Sofia University St. Kliment Ohridski
Faculty of Mathematics and Computer Science
Department of Mathematical Logic and Applications

Direct Construction of a Bimachine for Context-Sensitive Rewrite Rule

A thesis submitted for the degree of Master of Computer Science
by

Ivan Petrov Peikov

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Contents

1 Introduction

Context-sensitive rewrite rules are a well-known formalism practically useful in many fields of computational linguistics. They were first introduced in Chomsky's papers ([1]) and proved to be expressive enough to successfully model multiple linguistic phenomena.

In 1972 Johnson ([2]) notices that with the limitation to work only over their input the context-sensitive rewrite rules become as expressive as the regular relations. This purely theoretical result is later confirmed by Kaplan and Kay ([3]) who show the practical importance of context-sensitive rewrite rules when implemented with finite state transducers. Their paper gives rise to many consecutive works and remains one of the classics in the contemporary computational linguistics.

Bimachines as introduced by Schutzenberger ([4]) are deterministic abstract machines as expressive as the regular functions. They consume their input simultaneously from left to right and from right to left. On every position they produced output based on the left-hand prefix and the right-hand suffix. Bimachines are computationally very efficient which makes them applicable in practice.

Our purpose is to present a construction which by given context-sensitive rewrite rule builds directly a bimachine realizing its corresponding regular function.

There exist other methods which first construct a finite-state transducer realizing the context rule ([3]) and then translate the transducer into a bimachine ([5]). In 2004, Skut et al. ([6]) present a direct construction of a bimachine for a restricted form of rewrite rule. However, to the best of our knowledge, the construction presented in the current thesis is the first direct construction of a bimachine realizing a non-restricted context-sensitive rewrite rule.

The rest of the thesis is structured as follows. In Section 2 rewrite rules are introduced and the problem is formally defined. In Sections 3 and 4 finite state automata are introduced. In Section 5 bimachines are introduced together with two of their possible operational semantics. In Section 6 the direct construction is described and proved correct. The remaining Sections 7 and 8 present some of the algorithms used and study their complexity.

We denote concatenation of two words α and β with $\alpha \cdot \beta$ or $\alpha\beta$ (when no ambiguity could possibly occur). The fact that α is a subword of β is denoted as $\alpha \subset \beta$ or $\alpha \subseteq \beta$ in case when equality is possible. All other notations will either be introduced in the appropriate context or will be considered as widely adopted.

2 Contexts and Rules

Let's fix a finite alphabet Σ. Context-sensitive rewrite rule is any rule of the type

$$E \to \beta / L\_R \tag{1}$$

where $L, E, R \in \mathcal{RE}(\Sigma)$ are regular expressions over Σ, and $\beta \in \Sigma^*$. Application of such a rule over a fixed word $\alpha \in \Sigma$ is the simultaneous rewriting with β of all α's subwords which belong to the language $\mathcal{L}(E)$ of E and are found between a subword of $\mathcal{L}(L)$ – to the left and subword of $\mathcal{L}(R)$ – to the right:

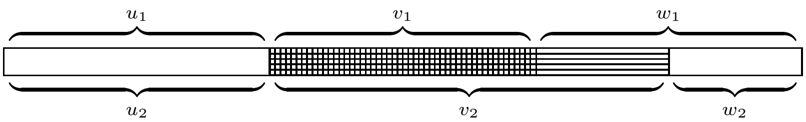
$$\alpha_1\alpha_2\dots\alpha_{i-1}\underbrace{\alpha_i\dots\alpha_{j-1}}_{\in\mathcal{L}(L)}\underbrace{\overbrace{\alpha_j\dots\alpha_{k-1}}^{\beta}}_{\in\mathcal{L}(E)}\underbrace{\alpha_k\dots\alpha_{l-1}}_{\in\mathcal{L}(R)}\alpha_l\dots\alpha_{n-1}\alpha_n$$

Definition 2.1. *Let $t \in \Sigma^*$, and $L, E, R \in \mathcal{RE}(\Sigma)$. A triple $\langle u, v, w\rangle$ is said to be a* rewrite context *(or simply* context*) if $u \in \mathcal{L}(\Sigma^* L)$, $v \in \mathcal{L}(E)$, $w \in \mathcal{L}(R\Sigma^*)$ and $uvw = t$. The words u, v and w are said to be respectively* prefix, focus *and* suffix *of the context. Additionally we denote*

$$\mathcal{C}(t; L, E, R) = \{\langle u, v, w\rangle | u \in \mathcal{L}(\Sigma^* L) \ \&\ v \in \mathcal{L}(E) \ \&\ w \in \mathcal{L}(R\Sigma^*) \ \&\ t = uvw\}$$

to be the set of all such triples or the context set *for any particular t and $E \to \beta \ / \ L\_R$.*

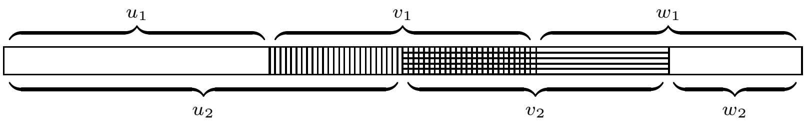
It becomes clear that there are certain ambiguities that might occur while applying a context rule over some word. For example, if two contexts share a common prefix but differ in their focuses (and respectively suffixes) the result of the rewriting will be ambiguous:



Such an ambiguity will be resolved by choosing for valid the context with the *longest* focus.

Example 2.1. *Let's apply the rule $a^+ \to A \ / \ b\_a$ over the word $\alpha = baaaab$. The result might ambiguously be defined as bAaaab, bAaab or bAab (rewriting respectively in context $\langle b, a, aaab\rangle$, $\langle b, aa, aab\rangle$ or $\langle b, aaa, ab\rangle$). By agreeing to always take the context with the longest focus for valid we resolve the ambiguity and obtain the single result bAab.*

Definition 2.2. *Two contexts $\langle u_i, v_i, w_i\rangle \in \mathcal{C}$ for $i = 1, 2$ are said to overlap (written $\langle u_1, v_1, w_1\rangle \prec \langle u_2, v_2, w_2\rangle$), if $u_1 \subset u_2 \subset u_1 v_1$.*



Whenever such a situation occurs while applying a context rule not more than one of the two contexts should be taken for valid. We'll resolve such an ambiguity by always taking the leftmost of all the possibilities and ignoring all the overlapped contexts.

We call such a strategy for resolving ambiguities *leftmost longest match* strategy.

Example 2.2. *Let's take the rule $xy|yz \to \epsilon$ / x_z. During its application over word like $xyzzxxyzz$ ambiguity arises. Possible rewriting contexts are $\langle x, yz, zxxyzz\rangle$, $\langle xyzzx, xy, zz\rangle$ and $\langle xyzzxx, yz, z\rangle$, the last two if which do overlap. The* leftmost longest match *strategy sorts out the first two contexts as valid for rewriting which produces the non-ambiguous result of $xzxzz$.*

There exist other strategies for resolving ambiguities. We'll concentrate on the described one because we believe it is the most natural and most useful in practice.

Let's now define formally the set of all rewriting contexts within a fixed word which we'll consider *valid* for rewriting.

Definition 2.3. *We define the following operators over sets of contexts:*

$$\begin{array}{lcl} \mathtt{OVER}(C, C') & = & \{\langle u, v, w\rangle_{\in C} | (\exists \langle u', v', w'\rangle_{\in C'})(\langle u', v', w'\rangle \prec \langle u, v, w\rangle)\} \\ \mathtt{LEFT}(C) & = & \{\langle u, v, w\rangle_{\in C} | \neg(\exists \langle u', v', w'\rangle_{\in C})(u' \subset u)\} \\ \mathtt{LONG}(C) & = & \{\langle u, v, w\rangle_{\in C} | \neg(\exists \langle u\ , v', w'\rangle_{\in C})(v \subset v')\} \end{array}$$

The first operator `OVER` defines all contexts in a given set which are overlapped by a context in another set. The second operator `LEFT` defines the *leftmost* contexts in a given set.

Considering the above definitions we may proceed with defining formally the valid contexts. We construct inductively the sequence $\{\mathcal{C}_i\}_{i=0}^{\infty}$, where $\mathcal{C}_i \subseteq \mathcal{C}$ for all $i \geq 0$:

- $\mathcal{C}_0 = \emptyset$
- $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \mathtt{LEFT}(\mathcal{C} - \mathcal{C}_i - \mathtt{OVER}(\mathcal{C}, \mathcal{C}_i))$

This is a monotonously increasing sequence and hence it has a least upper bound. We define $\mathcal{C}_v = \cup_{i=0}^{\infty} \mathcal{C}_i$ we call $\mathcal{C}_v$ the valid contexts set. We denote $\mathcal{C}_l = \mathtt{LONG}(\mathcal{C}_v)$ and we call $\mathcal{C}_l$ the set of the longest valid contexts.

Proposition 2.1. $\mathcal{C}_v \subseteq \mathcal{C}$

Proposition 2.2. *$\mathcal{C}_v$ doesn't contain any overlapping contexts.*

Proof. Let's assume $\langle u_i, v_i, w_i\rangle \in \mathcal{C}_v$ for $i = 1, 2$ and $\langle u_1, v_1, w_1\rangle \prec \langle u_2, v_2, w_2\rangle$. Then $u_1 \subset u_2 \subset u_1 v_1$. Let k_i are the least indices for which $\langle u_i, v_i, w_i\rangle \in \mathcal{C}_{k_i}$ (we're sure they exist because $\langle u_i, v_i, w_i\rangle \in \mathcal{C}_v = \cup_{k=0}^{\infty} \mathcal{C}_k$).

Let's assume that $k_1 \geq k_2$. Then

$$\langle u_2, v_2, w_2\rangle \in \mathcal{C}_{k_2} = \mathcal{C}_{k_2-1} \cup \mathtt{LEFT}(\mathcal{C} - \mathcal{C}_{k_2-1} - \mathtt{OVER}(\mathcal{C}, \mathcal{C}_{k_2-1}))$$

We denote $R = \mathcal{C} - \mathcal{C}_{k_2-1} - \texttt{OVER}(\mathcal{C}, \mathcal{C}_{k_2-1})$. As $\langle u_2, v_2, w_2\rangle \notin \mathcal{C}_{k_2-1}$, we conclude that $\langle u_2, v_2, w_2\rangle \in \texttt{LEFT}(R)$. On the other hand $k_1 \geq k_2$, hence $\langle u_1, v_1, w_1\rangle \in R$. This is a contradicts with the definition of LEFT so finally $k_1 < k_2$.

After we saw that $k_1 < k_2$, we may further conclude that $\langle u_2, v_2, w_2\rangle \notin \mathcal{C}_k$ for $k \leq k_1$. For $k \geq k_1$, $\langle u_1, v_1, w_1\rangle \in \mathcal{C}_k$ therefore $\langle u_2, v_2, w_2\rangle \in \texttt{OVER}(\mathcal{C}, \mathcal{C}_k)$, e.g. $\langle u_2, v_2, w_2\rangle \notin \mathcal{C}_{k+1}$. This contradicts with the fact that $\langle u_2, v_2, w_2\rangle \in \mathcal{C}_{k_2}$ ($k_2 > k_1$) and hence $\mathcal{C}_v$ doesn't contain any overlapping contexts. □

Proposition 2.3. *Let $\langle u, v, w\rangle \in \mathcal{C}$ is a non-valid context ($\langle u, v, w\rangle \notin \mathcal{C}_v$). There exists ($\langle u_0, v_0, w_0\rangle \in \mathcal{C}_v$), such that $\langle u_0, v_0, w_0\rangle \prec \langle u, v, w\rangle$*

Proof. Let's take the sequence $\{\mathcal{O}_i\}_{i=0}^{\infty}$, where $\mathcal{O}_i = \texttt{OVER}(\mathcal{C}, \mathcal{C}_i)$. One could easily prove that this is a monotonously increasing sequence whose least upper bound exists and is exactly $\texttt{OVER}(\mathcal{C}, \mathcal{C}_v)$.

