cdot (2y + 1) - 1$. Then for all t and z , $t < J(t, z)$ and $z < J(t, z)$.

Having $f_N : \omega \rightarrow N$, define a total surjective function $f^* : \omega \rightarrow N^*$, as follows:

$$\begin{cases} f^*(J(x, y)) &= \langle f^*(x), f^*(y) \rangle \\ f^*(2z + 1) &= f_N(z) \\ f^*(0) &= 0^* \end{cases}$$

The set $X = \{2z + 1 \mid z \in \omega\}$ is recursive and $f^*(X) = N$. Therefore, using Lemma 3.3, $f^{*-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_1) \oplus \dots \oplus f^{*-1}(\Sigma_k) \oplus E^*$.

There exists a recursive function g , such that $y^* = f^*(g(y))$, that we define as follow:

$$\begin{cases} g(0) &= 0 \\ g(y + 1) &= J(0, g(y)) \end{cases}$$

Therefore there exists a recursive function $h_{a,b}$, such that

$$\langle f_N(z_1) \dots f_N(z_a), y_1^* \dots y_b^* \rangle = \langle f^*(2z_a + 1), f^*(g * y_1) \dots f^*(g(y_b)) \rangle = f^*(J(2z_1 + 1, J(\dots, J(g(y_b)) \dots))) = f^*(h_{a,b}(\langle z_1 \dots z_a, y_1 \dots y_b \rangle)).$$

1) We can prove that for every predicate $\Sigma \subseteq N^a \times \omega^b$, $f^{*-1}(\Sigma) \leq E^* \oplus f_N^{-1}(\Sigma)$, from which follows that $f^{*-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*$. Indeed

$$\begin{aligned} x \in f^{*-1}(\Sigma) &\iff \hat{\Sigma}(f^*(x)) = 0^* \iff \\ \exists s_1 \dots s_a \in N \ \exists y_1 \dots y_b \in \omega & \end{aligned}$$

$(f^*(x) = \langle s_1 \dots s_a, y_1^* \dots y_b^* \rangle \ \& \ (s_1 \dots s_a, y_1 \dots y_b) \in \Sigma) \iff$
 $\exists z_1 z_a, y_1 \dots y_b \in \omega$
 $(f^*(x) = \langle f_N(z_1) \langle f_N(z_a), y_1^* \dots y_b^* \rangle \ \& \ (f_N(z_1) \dots f_N(z_a), y_1 \dots y_b) \in \Sigma) \iff$
 $\exists z_1 \dots z_a, y_1 \dots y_b$
 $(f^*(x) = f^*(h_{a,b}(\langle z_1 \dots z_a, y_1 \dots y_b \rangle)) \ \& \ \langle z_1 \dots z_a, y_1 \dots y_b \rangle \in f_N^{-1}(\Sigma)) \iff$
 $\exists z_1 \dots z_a, y_1 \dots y_b (\langle x, \langle z_1 \dots z_a, y_1 \dots y_b \rangle \rangle \in E^* \ \& \ \langle z_1 \dots z_a, y_1 \dots y_b \rangle \in f_N^{-1}(\Sigma)).$
 Therefore $f^{*-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_1) \oplus \dots \oplus f^{*-1}(\Sigma_k) \oplus E^* \leq_e$
 $f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*.$

2) We can prove that for every predicate $\Sigma \subseteq N^a \times \omega^b$, $f_N^{-1}(\Sigma) \leq_e f^{*-1}(\Sigma) \oplus E^*$. Indeed

$\langle x_1 \dots x_a, y_1 \dots y_b \rangle \in f_N^{-1}(\Sigma) \iff (f_N(x_1) \dots f_N(x_a), y_1 \dots y_b) \in \Sigma \iff$
 $\exists s \in N^* (\hat{\Sigma}(s) = 0^* \ \& \ 4s = \langle f_N(x_1) \dots f_N(x_a), y_1^* \dots y_b^* \rangle) \iff$
 $\exists x \in \omega (\hat{\Sigma}(f^*(x)) = 0^* \ \& \ f^*(x) = \langle f_N(x_1) \dots f_N(x_a), y_1^* \dots y_b^* \rangle) \iff$
 $\exists x (x \in f^{*-1}(\Sigma) \ \& \ f^*(x) = f^*(x) = f^*(h_{a,b}(\langle x_1 \dots x_a, y_1 \dots y_b \rangle))) \iff$
 $\exists x, z (x \in f^{*-1}(\Sigma) \ \& \ 4h(\langle x_1 \dots x_a, y_1 \dots y_b \rangle) = z \ \& \ \langle x, z \rangle \in E^*).$