We'll verify that $\mathcal{C}_v \cup \texttt{OVER}(\mathcal{C}, \mathcal{C}_v) = \mathcal{C}$. Let's assume that there exists $\langle u, v, w\rangle \in \mathcal{C}$, such that $\langle u, v, w\rangle \notin \mathcal{C}_v$ and $\langle u, v, w\rangle \notin \texttt{OVER}(\mathcal{C}, \mathcal{C}_v)$ and let's choose the one with shortest prefix. $\mathcal{C}$ is a finite set so there exists and index k, such that $\mathcal{C}_k = \mathcal{C}_v$. $\langle u, v, w\rangle \notin \texttt{OVER}(\mathcal{C}, \mathcal{C}_k)$ which implies that $\mathcal{C} - \mathcal{C}_k - \texttt{OVER}(\mathcal{C}, \mathcal{C}_k) \neq \emptyset$, and therefore $\mathcal{C}_v \subset \mathcal{C}_{k+1}$ is not a least upper bound of $\{\mathcal{C}_i\}$. This is a contradiction and hence $\mathcal{C}_v \cup \texttt{OVER}(\mathcal{C}, \mathcal{C}_v) = \mathcal{C}$.

Now back to the proposition. Let $\langle u, v, w\rangle \notin \mathcal{C}_v$. Then (according to everything already said) $\langle u, v, w\rangle \in \texttt{OVER}(\mathcal{C}, \mathcal{C}_v)$ which means that there is an index p ($\texttt{OVER}(\mathcal{C}, \mathcal{C}_v)$ is the least upper bound of $\{\mathcal{O}_i\}$) for which $\langle u, v, w\rangle \in \mathcal{O}_p$. Then by the definition of $\{\mathcal{O}_i\}$ it follows that there exists $\langle u_0, v_0, w_0\rangle \in \mathcal{C}_p$, for which $\langle u_0, v_0, w_0\rangle \prec \langle u, v, w\rangle$. □

Proposition 2.4. *Let $\langle u, v_i, w_i\rangle \in \mathcal{C}$ are contexts ($i = 1, 2$). Then $\langle u, v_1, w_1\rangle$ is a valid context iff $\langle u, v_2, w_2\rangle$ is also valid.*

Proof. Let $\langle u, v_1, w_1\rangle$ is a valid context and let's assume $\langle u, v_2, w_2\rangle$ is non-valid. Then there exists $k \geq 0$, such that $\langle u, v_2, w_2\rangle \in \texttt{OVER}(\mathcal{C}, \mathcal{C}_k)$. But this would mean that there exists $\langle u_0, v_0, w_0\rangle \in \mathcal{C}_k$, such that $u_0 \subset u \subset u_0v_0$. Therefore $\langle u_0, v_0, w_0\rangle \prec \langle u, v_1, w_1\rangle$. But according to Proposition 2.2 there are no overlapping contexts in $\mathcal{C}_v$ which contradicts with our assumption. □

Follows a definition of *result from the application of a context-sensitive rule over some given input text.*

Definition 2.4. *Let $\alpha \in \Sigma^*$ and $E \to \beta$ / L_R is a context-sensitive rewrite rule. Result from the application of $E \to \beta$ / L_R over α is the word*

$$u_1 \cdot \beta \cdot (u_1v_1)^{-1}u_2 \cdot \beta \cdot (u_2v_2)^{-1}u_3 \cdot \beta \cdot \ldots (u_{k-1}v_{k-1})^{-1}u_k \cdot \beta \cdot w_k$$

where $\mathcal{C}_l(\alpha; L, E, R) = \{\langle u_i, v_i, w_i\rangle | i = 1, \ldots, k\}$ and $i < j \to u_i \subset u_j$.

The direct usage of Definition 2.4 would greatly complicate our treatment of the problem, we introduce the following equivalent definition.

Definition 2.5. *Let* $\alpha = a_1 a_2 \dots a_n \in \Sigma^*$ *and* $E \to \beta \;/\; L\_R$ *is a context-sensitive rewrite rule/ Result from the application of* $E \to \beta \;/\; L\_R$ *over* α *is the word* $\omega_1 \cdot \omega_2 \dots \cdot \omega_n \cdot \omega_{n+1}$*, where for* $i = 1, 2, \dots, n$

$$\omega_i = \begin{cases} \epsilon & \textit{if there exists } \langle u, v, w\rangle \in \mathcal{C}_v(\alpha; L, E, R),\ |u| < i-1 < |uv| \\ \beta & \textit{if there exists } \langle u, v, w\rangle \in \mathcal{C}_v(\alpha; L, E, R),\ |u| = i-1 < |uv| \\ \beta a_i & \textit{if there exists } \langle u, v, w\rangle \in \mathcal{C}_v(\alpha; L, E, R),\ |u| = i-1 = |uv| \\ & \quad \textit{and there is no } \langle u, v', w'\rangle \in \mathcal{C}_v(\alpha; L, E, R),\ \textit{such that } v' \neq \epsilon \\ a_i & \textit{otherwise} \end{cases}$$

and $\omega_{n+1} \in \Sigma^*$ *is defined as follows:*

$$\omega_{n+1} = \begin{cases} \beta & \textit{if } \langle \alpha, \epsilon, \epsilon\rangle \in \mathcal{C}_v(\alpha; L, E, R) \\ \epsilon & \textit{otherwise} \end{cases}$$

In order to simplify some of the proofs we further modify Definition 2.5 and finally reach the equivalent

Definition 2.6. *Let* $\alpha = a_1 a_2 \dots a_n \in \Sigma^*$ *and* $E \to \beta \;/\; L\_R$ *is a context-sensitive rewrite rule. Result from the application of* $E \to \beta \;/\; L\_R$ *over* α *is the word* $(\omega_1 \pi_1) \cdot (\omega_2 \pi_2) \dots \cdot (\omega_n \pi_n)$*, where* ω_i *are defined as in Definition 2.5, and*

$$\pi_i = \begin{cases} \beta & \textit{if } i = n \textit{ and } \langle \alpha, \epsilon, \epsilon\rangle \in \mathcal{C}_v(\alpha; L, E, R) \\ \epsilon & \textit{otherwise} \end{cases}$$

The proof that Definitions 2.4, 2.5 and 2.6 are equivalent is trivial and too technical to be interesting for our treatment. Therefore we'll use only Definition 2.6.

3 Finite State Automata

Definition 3.1. *Finite state automaton (FSA) is a 5-tuple*

$$\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$$

where Σ is a finite alphabet, Q is a finite set of states, $S \subseteq Q$ is a set of initial (starting) states, $F \subseteq Q$ is a set of accepting states, and $\Delta \subseteq Q \times \Sigma \times Q$ is transition relation. We extend inductively Δ to Δ^*

- $\langle q, \epsilon, q \rangle \in \Delta^*$ *for each* $q \in Q$
- $\langle q_1, \alpha a, q_2 \rangle \in \Delta^*$ *if there exists* $q \in Q$, *such that* $\langle q_1, \alpha, q \rangle \in \Delta^*$ *and* $\langle q, a, q_2 \rangle \in \Delta$

Definition 3.2. *Let $\Delta \subseteq Q \times \Sigma \times Q$, $T \subseteq Q$ and $a \in \Sigma$. We introduce the following notation*

$$T \xrightarrow{\Delta, a} \{q | (\exists q'_{\in T})(\langle q', a, q \rangle \in \Delta)\}$$

Definition 3.3. *Let $\mathcal{A}$ is a FSA and $\alpha \in \Sigma^*$ is a word. We say that the sequence $q_0, q_1, \ldots q_{|\alpha|}$ is an execution $\mathcal{A}$ over α, if $q_i \in Q$ for each $i = 0 \ldots |\alpha|$, $q_0 \in S$ and $\langle q_{i-1}, \alpha_i, q_i \rangle \in \Delta$ for each $i = 1 \ldots |\alpha|$. One such execution will be assumed* successful *iff $q_{|\alpha|} \in F$ is accepting. Whenever the notation allows we'll also use an alternative definition of automaton execution, namely $\sigma : [0, |\alpha|] \to Q$. It is said that $\mathcal{A}$ accepts (or recognizes) α, if there exists successful execution of $\mathcal{A}$ over α.*

Definition 3.4 (Language of FSA)**.** *Let $\mathcal{A}$ is a FSA. We define the language of $\mathcal{A}$ to be the set $\mathcal{L}(\mathcal{A}) = \{\alpha | \mathcal{A}$ recognizes $\alpha\}$.*

Definition 3.5 (Equivalent Automata)**.** *Let $\mathcal{A}_{1,2}$ are two finite state automata. We say that $\mathcal{A}_1$ is equivalent to$\mathcal{A}_2$ (and write $\mathcal{A}_1 \equiv \mathcal{A}_2$), iff $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$.*

Definition 3.6 (Normal form)**.** *Let $\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$ is a FSA. We say that $\mathcal{A}$ is in* normal form *if for every transition $\langle q_1, a, q_2 \rangle \in \Delta$ it is the case that $q_1 \notin F$ and $q_2 \notin S$. Informally speaking, the* normality *of $\mathcal{A}$ consists of the fact that no transition leaves accepting or enters initial state.*

Proposition 3.1. *For every FSA $\mathcal{A}$, there exists an equivalent $\mathcal{A}^N$, which is in normal form.*

Proof. Let $\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$ is a FSA. We'll construct $\mathcal{A}^N$ from $\mathcal{A}$, by removing possible transitions which spoil its normality: $\mathcal{A}^N = \langle \Sigma, Q^N, S^N, F^N, \Delta^N \rangle$, where

$$Q^N = Q \times \{1\} \cup S \times \{2\} \cup F \times \{3\}$$

The other components of $\mathcal{A}^N$ are $S^N = S \times \{2\}$, $F^N = F \times \{3\}$

$$\begin{aligned} \Delta^N \quad = \quad & \{\langle \langle q_1, 1 \rangle, a, \langle q_2, 1 \rangle \rangle | \langle q_1, a, q_2 \rangle \in \Delta\} \cup \\ & \{\langle \langle q_1, 2 \rangle, a, \langle q_2, 1 \rangle \rangle | \langle q_1, a, q_2 \rangle \in \Delta \ \& \ q_1 \in S\} \cup \\ & \{\langle \langle q_1, 1 \rangle, a, \langle q_2, 3 \rangle \rangle | \langle q_1, a, q_2 \rangle \in \Delta \ \& \ q_2 \in F\} \end{aligned}$$

$\mathcal{A}^N$ is obviously in normal form. Let's show that it is equivalent to $\mathcal{A}$.