Therefore $f_N^{-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_0) \oplus E^*$, and since $f^{*-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*$, we obtain that $f_N^{-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*$.

3) We have $E^* \leq_e E_f$, from which follows that $f_N^{-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f$.

For the proof of that statement is necessary to know that $\langle x, y \rangle \in E^* \iff f^*(x) = f^*(y) \iff x = y = 0 \vee \exists z, t (x = 2z+1 \ \& \ y = 2t+1 \ \& \ f_N(z) = f_N(t)) \vee \exists x_1, x_2, y_1, y_2$ such that $(x_1 < x \ \& \ y_1 < y \ \& \ x_2 < x \ \& \ y_2 < y)$ and $(x = J(x_1, x_2) \ \& \ y = J(y_1, y_2) \ \& \ f^*(x_1) = f^*(y_1) \ \& \ f^*(x_2) = f^*(y_2))$. \square

Definition 3.6

For an N -enumeration $f_N : \omega \rightarrow N$ and a structure $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$, define:

$$f_N^{-1}(\mathfrak{A}) := \{\langle i, z \rangle \mid z \in f_N^{-1}(\Sigma_i)\} \cup \{\langle 0, z \rangle \mid z \in E_f^+\}.$$

Remark: $f_N^{-1}(\mathfrak{A}) \equiv_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f \oplus \overline{E_f}$.

Definition 3.7

- 1) N -string τ_N is a function $\tau_N : [0 \dots n-1] \rightarrow N$, with length $lh(\tau_N) = n$.
- 2) $\tau_N \subseteq \sigma_N$ iff $\forall x (x < lh(\tau_N) \Rightarrow \tau_N(x) = \sigma_N(x))$.
- 3) Code of the N -string τ_N define to be $\ulcorner \tau_N \urcorner = \langle n^*, \tau_N(0), \dots, \tau_N(n-1) \rangle$.

Remark: The functions h_0, h_1, h_2 , such that

$$\begin{aligned}
& \forall s \in N^*(h_0(s) \downarrow \Leftrightarrow \exists \tau_N (s = \ulcorner \tau_N \urcorner)), \\
& h_1(\langle \ulcorner \tau_N \urcorner, \ulcorner \sigma_N \urcorner \rangle) = 0^* \Leftrightarrow \tau_N \subseteq \sigma_N, \\
& h_2(\langle \ulcorner \tau_N \urcorner, x^*, y \rangle) = 0^* \Leftrightarrow \tau_N(x) = y
\end{aligned}$$

are prime computable.

Definition 3.8

For a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, a N -string τ_N and a formula $F_e(z)$ with $e, z \in \omega$, define $\tau_n \Vdash_{\mathfrak{A}} F_e(z)$ as follows:

$$\begin{aligned}
& \tau_n \Vdash_{\mathfrak{A}} F_e(z) \text{ iff } \exists v (\langle v, z \rangle \in W_e \ \& \ \tau_n \Vdash_{\mathfrak{A}} D_v) \\
& \text{and} \\
& \tau_n \Vdash_{\mathfrak{A}} D_v \text{ iff } \forall u \in D_v (u = \langle i, \langle x_1 \dots x_{a_i}, y_1 \dots y_{b_i} \rangle \rangle \ \& \\
& 1 \leq i \leq k \ \& \ x_1 \dots x_{a_i} \in \text{Dom}(\tau_N) \ \& \ (\tau_N(x_1) \dots \tau_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma_i \vee \\
& u = \langle 0, 2 \cdot \langle x, y \rangle \rangle \ \& \ x, y \in \text{Dom}(\tau_N) \ \& \\
& \tau_N(x) = \tau_N(y) \ \& \ u = \langle 0, 2 \cdot \langle x, y \rangle + 1 \rangle \ \& \ x, y \in \text{Dom}(\tau_N) \ \& \ \tau_N(x) \neq_N \tau_N(y)).
\end{aligned}$$