Let $\alpha \in \mathcal{L}(\mathcal{A})$ and $q_0, q_1, \ldots q_{|\alpha|-1}, q_{|\alpha|}$ is a successful execution of $\mathcal{A}$ over α. We'll verify that $\langle q_0, 2\rangle, \langle q_1, 1\rangle, \langle q_2, 1\rangle, \ldots, \langle q_{|\alpha|-1}, 1\rangle, \langle q_{|\alpha|}, 3\rangle$ is a successful execution of $\mathcal{A}^N$ over α. First, let's notice that $\langle q_0, 2\rangle \in S^N$, because $q_0 \in S$ and $\langle q_{|\alpha|}, 3\rangle \in F^N$, because $q_{|\alpha|} \in F$. It is clear by the construction of Δ^N that $\langle\langle q_{i-1}, 1\rangle, \alpha_i, \langle q_i, 1\rangle\rangle \in \Delta^N$ for $i = 2 \ldots |\alpha| - 1$, because $\langle q_{i-1}, \alpha_i, q_i\rangle \in \Delta$ for $i = 2 \ldots |\alpha| - 1$. It is clear that $\langle q_0, \alpha_0, q_1\rangle \in \Delta$ and $q_0 \in S$, and therefore by the definition of Δ^N it is true that $\langle\langle q_0, 2\rangle, \alpha_0, \langle q_1, 1\rangle\rangle \in \Delta^N$. By analogy, $\langle\langle q_{|\alpha|-1}, 1\rangle, \alpha_{|\alpha|}, \langle q_{|\alpha|}, 3\rangle\rangle \in \Delta^N$. Thus we showed that there exists a successful execution of $\mathcal{A}^N$ over α, i.e. $\alpha \in \mathcal{L}(\mathcal{A}^N)$.

The proof of in the other direction is analogical. Therefore, we showed that $\mathcal{A} \equiv \mathcal{A}^N$. ☐

Theorem 3.1 (Kleene). *For every regular expression $\mathcal{E}$, there exists a FSA $\mathcal{A}$, such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{E})$.*

The automaton from Kleene's Theorem is not unique and there are many constructions that build it directly from $\mathcal{E}$ (for example [7]). The particular construction is not significant to our purposes so whenever we need to construct a FSA by a regular expression $\mathcal{E}$, we'll write $\mathcal{A}(\mathcal{E})$ and will mean a FSA in normal form such that $\mathcal{L}(\mathcal{A}(\mathcal{E})) = \mathcal{L}(\mathcal{E})$.

Definition 3.7 (Mirror FSA). *Let $\mathcal{A} = \langle\Sigma, Q, S, F, \Delta\rangle$ is a FSA. The FSA $\widetilde{\mathcal{A}} = \langle\Sigma, Q, F, S, \widetilde{\Delta}\rangle$, where $\widetilde{\Delta} = \{\langle q_2, a, q_1\rangle | \langle q_1, a, q_2\rangle \in \Delta\}$ is said to be the* mirror FSA *of $\mathcal{A}$.*

Proposition 3.2. *Let $\mathcal{A} = \langle\Sigma, Q, S, F, \Delta\rangle$ is a FSA and $\widetilde{\mathcal{A}}$ is its mirror FSA. Then for every $\alpha \in \Sigma^*$, $q_0, q_1, \ldots q_{|\alpha|}$ is a successful execution of $\mathcal{A}$ iff $q_{|\alpha|}, q_{|\alpha|-1}, \ldots q_0$ is a successful execution of $\widetilde{\mathcal{A}}$.*

Proof. Obvious by the definition of mirror automaton. ☐

It is a well-known fact that the class of regular languages is closed under concatenation. In other words if the languages $L_{1,2}$ are recognizable by (respectively) $\mathcal{A}_{1,2}$, there exists an automaton $\mathcal{A}$, recognizing $L_1 \cdot L_2$.

We'll define two constructions which define the concatenation of automata and have the special property that every their execution contains as subexecution an execution of (respectively) the first or the second automaton.

Definition 3.8 (Concatenation to the Left). *Let $\mathcal{A}_i = \langle\Sigma, Q_i, S_i, F_i, \Delta_i\rangle$, for $i = 1, 2$ are FSA and $Q_1 \cap Q_2 = \emptyset$. We say that $\mathcal{A} = \langle\Sigma, Q, S, F, \Delta\rangle$ is the result from concatenating of $\mathcal{A}_1$* **to the left** *of $\mathcal{A}_2$ and we write $\mathcal{A} = \mathcal{A}_1 \cdot_l \mathcal{A}_2$, if:*

- $Q = Q_1 \cup Q_2$
- $S = \begin{cases} S_1 \cup S_2 & \textit{if } S_1 \cap F_1 \neq \emptyset \\ S_1 & \textit{if } S_1 \cap F_1 = \emptyset \end{cases}$
- $F = F_2$

- $\Delta = \Delta_1 \cup \Delta_2 \cup \{\langle q_1, a, q_2\rangle | (\exists q_{\in F_1})(\langle q_1, a, q\rangle \in \Delta_1) \;\&\; q_2 \in S_2\}$

In order to illustrate the *concatenation to the left*, let's again examine the example from Fig. 1. It is noticeable that every successful execution of the so constructed automaton $\mathcal{A}_1 \cdot_l \mathcal{A}_2$ contains a successful execution of $\mathcal{A}_2$.

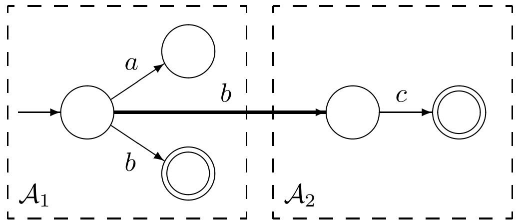


Figure 1: Concatenation to the left

Proposition 3.3. *Let $\mathcal{A}_{1,2}$ are FSA and $\mathcal{A} = \mathcal{A}_1 \cdot_l \mathcal{A}_2$. Then every successful execution of $\mathcal{A}$ over $\overline{a_1 a_2 \ldots a_n} \in \Sigma^*$ looks like*

$$q_0, \ldots, q_{p-1}, q_p, \ldots, q_n$$

where $p \in [0, n]$, $q_p, \ldots, q_n$ is a successful execution of $\mathcal{A}_2$ over $\overline{a_{p+1} \ldots a_n}$ and there exists a state $q \in F_1$, such that $q_0, \ldots, q_{p-1}, q$ is a successful execution of $\mathcal{A}_1$ over $\overline{a_1 \ldots a_p}$.

Proof. Let $q_0, \ldots, q_n$ is some successful execution of $\mathcal{A}$ over $\overline{a_1 a_2 \ldots a_n}$. It is successful and therefore, q_n is an accepting state, e.g. $q_n \in F = F_2$ (by definition of the concatenation to the left). Let $p \in [0, n]$ is the least index such that $q_p \in Q_2$ (it is certain that such a p exists, because $q_n \in Q_2$). Now we notice that for any $i \in [p, n]$, $q_i \in Q_2$. This is true because if it weren't we would be able to choose $i > p$, such that $q_i \in Q_1$ and it would follow that $\langle q_{i-1}, a_i, q_i\rangle \in \Delta$, where $q_{i-1} \in Q_2$, and $q_i \in Q_1$. By the construction of $\mathcal{A}$, this is impossible.

Therefore $q_0, \ldots, q_{p-1} \in Q_1$, and $q_p, \ldots, q_n \in Q_2$. Now if $p > 0$, $\langle q_{p-1}, a_p, q_p\rangle \in \Delta$ and by the definition of Δ it is clear that $q_p \in S_2$. On the other hand if $p = 0$, it would follow $q_p \in S_2$, because $q_0, \ldots, q_n$ is an execution of $\mathcal{A}$. Following the same reasoning we would deduce that $\langle q_{i-1}, a_i, q_i\rangle \in \Delta_2$ (for each $i \in [p+1, n]$), and consequently $q_p, \ldots, q_n$ is a successful execution of $\mathcal{A}_2$ over $\overline{a_{p+1} \ldots a_n}$.

If $p = 0$, then $\overline{a_1 \ldots a_p} = \epsilon$. By the definition of $\mathcal{A}$ and the fact that $q_0 \in S \cap S_2 \neq \emptyset$, it follows that $S_1 \cap F_1 \neq \emptyset$. We choose a state $q \in S_1 \cap F_1$ and thus we show a successful execution of $\mathcal{A}_1$ over ϵ. Now let $p > 0$. Then $q_0 \in Q_1$, and from here it follows that $q_0 \in S_1$. Because $\langle q_{i-1}, a_i, q_i\rangle \in \Delta$ for $i \in (0, p)$ and $q_i \in Q_1$ for $i \in [0, p)$, it follows that $\langle q_{i-1}, a_i, q_i\rangle \in \Delta_1$ for $i \in (0, p)$, and consequently $q_0, \ldots, q_{p-1}$ is an execution of $\mathcal{A}_1$ over $\overline{a_1 \ldots a_{p-1}}$. Now by the definition of $\mathcal{A}$ it follows that there exists $q \in F_1$, such that $\langle q_{p-1}, a_p, q\rangle \in \Delta_1$, and therefore $q_0, \ldots, q_{p-1}, q$ is a successful execution of $\mathcal{A}_1$ over $\overline{a_1 \ldots a_p}$. $\square$

Proposition 3.4. *Let $\mathcal{A}_{1,2}$ are FSA. Then $\mathcal{L}(\mathcal{A}_1 \cdot_l \mathcal{A}_2) = \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$.*

Proof. Let $\mathcal{A}_{1,2} = \langle \Sigma, Q_{1,2}, S_{1,2}, F_{1,2}, \Delta_{1,2} \rangle$ and the concatenation of $\mathcal{A}_1$ to the left of $\mathcal{A}_2$ is $\mathcal{A} = \mathcal{A}_1 \cdot_l \mathcal{A}_2 = \langle \Sigma, Q, S, F, \Delta \rangle$.

Let $\alpha \in \mathcal{L}(\mathcal{A})$. This means that there exists a successful execution $q_0, \ldots, q_n$ of $\mathcal{A}$ over α. According to Proposition 3.3 there exist successful executions of $\mathcal{A}_1$ over $\overline{a_1 \ldots a_p}$ and of $\mathcal{A}_2$ over $\overline{a_{p+1} \ldots a_n}$. This means that (respectively) $\overline{a_1 \ldots a_p} \in \mathcal{L}(\mathcal{A}_1)$ and $\overline{a_{p+1} \ldots a_n} \in \mathcal{L}(\mathcal{A}_2)$, and therefore$\alpha = \overline{a_1 \ldots a_p a_{p+1} \ldots a_n} \in \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$.

Conversely, $\alpha \in \mathcal{L}(\mathcal{A}_1)$, and $\beta \in \mathcal{L}(\mathcal{A}_2)$. Then there exist successful executions $q_{i_0}, \ldots, q_{i_n}$ of $\mathcal{A}_1$ over α and $q_{j_0}, \ldots, q_{j_m}$ of $\mathcal{A}_2$ over β. Obviously $q_{i_0}, \ldots, q_{i_{n-1}}, q_{j_0}, \ldots, q_{j_m}$ will be a successful execution of $\mathcal{A}$ over $\alpha\beta$, and therefore $\alpha\beta \in \mathcal{L}(\mathcal{A})$. Thus we showed that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$. $\square$

By analogy with the operation *concatenation to the left* we define its dual — *concatenation to the right*. We omit the proofs as they are no different from the ones already shown.

Definition 3.9 (Concatenation to the Right)**.** *Let $\mathcal{A}_i = \langle \Sigma, Q_i, S_i, F_i, \Delta_i \rangle$, for $i = 1, 2$ are FSA and $Q_1 \cap Q_2 = \emptyset$. We say that $\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$ is the result from concatenating of $\mathcal{A}_2$* **to the right** *of $\mathcal{A}_1$ and we write $\mathcal{A} = \mathcal{A}_1 \cdot_r \mathcal{A}_2$, if:*

- $Q = Q_1 \cup Q_2$
- $S = S_1$
- $F = \begin{cases} F_1 \cup F_2 & \text{if } S_2 \cap F_2 \neq \emptyset \\ F_2 & \text{if } S_2 \cap F_2 = \emptyset \end{cases}$
- $\Delta = \Delta_1 \cup \Delta_2 \cup \{\langle q_1, a, q_2 \rangle | (\exists q_{\in S_2})(\langle q, a, q_2 \rangle \in \Delta_2) \;\&\; q_1 \in F_1\}$

Proposition 3.5. *Let $\mathcal{A}_{1,2}$ are FSA and $\mathcal{A} = \mathcal{A}_1 \cdot_r \mathcal{A}_2$. Then every successful execution of $\mathcal{A}$ over $\overline{a_1 a_2 \ldots a_n} \in \Sigma^*$ looks like*

$$q_0, \ldots, q_p, q_{p+1}, \ldots, q_n$$

where for some $p \in [0, n]$, $q_0, \ldots, q_p$ is a successful execution of $\mathcal{A}_1$ over $\overline{a_1 \ldots a_p}$ and there exists a state $q \in S_2$, such that $q, q_{p+1}, \ldots, q_n$ is a successful execution of $\mathcal{A}_2$ over $\overline{a_{p+1} \ldots a_n}$.