Properties:

- 1) If $\tau_n \subseteq \sigma_N$ and $\tau_N \Vdash_{\mathfrak{A}} F_e(z)$, then $\sigma_N \Vdash_{\mathfrak{A}} F_e(z)$.
- 2) The function h_3 , such that $h_3(\langle \ulcorner \tau_N \urcorner, e^*, z^* \rangle) = 0^* \Leftrightarrow \tau_N \Vdash_{\mathfrak{A}} F_e(z)$ is $\mathbf{SC}(\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}_N, \hat{\neq}_N)$.

Definition 3.9

For an N -enumeration $f_N : \omega \rightarrow N$ and a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, define

$$f_n \models_{\mathfrak{A}} F_e(z) \text{ if and only if } z \in \Psi_e(f_N^{-1}(\mathfrak{A})).$$

Proposition 3.10 $f_N \models_{\mathfrak{A}} F_e(z)$ iff $\exists \tau_N \subseteq f_N(\tau_N \Vdash_{\mathfrak{A}} F_e(z))$.

Definition 3.11

The predicate $\Sigma \subseteq N^a \times \omega^b$ has *normal form* in the structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$, when there exist $e \in \omega$, an N -string δ_N and $x_1 \dots x_a \notin \text{Dom}(\delta_N)$, such that

$$\begin{aligned}
& (\forall s_1 \dots s_a \in N, \forall y_1 \dots y_b \in \omega) \\
& ((s_1 \dots s_a, y_1 \dots y_b) \in \Sigma \text{ iff } \exists \tau_N \supseteq \delta_N \text{ such that} \\
& (\tau_N(x_1) = s_1 \ \& \ \dots \ \& \ \tau_N(x_a) = s_a \ \& \ \tau_N \Vdash_{\mathfrak{A}} F_e(\langle x_1 \dots x_a, y_1 \dots y_b \rangle))).
\end{aligned}$$

Theorem 3.12 (The Normal form theorem)

Let $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ be a structure, with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$. Then for every predicate $\Sigma \subseteq N^a \times \omega^b$ the followings are equivalent:

- 1) Σ has in normal form in \mathfrak{A} .

- 2) $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$.
3) $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A})$ where f_N is an arbitrary N -enumeration.

PROOF:

1) (3) follows from (2).

Let $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$, i.e. $\Sigma \leq_{\mathbf{SC}} (\Sigma_1 \dots \Sigma_k, =_N, \neq_N)$. Therefore, using Corollary 7.6, for every N -enumeration f_N , we have that $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(=_N) \oplus f_N^{-1}(\hat{\neq}_N) \oplus E_f$, but $f_N^{-1}(=_N) = E_f$ and $f_N^{-1}(\hat{\neq}_N) = \overline{E}_f$, therefore $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f^+ \equiv_e f_N^{-1}(\mathfrak{A})$.

2) (1) follows from (3).

Let for every total surjective function $f_N : \omega \rightarrow N$, $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A})$, i.e. there exists e , such that $f_N^{-1}(\Sigma) = \Psi_e(f_N^{-1}(\mathfrak{A}))$. Assume that Σ has no normal form in \mathfrak{A} , i.e. for all $e \in \omega$, for all N -string δ_N and $x_1 \dots x_a \in \omega \setminus \text{Dom}(\delta_N)$, there exist $s_1 \dots s_a \in N$ and $y_1 \dots y_b \in \omega$, such that the following holds true:

$$(*) \quad (\overline{s}, \overline{y}) \notin \Sigma \text{ iff} \\
\exists \tau_N \supseteq \delta_N (\tau_N(x_1) = s_1 \& \dots \& \tau_N(x_a) = s_a \& \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x}, \overline{y} \rangle))$$