Proposition 3.6. *Let $\mathcal{A}_{1,2}$ are FSA. Then $\mathcal{L}(\mathcal{A}_1 \cdot_r \mathcal{A}_2) = \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$.*

4 Deterministic Finite Automata

Definition 4.1. *We call a finite state automaton $\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$* determin-istic *(DFA), if for every $q_1 \in Q$ and $a \in \Sigma$ there exists at most one $q_2 \in Q$, such that $\langle q_1, a, q_2 \rangle \in \Delta$, and $|S| = 1$. In other words an automaton is said to be deterministic if it has a single initial state and its transition relation is functional. Because of these properties we'll sometimes denote the deterministic automata as $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$, where $q_0 \in Q$ is an initial state, and $\delta : Q \times \Sigma \to Q$ is transition function.*

Construction 4.1 (Determinization)**.** *Let $\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle$ is a FSA. From $\mathcal{A}$ we construct $\mathcal{A}^D = \langle \Sigma, Q^D, S^D, F^D, \Delta^D \rangle$ by first constructing in parallel the sequences $\{Q_i\}_{i=0}^{\infty}$ and $\{\Delta_i\}_{i=0}^{\infty}$ (where $Q_i \subseteq 2^Q$, and $\Delta_i \subseteq 2^Q \times \Sigma \times 2^Q$) following this inductive scheme:*

- $Q_0 = \{S\}$, $\Delta_0 = \emptyset$
- $Q_{i+1} = Q_i \cup \{T | (\exists T'_{\in Q_i})(\exists a_{\in \Sigma})(T' \xrightarrow{\Delta, a} T)\}$
- $\Delta_{i+1} = \Delta_i \cup \{\langle T', a, T \rangle | T' \in Q_i \ \& \ T' \xrightarrow{\Delta, a} T\}$

It is immediately seen that both sequences are monotonously increasing and therefore converge to their least upper bounds. We define

- $Q^D = \cup Q_i$
- $S^D = \{S\}$
- $F^D = \{T | T \in Q^D \ \& \ T \cap F \neq \emptyset\}$
- $\Delta^D = \cup \Delta_i$

We say that $\mathcal{A}^D$ is received from $\mathcal{A}$ by determinization.

Lemma 4.1. *Let $\mathcal{A}$ is FSA and $\mathcal{A}^D$ is received from it by determinization (Construction 4.1). Then $\mathcal{A}^D$ is deterministic.*

Proof. We'll verify that the relation Δ^D is functional. Let's assume that there exists $T', T_1, T_2 \subseteq Q^D$ and $a \in \Sigma$, such that $\langle T', a, T_1 \rangle \in \Delta^D$, $\langle T', a, T_2 \rangle \in \Delta^D$ and $T_1 \neq T_2$. Δ^D is the least upper bound of $\{\Delta_i\}$ and therefore there exists $n \geq 0$, such that $\langle T', a, T_1 \rangle \in \Delta_n$ and $\langle T', a, T_2 \rangle \in \Delta_n$, and it follows (by the definition of the sequence) that $T' \xrightarrow{\Delta, a} T_1$ and $T' \xrightarrow{\Delta, a} T_2$. In other words $T_1 = \{q | (\exists q'_{\in T'})(\langle q', a, q \rangle \in \Delta)\} = T_2$. Which is a contradiction and that's how we showed that Δ^D is functional relation.

Clearly $|S^D| = |\{S\}| = 1$ and finally $\mathcal{A}^D$ is deterministic.

We denote the initial state of $\mathcal{A}^D$ with $q_0^D = Q^D$, and the transition function with $\delta^D : Q^D \times \Sigma \longrightarrow Q^D$ (defined by its graph Δ^D). ☐

Lemma 4.2. *Let $\mathcal{A}$ is a FSA, $\alpha \in \Sigma^*$ and $q_0, q_1, \ldots, q_{|\alpha|}$ is an execution of $\mathcal{A}$ over α. Let $\mathcal{A}^D$ is received from $\mathcal{A}$ by determinization. Then there exists an execution of $\mathcal{A}^D$: $T_0, T_1, \ldots, T_{|\alpha|}$, such that $q_i \in T_i$ for each $i = 0, 1, \ldots, |\alpha|$.*

Proof. The proof goes by induction on the length of α.

- $|\alpha| = 0$. From $\alpha = \epsilon$ it follows that the execution of $\mathcal{A}$ contains a single state $q_0 \in S$. The searched execution of $\mathcal{A}^D$ also has a single state $T_0 = q_0^D = S$. Obviously $q_0 \in T_0$.

- $|\alpha| > 0$. Let $\alpha = \alpha' a$. By induction hypothesis the lemma should hold for α', and therefore there exists an execution $T_0, T_1, \ldots, T_{|\alpha'|}$ of $\mathcal{A}^D$ over α', for which $q_i \in T_i$ for every $i = 0, \ldots, |\alpha'|$. As $|\alpha'| = |\alpha| - 1$ and $q_0, q_1, \ldots, q_{|\alpha|-1}, q_{|\alpha|}$ is an execution of $\mathcal{A}$ over $\alpha' a$, it should be true that $\langle q_{|\alpha'|}, a, q_\alpha \rangle \in \Delta$. Consequently, if $T_{|\alpha'|} \xrightarrow{\Delta, a} T$, then $q_{|\alpha|} \in T$. Let's now take from the construction of $\mathcal{A}^D$ the least $i \geq 0$, such that $T_{|\alpha'|} \in Q_i$ (we are certain such execution exists because $T_{|\alpha'|} \in \cup Q_i$). Then on the next step of the construction $Q_{i+1} = Q_i \cup \{T | (\exists T'_{\in Q_i})(\exists a_{\in \Sigma})(T' \xrightarrow{\Delta, a} T)\}$, and because $T_{|\alpha'|} \in Q_i$ and $T_{|\alpha'|} \xrightarrow{\Delta, a} T$, it follows that $T \in Q_{i+1}$. Analogically we show that $\langle T_{|\alpha'|}, a, T \rangle \in \Delta_{i+1}$. Therefore $\delta^D(T_{|\alpha'|}, a) = T$ and $T_0, T_1, \ldots, T_{|\alpha'|}, T$ is an execution of $\mathcal{A}^D$ over α complying with the requirements.

□

Lemma 4.3. *Let $\mathcal{A}$ is a FSA and $\mathcal{A}^D$ is received from it by determinization. Let $\alpha \in \Sigma^*$, $T_0, T_1, \ldots, T_{|\alpha|}$ is an execution of $\mathcal{A}^D$ and $q \in T_{|\alpha|}$. There exists an execution of $\mathcal{A}$ — $q_0, q_1, \ldots, q_{|\alpha|} = q$, such that $q_i \in T_i$ for every $i = 0, 1, \ldots, |\alpha|$.*

Proof. The proof again follows induction on the length of α.

- $|\alpha| = 0$. As $\alpha = \epsilon$, the execution of $\mathcal{A}^D$ is a sequence of a single state $T_0 = q_0^D = S$. The sequence of the single state q complies with the requirements because $q \in T_0 = S$.

- $|\alpha| > 0$. Let $\alpha = \alpha' a$. As $T_0, T_1, \ldots, T_{|\alpha'|}, T_{|\alpha|}$ is an execution of $\mathcal{A}^D$, it follows that $\langle T_{|\alpha'|}, a, T_{|\alpha|} \rangle \in \Delta^D$, and therefore $\langle T_{|\alpha'|}, a, T_{|\alpha|} \rangle \in \Delta_n$ for some $n \geq 0$ (Δ^D is the least upper bound of the sequence $\{\Delta_i\}_{i=0}^{\infty}$). Let $m < n$ is the largest index such that $\langle T_{|\alpha'|}, a, T_{|\alpha|} \rangle \notin \Delta_m$. Then on step $m+1$ of the construction, this triple was added to Δ_{m+1}, because $T_{|\alpha'|} \xrightarrow{\Delta, a} T_{|\alpha|}$. Now because $q \in T_{|\alpha|}$, there is $q' \in T_{|\alpha'|}$, such that $\langle q', a, q \rangle \in \Delta$. By the induction hypothesis the lemma holds for $|\alpha'|$, which implies that there exists an execution $q_0, q_1, \ldots, q'$ of $\mathcal{A}$ over α', complying with the requirements. We append q and receive the execution of $\mathcal{A}$ over α we were looking for.

□

Proposition 4.1. *Let $\mathcal{A}$ is a FSA and $\mathcal{A}^D$ is received from it by determinization. Then $\mathcal{A}$ and $\mathcal{A}^D$ are equivalent.*

Proof. Let $\alpha \in \mathcal{L}(\mathcal{A})$. Then there exists a successful execution $q_0, q_1, \dots, q_{|\alpha|} \in F$ of $\mathcal{A}$ over α. According to Lemma 4.2 there exists an execution $T_0, T_1, \dots, T_{|\alpha|}$ of $\mathcal{A}^D$, such that $q_i \in T_i$ for $i \in [0, |\alpha|]$. On the other hand $q_{|\alpha|} \in T_{|\alpha|}$ and $T_{|\alpha|} \cap F \neq \emptyset$, and therefore $T_{|\alpha|} \in F^D$, e.g. $\mathcal{A}^D$ has a successful execution over α e.g. $\alpha \in \mathcal{L}(\mathcal{A}^D)$.

Conversely, let $\alpha \in \mathcal{L}(\mathcal{A}^D)$. This means that there exists a successful execution $T_0, T_1, \dots, T_{|\alpha|}$ of $\mathcal{A}^D$ over α. On the other hand $T_{|\alpha|} \in F^D$ and that's why $T_{|\alpha|} \cap F \neq \emptyset$, e.g. there exists $q \in T_{|\alpha|}$ which is a final state of $\mathcal{A}$. Using Lemma 4.3 we conclude that there is an execution $q_0, q_1, \dots, q$ of $\mathcal{A}$ over α, which shows to be successful ($q \in F$) and it follows that $\alpha \in \mathcal{L}(\mathcal{A})$. $\square$

Corollary 4.1. *For every FSA $\mathcal{A}$ there exists a DFA $\mathcal{A}^D$, such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}^D)$.*

5 Bimachines

Definition 5.1. *We define a bimachine as*

$$\mathcal{B} = \langle \mathcal{A}_L, \mathcal{A}_R, \psi \rangle$$

where $\mathcal{A}_{L,R} = \langle \Sigma, Q_{L,R}, q_{L,R}, \delta_{L,R} \rangle$ *are deterministic finite state automata without any accepting states (respectively left and right), and* $\psi : Q_L \times \Sigma \times Q_R \to \Sigma^*$ *is an output function.*

The bimachines are abstract machines which work over input word (bidirectionally) and produce an output word based on executions of their automata and the input word. Their operational semantics is defined by the transitive closure of ψ, $\psi^* : Q_L \times \Sigma^* \times Q_R \to \Sigma^*$, defined as follows

- $\psi^*(q_1, \epsilon, q_2) = \epsilon$
- $\psi^*(q_1, a\alpha, q_2) = \psi(q_1, a, \delta_R^*(q_2, \widetilde{\alpha})) \cdot \psi^*(\delta_L(q_1, a), \alpha, q_2)$

Based on the functional nature of the bimachines we'll often use the $\mathcal{B} : \Sigma^* \to \Sigma^*$ notation, in which for any $\alpha \in \Sigma^*$ we define as $\mathcal{B}(\alpha) = \psi^*(q_L, \alpha, q_R)$. We'll say that for any particular word $\alpha \in \Sigma^*$ over the bimachine's input alphabet, $\mathcal{B}(\alpha)$ is the result from $\mathcal{B}$'s execution over α.