We can define an N -enumeration $f_N : \omega \rightarrow N$, such that $f_N^{-1} \not\leq_e f_N^{-1}(\mathfrak{A})$, i.e. such that for every e , $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$. First we define by induction N -strings σ_N^q , such that $\sigma_N^0 \subseteq \dots \subseteq \sigma_N^q \subseteq \dots$, so that at stage $2e + 1$ we insure f_N to be total and surjective, and at stage $2e + 2$ we insure $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$, for $f_n := \bigcup_q \sigma_N^q$, as follows:

Stage 0

$$\sigma_N^0 = \emptyset$$

Stage $2e + 1$

Suppose we have already defined σ_N^q , for $q = 2e$. Let $x = lh(\sigma_N^q)$, for the least $n \notin N$, such that $n \notin \text{Range}(\sigma_N^q)$.

$$\text{Define } \sigma_N^{q+1} \supseteq \sigma_N^q, lh(\sigma_N^{q+1}) = lh(\sigma_N^q) + 1 \text{ and } \sigma_N^{q+1}(x) = n.$$

Stage $2e + 2$

Suppose we have already defined σ_N^q , where $q = 2e + 1$.

Let $x_1 \dots x_a \in \omega$ are such that $x_j = lh(\sigma_N^q) + j - 1$, $1 \leq j \leq a$. Therefore $x_1 \dots x_a \notin \text{dom}(\sigma_N^q)$ are the first a elements, where σ_N^q is not defined.

Case 1. There exist $s_1 \dots s_a \in N$ and $y_1 \dots y_b \in \omega$, such that **(*)** holds true, i.e. $(\overline{s}, \overline{y}) \notin \Sigma$ iff $\exists \tau_N \supseteq \sigma_N^q (\tau_N(x_1) = s_1 \& \dots \& \tau_N(x_a) = s_a \& \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x}, \overline{y} \rangle))$, i.e. for $\delta_N \supseteq \sigma_N^q$, such that $\delta_N(x_1) = s_1 \& \dots \& \delta_N(x_a) = s_a \&$

$lh(\delta_N(x_a) = lh(\sigma_N^q) + a$, the following holds: $(\bar{s}, \bar{y}) \notin \Sigma$ iff $\exists \tau_N \supseteq \delta_N$ ($\tau_N \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$).

Subcase 1.1. $\exists \tau_N \supseteq \delta_N$ ($\tau_N \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$). Define $\sigma_N^{q+1} := \tau_N$.

Subcase 1.2. $\forall \tau_N \supseteq \delta_N$ ($\tau_N \not\Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$). Define $\sigma_N^{q+1} := \delta_N$.

Case 2. Otherwise, define $\sigma_N^{q+1} := \sigma_N^q$.

Define $f := \bigcup_q \sigma_N^q$, which is total and surjective, i.e. an N -enumeration.

Using the assumption that Σ has no normal form in \mathfrak{A} it follows that at stage $2e + 2$ the *Case 2* never happens, so we have the following two subcases only:

In *Subcase 1.1*, $\sigma_N^{q+1} \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$ and $(\bar{s}, \bar{y}) \notin \Sigma$, and $\sigma_N^{q+1}(x_j) = s_j$ for $1 \leq j \leq a$ (from the last one it follows that $f_N(d_j) = s_j$). From $\sigma_N^{q+1} \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$, using *Proposition 3.10*, we obtain that $f_N \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$, i.e. $\langle \bar{x}, \bar{y} \rangle \in \Psi_e(f_N^{-1}(\mathfrak{A}))$. But $(f_N(x_1) \dots f_N(x_a), y_1 \dots y_b) = (\bar{s}, \bar{y}) \notin \Sigma$, i.e. $\langle \bar{x}, \bar{y} \rangle \notin f_N^{-1}(\Sigma)$ and therefore $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$.