Behind this long definition of bimachine's execution result lies an intuitively simple strategy. We might assume that the bimachine reads its input word and for any character outputs a word over its alphabet. The result from the bimachine's execution is the concatenation of all output words. On every step the output function decides on its output according to the current character and the two states which would have been reached respectively by the left and right automata exactly before they consume this same character.

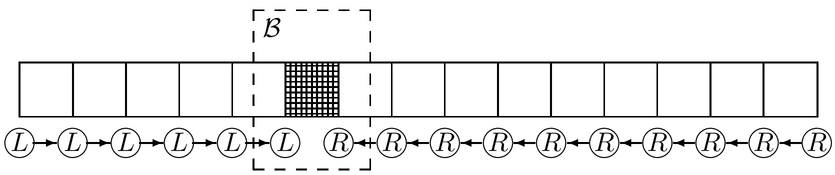


Figure 2: Execution of a bimachine

It becomes clear that the so defined bimachines work only over their input word. Let's examine a modified version of this strategy which only differs in the operational semantics used.

Definition 5.2. *(Left) Output-driven bimachine is*

$$\mathcal{B} = \langle \mathcal{A}_L, \mathcal{A}_R, \psi \rangle$$

where $\mathcal{A}_{L,R}$ *are again DFA,* ψ *is an output function and the operational semantics* $\mathcal{B}$ *of the bimachine is defined as* $\mathcal{B}(\alpha) = \psi^{**}(q_L, \alpha, q_R)$*, where*

- $\psi^{**}(q_1, \epsilon, q_2) = \epsilon$
- $\psi^{**}(q_1, a\alpha, q_2) = \psi(q_1, a, \delta_R^*(q_2, \widetilde{\alpha})) \cdot \psi^{**}(\delta_L^*(q_1, \psi(q_1, a, \delta_R^*(q_2, \widetilde{\alpha}))), \alpha, q_2)$

An important property of the output-driven bimachines is the fact that their left automaton doesn't read the input word directly but the output produced by the output function instead. Absolutely symmetrically one might define a right output-driven bimachine.

Now we proceed to demonstrate that every output-driven bimachine can be simulated by an equivalent *classical* one (working purely over its input).

Construction 5.1. *Let $\mathcal{B}$ is an output-driven bimachine. We define a FSA*

$$\mathcal{A}_L^N = \langle \Sigma, Q_L \times Q_R, \{q_L\} \times Q_R, \Delta_L \rangle$$

where the transition relation is defined as follows:

$$\langle \langle p_1, q_1 \rangle, a, \langle p_2, q_2 \rangle \rangle \in \Delta_L \iff \delta_R(q_2, a) = q_1 \,\&\, \delta_L^*(p_1, \psi(p_1, a, q_2)) = p_2$$

Now we construct the left DFA $\mathcal{A}'_L$ by determinizing $\mathcal{A}_L^N$ (Construction 4.1). We take $\mathcal{A}'_R = \mathcal{A}_R$ for right DFA of the bimachine and define the output function $\psi' : Q'_L \times \Sigma \times Q'_R$ as follows:

$$\psi'(L, a, r) = \beta \iff (\exists \langle p, q \rangle \in L)(\delta_R(r, a) = q \,\&\, \psi(p, a, r) = \beta)$$

Thus we completed the construction of $\mathcal{B}' = \langle \mathcal{A}'_L, \mathcal{A}'_R, \psi' \rangle$.

In order to be sure that $\mathcal{B}'$ is a classical bimachine, equivalent to $\mathcal{B}$, we should first check that it is correctly defined and that for any $\alpha \in \Sigma^*$, $\mathcal{B}'(\alpha) = \mathcal{B}(\alpha)$. The correctness proof requires only to show that ψ' is actually a function.

Proposition 5.1. *Let $L \in Q'_L$ is a state of $\mathcal{A}'_L$ and $\langle p_k, q_k \rangle \in L (k = 1, 2)$. Then $q_1 = q_2 \rightarrow p_1 = p_2$*

Proof. Let $Q'_L = \cup Q_i$ is the least upper bound of the sequence $\{Q_i\}_{i=0}^{\infty}$ from Construction 4.1. We demonstrate a proof based on complete induction on the least index i, such that $L \in Q_i$ (i exists because Q'_L is the least upper bound of the sequence).

$i = 0$ Obviously $p_1 = p_2 = q_L$, because $L = \{q_L\} \times Q_R$.

$i + 1$ Let $i + 1$ is the least index such that $L \in Q_{i+1}$. Then (from Construction 4.1) there exists $L' \in Q_i$, such that $\delta'_L(L', a) = L$ for some $a \in \Sigma$. Let $\langle p_k, q_k \rangle \in L$. Hence there exist $\langle p'_k, q'_k \rangle \in L'$, such that $\langle \langle p'_k, q'_k \rangle, a, \langle p_k, q_k \rangle \rangle \in \Delta_L$ (for $k = 1, 2$). Now let's assume that $q_1 = q_2$. Therefore $q'_1 = \delta_R(q_1, a) = \delta_R(q_2, a) = q'_2$. Let $i' < i + 1$ is the least index such that $L' \in Q_{i'}$. From the induction hypothesis for i' we deduce that $p'_1 = p'_2$. Finally, $p_1 = \delta_L^*(p'_1, \psi(p'_1, a, q_1)) = \delta_L^*(p'_2, \psi(p'_2, a, q_2)) = p_2$, which is what we wanted to show.

□

Corollary 5.1. *ψ' is correctly defined function*

Proof. Let's assume that $\psi'(L, a, r) = \beta_{1,2}$. By the definition of ψ' it would follow that there exist $\langle p_i, q_i\rangle \in L$ for $i = 1, 2$, such that $\delta_R(r, a) = q_i$, and additionally $\psi(p_i, a, r) = \beta_i$. But then $q_1 = \delta_R(r, a) = q_2$ and from Proposition 5.1 it follows that $p_1 = p_2$. This leads us to the fact that $\beta_1 = \psi(p_1, a, r) = \psi(p_2, a, r) = \beta_2$. Thus, we showed that ψ' is correctly defined function. ☐

It remains to show that the constructed classical bimachine is equivalent to the output-driven bimachine.

Proposition 5.2. *Let $t = \alpha\beta \in \Sigma^*$, $\delta_L'^*(q_L', \alpha) = L$. Then*

$$\langle \delta_L^*(q_L, \psi^{**}(q_L, \alpha, \delta_R^*(q_R, \widetilde{\beta}))), \delta_R^*(q_R, \widetilde{\beta})\rangle \in L$$

Proof. The proof uses a simple induction on α.

$\alpha = \epsilon$ When $\alpha = \epsilon$, by the definition of ψ^{**} we have that $\psi^{**}(q_L, \alpha, \delta_R^*(q_R, \widetilde{\beta})) = \epsilon$ therefore $\delta_L^*(q_L, \psi^{**}(q_L, \alpha, \delta_R^*(q_R, \widetilde{\beta}))) = q_L$. Now $L = q_L' = \{q_L\} \times Q_R$ hence $\langle q_L, \delta_R^*(q_R, \widetilde{\beta})\rangle \in L$.

$\alpha = \alpha_1 a$ Let's assume that the proposition is true for α_1 and examine the $\alpha = \alpha_1 a$ case. Let $\delta_L^*(q_L, \psi^{**}(q_L, \alpha_1, \delta_R^*(q_R, \widetilde{\beta}))) = p_1$, $\delta_R^*(q_R, \widetilde{\beta}) = r$, and $\delta_R^*(r, a) = r_1$. By induction hypothesis we might deduce that $\langle p_1, r_1\rangle \in L'$. On the other hand $L = \delta_L'(L', a)$, and because $\delta_R(r, a) = r_1$ we show that $\langle\langle p_1, r_1\rangle, a, \langle \delta_L^*(p_1, \psi(p_1, a, r)), r\rangle\rangle \in \Delta_L$. Hence (by Construction 4.1) it follows that $\langle \delta_L^*(p_1, \psi(p_1, a, r)), r\rangle \in L$. Now by the definition of ψ^{**}, $\delta_L^*(p_1, \psi(p_1, a, r)) = \delta_L^*(q_L, \psi^{**}(q_L, \alpha, \delta_R^*(q_R, \widetilde{\beta})))$, which proves the proposition.

☐

Corollary 5.2. *For any $\alpha \in \Sigma^*$, $\mathcal{B}(\alpha) = \mathcal{B}'(\alpha)$*

Proof. Let's start with the case when $\alpha = \epsilon$. By the definitions of ψ and ψ' it follows that $\mathcal{B}(\alpha) = \psi^{**}(q_L, \epsilon, q_R) = \epsilon = \psi'^*(q_L', \alpha, q_R') = \mathcal{B}'(\alpha)$.

If $\alpha = \overline{a_1 a_2 \dots a_n}$ we should prove that for $i = 1, 2, \dots, n$ and $r = \delta_R^*(q_R, \overline{a_n a_{n-1} \dots a_{i+1}})$

$$\psi(\delta_L^*(q_L, \psi^{**}(q_L, \overline{a_1 a_2 \dots a_{i-1}}, \delta_R(r, a_i))), a_i, r) = \psi'(\delta_L'^*(q_L', \overline{a_1 a_2 \dots a_{i-1}}), a_i, r)$$

But this is exactly the case because according to the already proven Proposition 5.2 and the definition of ψ':

$$\langle \delta_L^*(q_L, \psi^{**}(q_L, \overline{a_1 a_2 \dots a_{i-1}}, \delta_R(r, a_i))), \delta_R(r, a_i)\rangle \in \delta_L'^*(q_L', \overline{a_1 a_2 \dots a_{i-1}})$$

This is how we showed that in every position of the input word $\mathcal{B}$ and $\mathcal{B}'$ output equal results. Hence $\mathcal{B}(\alpha) = \mathcal{B}'(\alpha)$. ☐

6 Direct Construction

Let $E \to \beta \ / \ L\_R$ is a context-sensitive rewrite rule over finite alphabet Σ. We construct the finite automata $\mathcal{A}_L$, $\mathcal{A}_E$ and $\mathcal{A}_R$, such that

$$\begin{array}{lclcll}
\mathcal{A}_L & = & \mathcal{A}(\Sigma^* L) & = & \langle \Sigma, Q_L, S_L, F_L, \Delta_L \rangle & , \quad \mathcal{L}(\mathcal{A}_L) = \mathcal{L}(\Sigma^* L) \\
\mathcal{A}_E & = & \mathcal{A}(E) & = & \langle \Sigma, Q_E, S_E, F_E, \Delta_E \rangle & , \quad \mathcal{L}(\mathcal{A}_E) = \mathcal{L}(E) \\
\mathcal{A}_R & = & \mathcal{A}(R\Sigma^*) & = & \langle \Sigma, Q_R, S_R, F_R, \Delta_R \rangle & , \quad \mathcal{L}(\mathcal{A}_R) = \mathcal{L}(R\Sigma^*)
\end{array}$$

Even though we haven't fixed a particular construction of FSA from regular expression we assume the implicit requirement that $\mathcal{A}_L$, $\mathcal{A}_E$ and $\mathcal{A}_R$ are in normal form.