In *Subcase 1.2*, $\forall \tau_N \supseteq \sigma_N^{q+1}$ ($\tau_N \not\Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$), $(\bar{s}, \bar{y}) \in \Sigma$, and $\sigma_N^{q+1}(d_j) = s_j$ for $1 \leq j \leq a$ (then we have $f_N(x_j) = s_j$ and therefore $\langle \bar{x}, \bar{y} \rangle \in f_N^{-1}(\Sigma)$). From $\forall \tau_N \supseteq \sigma_N^{q+1}$ ($\tau_N \not\Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$), using *Proposition 3.10*, it follows that $f_N \not\Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$, i.e. $\langle \bar{x}, \bar{y} \rangle \notin \Psi_e(f_N^{-1}(\mathfrak{A}))$. Therefore $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$.

We have proven that for all e , $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$, i.e. $f_N^{-1}(\Sigma) \not\leq_e f_N^{-1}(\mathfrak{A})$, which is a contradiction. Therefore our assumption is false, i.e. Σ has normal form in \mathfrak{A} .

2) (2) follows from (1).

Suppose Σ has normal form in \mathfrak{A} .

We can prove that $\hat{\Sigma} \in \mathbf{SC}(\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}_N, \hat{\neq}_N)$, which according to the definition means $\Sigma \leq_{\mathbf{SC}} (\Sigma_1 \dots \Sigma_k, =_N, \neq_N)$, i.e. that $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$. Since Σ has normal form in \mathfrak{A} ,

($\exists e \in \omega, \exists \delta$ and N -string, $\exists x_1 \dots x_a \notin \text{Dom}(\delta_N)$)

($\forall s_1 \dots s_a \in N, \forall y_1 \dots y_b \in \omega$)

($(\bar{s}, \bar{y}) \in \Sigma$ iff

$\exists \tau_N \supseteq \delta_N$ ($\tau_N(x_1) = s_1 \ \& \ \dots \ \& \ \tau_N(x_a) = s_a \ \& \ \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \bar{x}, \bar{y} \rangle)$),

therefore $\hat{\Sigma}(s) = 0^*$ iff $\exists s_1 \dots s_a \in N \exists y_1 \dots y_b \in \omega$ ($s = \langle s_1 \dots s_a, y_1^* \dots y_b^* \rangle \ \& \ (\bar{s}, \bar{y}) \in \Sigma$) iff ($s = \langle \bar{s}, \bar{y}^* \rangle \ \& \ s_j \in N \ \& \ y_i^* \in \omega^* \ \& \ \exists \tau \supseteq \delta$ ($\tau_N(x_1) = s_1 \ \& \ \dots \ \& \ \tau_N(x_a) = s_a \ \& \ \exists v(\langle v, \langle \bar{x}, \bar{y} \rangle \rangle \in W_e \ \& \ \tau_N \Vdash_{\mathfrak{A}} D_v)$)).

Note that the functions $\chi_{\omega^*}, \chi_N, I_j^m$, (where $I_j^m(r) = r_j \Leftrightarrow r = \langle r_1 \dots r_m \rangle$), h_0 , (where $h_0(s) = 0^* \Leftrightarrow s = \ulcorner \tau_N \urcorner$), h_1 , (where $h_1(\langle \ulcorner \tau_N \urcorner, \ulcorner \sigma_N \urcorner \rangle) = 0^* \Leftrightarrow \tau_N \subseteq \sigma_N$), h_2 , (where $h_2(\langle \ulcorner \tau_N \urcorner, x^*, s \rangle) = 0^* \Leftrightarrow \tau_N(x) = s$) are **PC**. The function φ (the semicharacteristic function of the set W_e), is partial recur-

sive and therefore (see Proposition 1.4) the function φ^* is **PC**, for the same reason the function $\langle \cdot, \cdot \rangle^*$, the coding of pair of natural numbers (where $\langle x^*, y^* \rangle = z^* \Leftrightarrow \langle x, y \rangle = z$) is **PC**.