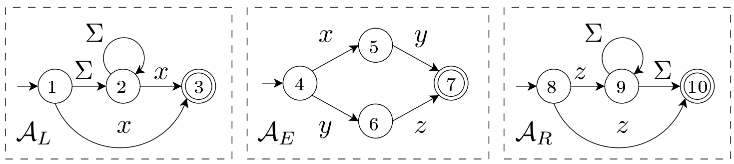


Figure 3: Finite automata in normal form constructed respectively from the regular expressions $\Sigma^* x$, $xy|yz$ and $z\Sigma^*$

After we constructed $\mathcal{A}_L$, $\mathcal{A}_E$ and $\mathcal{A}_R$, we concatenate them to obtain

$$\mathcal{A} = \langle \Sigma, Q, S, F, \Delta \rangle = \mathcal{A}_L \cdot_l \mathcal{A}_L \cdot_r \mathcal{A}_R$$

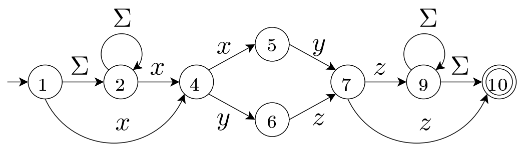


Figure 4: Finite automaton obtained by the concatenation $\mathcal{A} = \mathcal{A}_L \cdot_l \mathcal{A}_E \cdot_r \mathcal{A}_R$, from the rewrite rule $xy|yz \to \epsilon \ / \ x\_z$ from Example 2.2

From the properties of the FSA's normal form and the concatenation to the left/right we can deduce the following

Property 6.1. *Every successful execution of $\mathcal{A}$ over $t \in \Sigma^*$ is of the type*

$$q_{l_0}, q_{l_1}, \ldots, q_{l_{|u|-1}}, q_{e_0}, q_{e_1}, \ldots, q_{e_{|v|}}, q_{r_1}, q_{r_2}, \ldots, q_{r_{|w|}}$$

where $\langle u, v, w \rangle \in \mathcal{C}(t; L, E, R)$, $q_{e_0} \in S_E$, $q_{e_{|v|}} \in F_E$, $q_{e_i} \in Q_E - S_E - F_E$ for every $i \in (0, |v|)$, $q_{l_i} \in Q_L$ for every $i \in [0, |u|)$, $q_{r_i} \in Q_R - S_R - F_R$ for every $i \in (|uv|, |uvw|)$, and $q_{r_{|uvw|}} \in F_R$. We'll call such an execution of $\mathcal{A}$ — an execution for the context $\langle u, v, w \rangle$. We denote $\mathcal{X}(t; L, E, R) = \{\sigma | \sigma$ is an execution for some $\langle u, v, w \rangle \in \mathcal{C}(t; L, E, R)\}$

Property 6.2. *For every context* $\langle u, v, w\rangle \in \mathcal{C}(t; L, E, R)$ *there exists and execution* $\sigma \in \mathcal{X}(t; L, E, R)$ *for* $\langle u, v, w\rangle$.

We'll denote $\mathcal{X}_v(t; L, E, R) = \{\sigma | \sigma \text{ is an execution for some } \langle u, v, w\rangle \in \mathcal{C}_v(t; L, E, R)\}$

In order to construct the classical bimachine in question, we first construct an output-driven bimachine $\mathcal{B} = \langle \mathcal{A}_L, \mathcal{A}_R, \psi\rangle$, working only over its output. It will resolve any possible context ambiguities.

6.1 The Left Automaton

We extend the input alphabet Σ into $\Sigma^c = \Sigma \cup \{a^c | a \in \Sigma\}$. Informally speaking, Σ^c extends Σ by adding a cloned version of every character in Σ. In order to construct the left automaton we first define

$$\mathcal{A}_1^N = \langle \Sigma^c, Q, S, F, \Delta_1^N\rangle$$

where $\Delta_1^N = \Delta \cup \{\langle q_1, a^c, q_2\rangle | \langle q_1, a, q_2\rangle \in \Delta \;\&\; q_2 \notin S_E\} - Q \times \Sigma^c \times Q_R$.

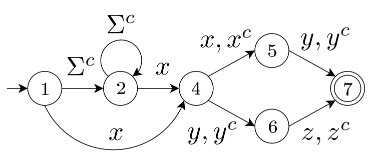


Figure 5: Non-deterministic version $\mathcal{A}_1^N$ of the left automaton $\mathcal{A}_1$, constructed from the rewrite rule $xy|yz \rightarrow \epsilon \;/\; x\_z$

Intuitively, $\mathcal{A}_1^N$ behaves exactly as $\mathcal{A}_L \cdot_l \mathcal{A}_E$ with the only difference that whenever a cloned character is consumed no transitions into initial states of $\mathcal{A}_E$ are allowed. Thus, no executions are possible for contexts, whose focus is about to be processed. On the other hand, the output function of the bimachine will be responsible to output cloned characters only when focuses of valid contexts are processed.

We then determinize $\mathcal{A}_1^N$ (Construction 4.1) and receive the left automaton $\mathcal{A}_1$ of the bimachine.

6.2 The Right Automaton

We first reverse $\mathcal{A}$ to receive its mirror $\mathcal{A}_2^N = \widetilde{\mathcal{A}}$. Then we construct the right automaton $\mathcal{A}_2$ of the bimachine by determinizing $\mathcal{A}_2^N$ (Construction 4.1).

6.3 The Output Function

The output function ψ of the bimachine $\mathcal{B}$ is defined as follows:

$$\psi(L, a, R) = \begin{cases} a^c & \text{if } \delta_1(L, a) \cap R \cap (Q_E - S_E - F_E) \neq \emptyset \\ a & \text{if } \delta_1(L, a) \cap R \cap (Q_E - S_E - F_E) = \emptyset \end{cases}$$

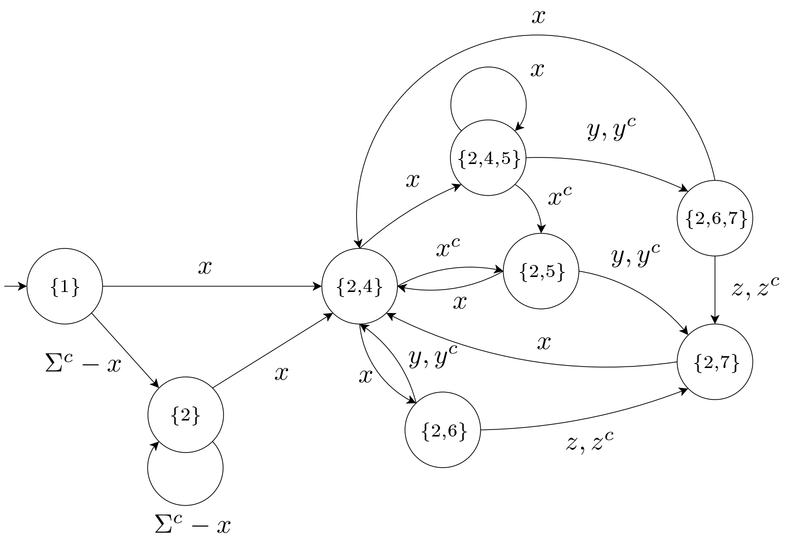


Figure 6: Deterministic $\mathcal{A}_1$ constructed from $\mathcal{A}_1^N$ of Figure 5. For greater readability in the above example we have omitted transitions from every state with every character in Σ^c to $\{2\}$

This finishes the definition of $\mathcal{B} = \langle \mathcal{A}_L, \mathcal{A}_R, \psi \rangle$ - an output-driven bimachine. Before we finish the construction of a bimachine for context-sensitive rewrite rule, let's show several interesting properties of $\mathcal{B}$.

Lemma 6.1. *Let $t = \alpha\beta$, $\delta_2^*(q_2, \widetilde{\beta}) = R$ and $\delta_1^*(q_1, \psi^{**}(q_1, \alpha, R)) = L$. Also let σ is an execution of $\mathcal{A}$ over α, such that $\sigma(|\alpha|) \in Q_L$. Then $\sigma(|\alpha|) \in L$.*

Proof. Induction on the length of α.

$\alpha = \epsilon$ It is obvious that $L = \delta_1^*(q_1, \psi^{**}(q_1, \epsilon, R)) = \delta_1^*(q_1, \epsilon) = q_1 = S$. Hence $\sigma(|\alpha|) = \sigma(0) \in S = L$.

$\alpha = \alpha' a$ Since $\sigma(|\alpha|) \in Q_L$ and because of Property 6.1 we might assert that $\sigma(|\alpha'|) \in Q_L$. Let's denote $R' = \delta_2(R, a)$ and $L' = \delta_1^*(q_1, \psi^{**}(q_1, \alpha', R))$. Then by the induction hypothesis it follows that $\sigma(|\alpha'|) \in L'$. As σ is an execution of $\mathcal{A}$, it is true that $\langle \sigma(|\alpha'|), a, \sigma(|\alpha|) \rangle \in \Delta$. It remains to note that $L = \delta_1^*(L', \psi(L', a, R))$, moreover $\sigma(|\alpha|) \notin S_E \cup Q_R$ and from the definition of δ_1 we may conclude that $\sigma(|\alpha|) \in L$.

□

Lemma 6.2. *Let $t = \alpha\beta$ and $\delta_2^*(q_2, \widetilde{\beta}) = R$ and $\delta_1^*(q_1, \psi^{**}(q_1, \alpha, R)) = L$. Then if $q \in L$ then there exists σ – an execution of $\mathcal{A}$ over α, such that $\sigma(|\alpha|) = q$.*

Proof. Induction on α:

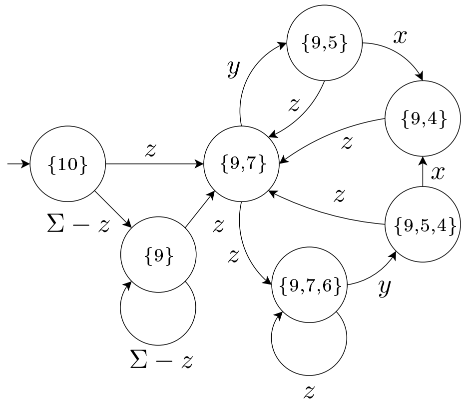


Figure 7: The right automaton $\mathcal{A}_R$ constructed from the rewrite rule $xy|yz \to \epsilon\ /\ x\_y$. For simplicity transitions going into $\{9\}$ have been omitted

		L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7
		{1}	{2}	{2,4}	{2,4,5}	{2,5}	{2,6}	{2,6,7}	{2,7}
R_0	{10}	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_1	{9}	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_2	{9,7}	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_3	{9,5}	$\cdot/\cdot$	$\cdot/\cdot$	x/x^c	x/x^c	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_4	{9,7,6}	$\cdot/\cdot$	$\cdot/\cdot$	y/y^c	y/y^c	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_5	{9,4}	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R_6	{9,5,4}	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$

Figure 8: The output function of $\mathcal{B}$ from the rewrite rule $xy|yz \to \epsilon\ /\ x\_z$. Only non-trivial rewritings have been shown

$\alpha = \epsilon$ Then $L = q_1 = S$. Now if $q \in L$ we examine an execution σ consisting of a single state q. σ is obviously an execution of $\mathcal{A}$ and $\sigma(|\alpha|) = q$.