It is not difficult to check that the function g , such that $g(\ulcorner \tau_N \urcorner, v^*) = 0^*$
 $\Leftrightarrow \tau_N \Vdash_{\mathfrak{A}} D_v \Leftrightarrow \forall u \in D_v (u = \langle i, \langle x_1 \dots x_{a_i}, y_1 \dots y_{b_i} \rangle \rangle \ \& \ 1 \leq i \leq k \ \& \ x_1 \dots x_{a_i} \in \text{Dom}(\tau_N) \ \& \ (\tau_N(x_1) \dots \tau_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma_i \vee u = \langle 0, 2\langle x, y \rangle \rangle \ \& \ x, y \in \text{Dom}(\tau_N) \ \& \ \tau_N(x) =_N \tau_N(y) \vee u = \langle 0, 2\langle x, y \rangle + 1 \rangle \ \& \ x, y \in \text{Dom}(\tau_N) \ \& \ \tau_N(x) \neq \tau_N(y))$, is **PC**($\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}^N, \hat{\neq}$).

Therefore $\Sigma \leq_{\text{SC}} \mathfrak{A}$.

□

4 Enumerations

Definition 4.1

An enumeration is a pair $\alpha = (f_N, R_\alpha)$, where $f_N : \omega \rightarrow N$ is a total surjective function and $R_\alpha \subseteq \omega$.

The structure of the enumeration α is the structure $\mathfrak{A}_\alpha = \mathfrak{A}(Gr(f_N), R_\alpha)$, where $Gr(f_N) = \{(s, x) \mid f_N(x) = s\} \subseteq N \times \omega$.

Definition 4.2

For every enumeration $\alpha = (f_N, R_\alpha)$ we define the following sets:

1. $E_\alpha := \{\langle x, y \rangle \mid f_N(x) =_N f_N(y)\} = E_f = f_N^{-1}(=_N) \subseteq \omega$.
2. $D(\alpha) = E_\alpha^+ \oplus R_\alpha$.

Proposition 4.3

Let a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ and an enumeration $\alpha = (f_N, R_\alpha)$ and a predicate $\Sigma \subseteq N^a \times \omega^b$ be given. Then $\Sigma \leq_{\text{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$ if and only if $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A}) \oplus R_\alpha$.

PROOF:

(\Rightarrow) Let $\Sigma \leq_{\text{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$, where $\mathfrak{A} \oplus \mathfrak{A}_\alpha = (\Sigma_1 \dots \Sigma_k, Gr(f_N), R_\alpha)$. Then from *The Normal Form Theorem*(3.12) it follows that $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$, i.e.

$f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(Gr(f_N)) \oplus f_N^{-1}(R_\alpha) \oplus E_f^+$, but $f_N^{-1}(Gr(f_N)) = \{\langle x, y \rangle \mid (f_N(x), y) \in Gr(f_N)\} = \{\langle x, y \rangle \mid f_N(y) =_N f_N(x)\} = f_N^{-1}(=_N) = E_\alpha$. Then $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(R_\alpha) \oplus E_\alpha^+$, but $f_N^{-1}(R_\alpha) = R_\alpha \subseteq \omega$, and therefore $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A}) \oplus R_\alpha$.

(\Leftarrow) Let

- (1) $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A}) \oplus R_\alpha$.

Consider an arbitrary N -enumeration $g_N : \omega \rightarrow N$, for which we can prove that $g_N^{-1}(\Sigma) \leq_e g_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$, from which, using the *Normal form theorem* (3.12) follows that $\Sigma \leq_{\mathbf{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$.

We know that $g_N^{-1}(Gr(f_N)) = \{\langle x, y \rangle \mid (g_N(x), y) \in Gr(f_N)\} = \{\langle x, y \rangle \mid g_N(x) = f_N(y)\}$, and $g_N^{-1}(R_\alpha) = R_\alpha$.

(2) $g_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma) \oplus g_N^{-1}(Gr(f_N))$, since
 $\langle \bar{x}, \bar{y} \rangle \in g_N^{-1}(\Sigma) \Leftrightarrow (g_N(x) \dots g_N(x_a), y_1 \dots y_b) \in \Sigma \Leftrightarrow \exists z_1 \dots z_a$
 $(\langle x_1, z_1 \rangle, \dots, \langle x_a, z_a \rangle \in g_N^{-1}(Gr(f_N)) \ \& \ (f_N(z_1), \dots, f_N(z_a), y_1 \dots y_b) \in \Sigma)$
 $\Leftrightarrow \exists z_1 \dots z_a (\langle x_1, z_1 \rangle, \dots, \langle x_a, z_a \rangle \in g_N^{-1}(Gr(f_N)) \ \& \ \langle \bar{z}, \bar{y} \rangle \in f_N^{-1}(\Sigma)).$