$\alpha = \alpha' a$ Let $\delta_2(R, a) = R'$ and $\delta_1^*(q_1, \psi^{**}(q_1, \alpha', R')) = L'$. Now $L = \delta_1^*(L', \psi(L', a, R))$. We notice that $\psi(L', a, R) \in \{a, a^c\}$, and therefore $L = \delta_1(L', \psi(L', a, R))$.

Let $q \in L$. There exists $q' \in L'$, such that $\langle q', \psi(L', a, R), q\rangle \in \Delta_1^N$, hence by the definition of Δ_1^N it follows that $\langle q', a, q\rangle \in \Delta$. By the induction hypothesis it follows that there exists σ' – an execution of $\mathcal{A}$ over α', such that $\sigma'(|\alpha'|) = q'$. We define

$$\sigma(i) = \begin{cases} \sigma'(i) & i \le |\alpha'| \\ q & i = |\alpha| \end{cases}$$

It is easily shown that σ' is exactly the wanted execution of $\mathcal{A}$ over α.

$\square$

Proposition 6.1. *Let $t = \alpha\beta \in \Sigma^*$, $\delta_2^*(q_2, \widetilde{\beta}) = R$, and $\delta_1^*(q_1, \psi^{**}(q_1, \alpha, R)) = L$. Then for every $q_{\in Q_E}$, $q \in L \cap R$ iff $(\exists \sigma_{\in \mathcal{X}_v})(\sigma(|\alpha|) = q)$.*

Proof. Before we proceed with the actual proof, let's verify that the proposition might be reduced to those $q \in Q_E$ for which there exists an execution $\sigma \in \mathcal{X}$, such that $\sigma(|\alpha|) = q$. Indeed, if $q \in L \cap R$, then $q \in L$ and by Lemma 6.2 an execution σ_L of $\mathcal{A}$ over α would exist, such that $\sigma_L(|\alpha|) = q$. On the other hand $q \in R$ hence by Lemma 4.3 there exists an execution σ_R of $\widetilde{\mathcal{A}}$ over $\widetilde{\beta}$, such that $\sigma_R(|\beta|) = q$. Then σ, defined as follows:

$$\sigma(i) = \begin{cases} \sigma_L(i) & i \leq |\alpha| \\ \sigma_R(|t| - i) & i > |\alpha| \end{cases}$$

is a successful execution of $\mathcal{A}$ and therefore $\sigma \in \mathcal{X}$.

The proof in the opposite direction is obvious because $\mathcal{X}_v \subseteq \mathcal{X}$.

Let us now define the set

$$A = \{\langle \alpha, q \rangle | \alpha \subseteq t \ \& \ q \in Q_E \ \& \ (\exists \sigma_{\in \mathcal{X}})(\sigma(|\alpha|) = q)\}$$

and a relation "$\prec$" in A, such that

$$\langle \alpha_1, q_1 \rangle \prec \langle \alpha_2, q_2 \rangle \iff \alpha_1 \subset \alpha_2 \vee \alpha_1 = \alpha_2 \ \& \ q_1 \in (Q_E - S_E) \ \& \ q_2 \in S_E$$

This is a partial ordering of A which turns A into a well-founded set.

We proof the proposition by *induction on the structure of* A. Let's fix an element $\langle \alpha, q \rangle \in A$ and assume that for any preceding element $\langle \alpha', q' \rangle \in A$ ($\langle \alpha', q' \rangle \prec \langle \alpha, q \rangle$) the proposition is valid. We show that it remains valid for $\langle \alpha, q \rangle$.

$\alpha = \epsilon$

$\Rightarrow$ Let $q \in L \cap R$. But $\alpha = \epsilon$, therefore $q = \sigma(|\alpha|) = \sigma(0) \in S \cap Q_E$, and hence $q \in S_E$. This means that there exists a context $\langle \epsilon, v, w \rangle \in \mathcal{C}$ and σ is an execution for it. But $\langle \epsilon, v, w \rangle \in \mathcal{C}_1$ (by the construction of $\mathcal{C}_v$), and therefore $\langle \epsilon, v, w \rangle \in \mathcal{C}_v$, and $\sigma \in \mathcal{X}_v$.

$\Leftarrow$ Let $\sigma \in \mathcal{X}_v$. Then $q = \sigma(|\alpha|) = \sigma(0) \in S = q_1 = L$. On the other hand $\widetilde{\sigma}$ is an execution of $\widetilde{\mathcal{A}}$ and according to Lemma 4.2 $q \in \delta_2^*(q_2, \widetilde{\beta}) = R$, which leads us to $q \in L \cap R$.

$\alpha = \alpha' a$ Let's denote $R' = \delta_2(R, a)$, $L' = \delta_1^*(q_1, \psi^{**}(q_1, \alpha', R'))$.

$\Rightarrow$ Let $q \in L \cap R$. We have two options for q:

$\mathbf{q \in Q_E - S_E}$ Then (by Construction 4.1) there exists $q' \in L'$, such that $\langle q', a, q \rangle \in \Delta$ and hence $q' \in \delta_2(R, a) = R'$. By property 6.1 we may assert that $q' \in Q_E$ hence $q' \in L' \cap R' \cap Q_E$. The induction hypothesis for $\langle \alpha', q' \rangle$ states that $\sigma' \in \mathcal{X}_v$ and $\sigma'(|\alpha'|) = q'$. As $q \in R$ there exists an execution σ_R of $\widetilde{\mathcal{A}}$ over $\widetilde{\beta}$, such that $\sigma_R(0) \in F$ and

$\sigma(|\widetilde{\beta}|) = q$ (Lemma 4.3). Now we may define σ – a successful execution of $\mathcal{A}$ as follows:

$$\sigma(i) = \begin{cases} \sigma'(i) & \text{if } i < |\alpha| \\ \sigma_R(|t| - i) & \text{if } i \geq |\alpha| \end{cases}$$

It is easily seen that if σ' was an execution for $\langle u_0, v_0, w_0\rangle \in \mathcal{C}_v$ then σ is an execution for $\langle u_0, v, w\rangle \in \mathcal{C}$. Now from Proposition 2.4 it follows that $\langle u_0, v, w\rangle \in \mathcal{C}_v$ is also a valid context, e.g. $\sigma \in \mathcal{X}_v$.

$\mathbf{q \in S_E}$ The first fact, we immediately notice is that $\psi(L', a, R) \notin (\Sigma^c - \Sigma)$ (this is true because if it weren't one could possibly have stated that $L \cap S_E = \emptyset$). As a direct consequence from the definition of ψ it follows that $L \cap R \cap (Q_E - S_E - F_E) = \emptyset$. Now let $\sigma \in \mathcal{X}$ is such that $\sigma(|\alpha|) = q$, and this is an execution for the context $\langle u, v, w\rangle \in \mathcal{C}$. We show that this context is a valid one ($\langle u, v, w\rangle \in \mathcal{C}_v$). Indeed, let's assume that $\langle u, v, w\rangle \notin \mathcal{C}_v$. Then there certainly exists $\langle u', v', w'\rangle \in \mathcal{C}_v$, such that $u' \subset u \subset u'v'$ (Proposition 2.3), and therefore there is an execution $\sigma' \in \mathcal{X}_v$ for $\langle u', v', w'\rangle$, such that $\sigma'(|\alpha|) \in Q_E - S_E - F_E$ (Property 6.1). Then $\langle \alpha, \sigma'(|\alpha|)\rangle \prec \langle \alpha, q\rangle$ and by the induction hypothesis $\sigma'(|\alpha|) \in L \cap R$, which leads us to $L \cap R \cap (Q_E - S_E - F_E) \neq \emptyset$, which is a contradiction with our assumption that $\langle u, v, w\rangle \notin \mathcal{C}_v$. Therefore $\sigma \in \mathcal{X}_v$.

$\Leftarrow$ Let $\sigma \in \mathcal{X}_v$ and $\sigma(|\alpha|) = q$. As σ is a successful execution of $\mathcal{A}$ over $\alpha\beta$, there should exist another execution σ_R of $\widetilde{\mathcal{A}}$ over $\widetilde{\beta}$ and $\sigma_R(|\beta|) = q$. This means that $q \in R$ (Lemma 4.2). It only remains to show that $q \in L$.

We again examine two cases:

$\mathbf{q \in Q_E - S_E}$ From Property 6.1 it follows that $\sigma(|\alpha'|) \in Q_E$ and then by induction hypothesis one might deduce that $\sigma(|\alpha'|) \in L' \cap R'$. Since $q \notin S_E \cup Q_R$ whatever the value of $\psi(L', a, R)$ is we'll have $q \in \delta_1(L', \psi(L', a, R)) = L$.

$\mathbf{q \in S_E}$ Having in mind how the executions of $\mathcal{A}$ look like (Property 6.1) we have that $\sigma(|\alpha'|) \in Q_L$. Then by Lemma 6.1, $\sigma(|\alpha'|) \in L'$. Let's assume that $q \notin L$. This could only be possible if $\psi(L', a, R) = a^c$ which on the other hand can happen only if $\delta_1(L', a) \cap R \cap (Q_E - S_E - F_E) \neq \emptyset$. Let q_0 is an evidence for that ($q_0 \in \delta_1(L', a) \cap R \cap (Q_E - S_E - F_E)$). Therefore $q_0 \in L \cap R$ and $q_0 \in (Q_E - S_E - F_E)$. Since $\langle \alpha, q_0\rangle \prec \langle \alpha, q\rangle$ we can apply the induction hypothesis for $\langle \alpha, q_0\rangle$ and conclude that there exists $\sigma_0 \in \mathcal{X}_v$, such that $\sigma_0(|\alpha|) = q_0$. From Property 6.1 and the fact that $q_0 \in (Q_E - S_E - F_E)$, we deduce that$\langle u_0, v_0, w_0\rangle \in \mathcal{C}_v$ and $u_0 \subset \alpha \subset u_0v_0$. But if σ is an execution for $\langle u, v, w\rangle \in \mathcal{C}_v$ it easily seen that $\alpha = u$ hence $\langle u, v, w\rangle \notin \mathcal{C}_v$ (there exists a valid context overlapping it and Proposition 2.2). This contradicts our assumption that $q \notin L$. Thus we proved that $q \in L \cap R$.

$\square$

6.4 Bimachine For Context-Sensitive Rewrite Rule

Finally, we construct $\mathcal{B}^I$ – a classical bimachine, working over its input, equivalent to $\mathcal{B}$. Based on $\mathcal{B}^I = \langle \mathcal{A}_1^I, \mathcal{A}_2^I, \psi^I \rangle$ we construct the final bimachine for $E \rightarrow \beta \ / \ L\_R$:

$$\mathcal{B}' = \langle \mathcal{A}_1', \mathcal{A}_2', \psi' \rangle$$

where $\mathcal{A}_1' = \mathcal{A}_1^I$, $\mathcal{A}_2' = \mathcal{A}_2^I$, and the only difference with $\mathcal{B}^I$ is the output function ψ', defined as follows

$$\psi'(L, a, R) = \psi_1'(L, a, R) \cdot \psi_2'(L, a, R)$$

where

$$\psi_1'(L, a, R) = \begin{cases} \epsilon & \text{if } \langle L', \delta_2'(R, a) \rangle \in L \ \& \ L' \cap \delta_2'(R, a) \cap (Q_E - S_E - F_E) \neq \emptyset \\ \beta & \text{if } \langle L', \delta_2'(R, a) \rangle \in L \ \& \ L' \cap \delta_2'(R, a) \cap (S_E - F_E) \neq \emptyset \\ \beta a & \text{if } \langle L', \delta_2'(R, a) \rangle \in L \ \& \ L' \cap \delta_2'(R, a) \cap (S_E \cap F_E) \neq \emptyset \\ a & \text{otherwise} \end{cases}$$

$$\psi_2'(L, a, R) = \begin{cases} \beta & \text{if } \langle L', R \rangle \in \delta_1'(L, a) \ \& \ L' \cap R \cap S_E \neq \emptyset \ \& \ R = q_2' \\ \epsilon & \text{otherwise} \end{cases}$$

Lemma 6.3. *Let $t = \alpha\beta \in \Sigma^*$, $\delta_1'^*(q_1', \alpha) = L$, $\delta_2'^*(q_2', \widetilde{\beta}) = R'$. Then $\langle L', R' \rangle \in L$ and $q \in L' \cap R' \cap Q_E$ iff there exists a context $\langle u, v, w \rangle \in \mathcal{C}_v(t; L, E, R)$ and execution for it $\sigma \in \mathcal{X}_v(t; L, E, R)$, such that $\sigma(|\alpha|) = q$.*

Proof. Let $\langle L', R' \rangle \in L$ and $q \in L' \cap R' \cap Q_E$. Then from Propositions 5.1 and 5.2 it immediately follows that $R' = \delta_2^*(q_2, \widetilde{\beta})$ and $L' = \delta_1^*(q_1, \psi^{**}(q_1, \alpha, R'))$. Now since $q \in L' \cap R' \cap Q_E$, from Proposition 6.1 we may deduce that there exists a valid context $\langle u, v, w \rangle \in \mathcal{C}_v$ and execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(|\alpha|) = q$.