(3) $f_N^{-1}(\Sigma_i) \leq_e g_N^{-1}(\Sigma_i) \oplus Gr(f_N)$, for all $1 \leq i \leq k$, since
 $\langle \bar{x}, \bar{y} \rangle \in f_N^{-1}(\Sigma_i) \Leftrightarrow (f_N(x_1) \dots f_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma \Leftrightarrow \exists z_1 \dots z_{a_i}$
 $(\langle z_1, x_1 \rangle \dots \langle z_{a_i}, x_{a_i} \rangle \in g_N^{-1}(Gr(f_N)) \ \& \ \langle \bar{z}, \bar{y} \rangle \in g_N^{-1}(\Sigma_i)).$

Using the same reasoning as in (3), we obtain $E_\alpha = E_f = f_N^{-1}(=N) \leq g_N^{-1}(=N) \oplus g_N^{-1}(Gr(f_N)) = E_g \oplus g_N^{-1}(Gr(f_N))$ and $\bar{E}_\alpha \leq_e \bar{E}_g \oplus g_N^{-1}(Gr(f_N))$.

From (1), (2) and (3) it follows that $g_N^{-1}(\Sigma) \leq_e g_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$. □

Corollary 4.4

For any enumeration $\alpha = (f_N, R_\alpha)$ and predicate $\Sigma \subseteq N^a \times \omega^b$, $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}_\alpha$ if and only if $f_N^{-1}(\Sigma) \leq_e D(\alpha)$.

PROOF:

Let $\emptyset = (N, \omega; =_N, \neq_N)$ be the empty structure. From *Proposition 4.3* it follows that $\Sigma \leq_{\mathbf{SC}} \emptyset \oplus \mathfrak{A}_\alpha \Leftrightarrow f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\emptyset) \oplus R_\alpha$, but $f_N^{-1}(\emptyset) = E_\alpha^+$, i.e. $\Sigma \leq_{\mathbf{SC}} \emptyset \oplus \mathfrak{A}_\alpha \Leftrightarrow f_N^{-1}(\Sigma) \leq_e D(\alpha)$. □

Corollary 4.5

For any structure $\mathfrak{A} = (\Sigma_1 \dots \Sigma_k)$ and enumeration $\alpha = (f_N, R_\alpha)$, $\mathfrak{A} \leq \mathfrak{A}_\alpha$ if and only if $f_N^{-1}(\mathfrak{A}) \leq_e D(\alpha)$.

PROOF:

Form *Corollary 4.4* it follows that $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{A}_\alpha \Leftrightarrow \forall i_{1 \leq i \leq k} \Sigma_i \leq_{\mathbf{SC}} \mathfrak{A}_\alpha \Leftrightarrow \forall i_{1 \leq i \leq k} f_N^{-1}(\Sigma_i) \leq_e D(\alpha) \Leftrightarrow f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_\alpha^+ \leq_e D(\alpha) \Leftrightarrow f_N^{-1}(\mathfrak{A}) \leq_e D(\alpha)$. □

Definition 4.6

The pair $f = (f_N, f_\omega)$, where $f_N : \omega \rightarrow N$ and $f_\omega : \omega \rightarrow \omega$ are total functions, is called an N - ω -enumeration.

Notation: $Gr(f_\omega) = \{\langle x, y \rangle \mid f_\omega(x) = y\} \subseteq \omega$ is the graph of f_ω .

Remark: The N - ω -enumerations $f = (f_N, f_\omega)$ are enumerations $\alpha = (f_N, R_\alpha)$ with $R_\alpha = Gr(f_\omega)$, $D(\alpha) = D(f) = E_f^+ \oplus Gr(f_\omega)$;
 $\mathfrak{A}_\alpha = \mathfrak{A}_f = \mathfrak{A}(Gr(f_N), Gr(f_\omega))$; $E_\alpha = E_f = f_N^{-1}(=N)$.