The proof is analogical in the opposite direction. $\square$

Now we are ready to show that the so constructed bimachine actually works as expected, namely realizes substitution according to the context-sensitive rewrite rule $E \rightarrow \beta \ / \ L\_R$.

Proposition 6.2. *Let $E \rightarrow \beta \ / \ L\_R$ is a rewrite rule and $\mathcal{B}'$ is the bimachine constructed from it. Let also $\alpha \in \Sigma^*$ is a word over the rule's alphabet Σ. Then $\mathcal{B}'(\alpha)$ is exactly the result from the application of the context rule over α.*

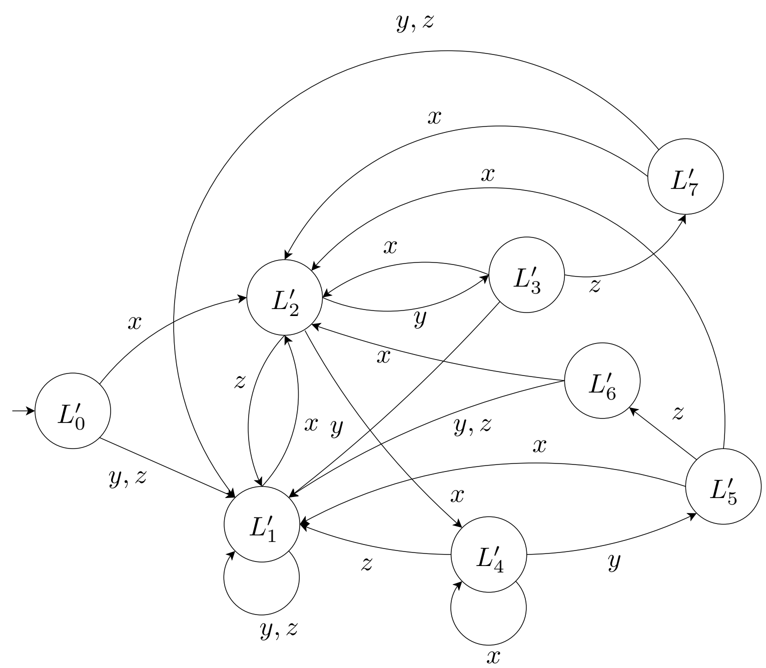


Figure 9: The left automaton of $\mathcal{B}'$ for the rule $xy|yz \to \beta \ / \ x\_z$. Incoming transitions of L'_0 have been omitted for brevity

Proof. In the case when $\alpha = \epsilon$, by Definition 2.6 the result from the application of $E \to \beta \ / \ L\_R$ is exactly $\epsilon = \mathcal{B}'(\alpha)$.

Let $\alpha = \overline{a_1 a_2 \dots a_n}$ where $a_i \in \Sigma$ for $1 \leq i \leq n$. Let $(\omega_1 \pi_1) \cdot (\omega_2 \pi_2) \dots \cdot (\omega_n \pi_n)$ is the result from applying the rule defined by Definition 2.6. We'll show that for any $i \in [1, n]$ if $\delta_1'^*(q_1', \overline{a_1 a_2 \dots a_{i-1}}) = L$ and $\delta_2'^*(q_2', \overline{a_n a_{n-1} \dots a_{i+1}}) = R$ then $\psi_1'(L, a_i, R) = \omega_i$ and $\psi_2'(L, a_i, R) = \pi_i$.

According to ψ_1''s definition four cases have to be examined:

I case Assume that $\langle L', \delta_2'(R, a) \rangle \in L \;\&\; L' \cap \delta_2'(R, a) \cap (Q_E - S_E - F_E) \neq \emptyset$. Then by Lemma 6.3 there exists a valid context $\langle u, v, w \rangle \in \mathcal{C}_v$ and execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(i-1) \in (Q_E - S_E - F_E)$. Based on the type of the executions of $\mathcal{A}$ (Property 6.1) we deduce that $|u| < i - 1 < |uv|$. This (according to Definition 2.6) means that $\omega_i = \epsilon$.

II case Assume that $\langle L', \delta_2'(R, a) \rangle \in L \;\&\; L' \cap \delta_2'(R, a) \cap (S_E - F_E) \neq \emptyset$. Then by Lemma 6.3 there exists a valid context $\langle u, v, w \rangle \in \mathcal{C}_v$ and execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(i-1) \in (S_E - F_E)$. Based on the type of the executions of $\mathcal{A}$ (Property 6.1) we deduce that $|u| = i - 1 < |uv|$. This (according to Definition 2.6) means that $\omega_i = \beta$.

III case Assume that $\langle L', \delta_2'(R, a) \rangle \in L \;\&\; L' \cap \delta_2'(R, a) \cap (S_E \cap F_E) \neq \emptyset$. Then by

	L'_0	L'_1	L'_2	L'_3	L'_4	L'_5	L'_6	L'_7
R'_0	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_1	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_2	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	z/ϵ	y/ϵ	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_3	$\cdot/\cdot$	$\cdot/\cdot$	x/β	$\cdot/\cdot$	x/β	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_4	$\cdot/\cdot$	$\cdot/\cdot$	y/β	z/ϵ	y/ϵ	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_5	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$
R'_6	$\cdot/\cdot$	$\cdot/\cdot$	x/β	$\cdot/\cdot$	x/β	$\cdot/\cdot$	$\cdot/\cdot$	$\cdot/\cdot$

Figure 10: Output function of $\mathcal{B}'$ for the rewrite rule $xy|yz \to \beta \ / \ x\_z$

Lemma 6.3 there exists a valid context $\langle u, v, w\rangle \in \mathcal{C}_v$ and execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(i-1) \in (S_E \cap F_E)$. Based on the type of executions of $\mathcal{A}$ (Property 6.1) we deduce that $|u| = i - 1$. Since haven't fallen into case II it follows that there doesn't exists $\langle u, v, w\rangle \in \mathcal{C}_v$, such that $|v| \neq \epsilon$ and therefore (by Definition 2.6) $\omega_i = \beta a_i$.

IV case Being here means we didn't fall into any of the above cases. Now let's assume there exists $\langle u, v, w\rangle \in \mathcal{C}_v$, such that $|u| \leq i-1 < |uv|$ or $|u| = i-1 = |uv|$. Then there exists an execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(i-1) \in Q_E - F_E$ or $\sigma(|i-1|) \in S_E \cap F_E$. Then (by Lemma 6.3) $\langle L', \delta'_2(R,a)\rangle \in L$ and $\sigma(i-1) \in L' \cap \delta'_2(R,a) \cap Q_E - F_E$ or $\sigma(i-1) \in L' \cap \delta'_2(R,a) \cap S_E \cap F_E$, which contradicts to the fact that we didn't fall into any of the above cases. Thus we conclude that $\omega_i = a_i$.

We showed that $\psi'_1(L, a_i, R) = \omega_i$ for $i = 1, 2, \ldots, n$. It remains to show that $\psi'_2(L, a_i, R) = \pi_i$ for any $i = 1, 2, \ldots, n$.

Let $\psi'_2(L, a_i, R) = \beta$. Then $\langle L', R\rangle \in \delta'_1(L, a)$ and $L' \cap R \cap S_E \neq \emptyset$. According to Lemma 6.3 there will exist $\langle u, v, w\rangle \in \mathcal{C}_v$ and execution for it $\sigma \in \mathcal{X}_v$, such that $\sigma(i) \in S_E$. On the other hand $R = q'_2$ and because of $\mathcal{A}$'s normal form it follows that $\alpha = t$ and hence $\langle u, v, w\rangle = \langle \alpha, \epsilon, \epsilon\rangle \in \mathcal{C}_v$. Now we can immediately conclude that $\pi_i = \beta$.

By analogy we show that if $\pi_i = \beta$ then $\psi'_2(L, a_i, R) = \beta$.

Thus we finished our proof that $\psi'(L, a_i, R) = \omega_i \pi_i$ and therefore $\mathcal{B}'(\alpha)$ is exactly the result from the application of $E \to \beta \ / \ L\_R$ over α. □

Example 6.1. *Let's run the bimachine constructed for $xy|yz \to B \ / \ x\_z$ over the input word $xyzzxxyzz$ from Example 2.2. The execution of $\mathcal{A}'_1$ will be $L'_0, L'_2, L'_3, L'_7, L'_1, L'_2, L'_4, L'_5, L'_6, L'_1$. The right automaton $\mathcal{A}'_2$ will produce $R'_0, R'_2, R'_4, R'_6, R'_5, R'_1, R'_2, R'_4, R'_6, R'_5$, reading the input in the reverse direction. According to Fig. 10 the output of $\mathcal{B}'$ is exactly $xBzxBzz$.*

7 Algorithms

In this section we present some the most interesting algorithms we need for the bimachine construction.

The algorithms are all presented without proof of correctness as they follow without any modifications the respective constructions shown earlier.

7.1 Concatenation to the Left

The algorithm `lconcat` for concatenation to the left corresponds to Definition 3.8.

```
subroutine lconcat(Automaton A, Automaton B)
   let C = new Automaton

   foreach state in A.states
      let cloned_state = C.add_state(state)
      if state.is_start then
         cloned_state.set_start(true)
      end if
   end foreach

   foreach state in B.states
      let cloned_state = C.add_state(state)
      if state.is_start and A.accepts("") then
         cloned_state.set_start(true)
      end if
      if state.is_final then
         cloned_state.set_final(true)
      end if
   end foreach

   foreach trans in A.transitions
      C.add_transition(trans.source, trans.char, trans.target)
      if trans.target.is_final then
         foreach state in B.start_states
            C.add_transition(trans.source, trans.char, state)
         end foreach
      end if
   end foreach
   foreach trans in B.transitions
      C.add_transition(trans.source, trans.char, trans.target)
   end foreach

   return C
end subroutine
```

7.2 Translation of an Output-Driven Bimachine

The following algorithm `output_to_input` corresponds to Construction 5.1 which constructs a classical input-driven bimachine from an equivalent output-driven one.

```
subroutine output_to_input(Bimachine B)
   let psi = B.output_function
   let AL  = B.left_automaton
   let AR  = B.right_automaton

   let AN = new Automaton
   let states = new Table
   foreach state1 in AL.states
      foreach state2 in AR.states
         states[state1][state2] = AN.add_state()
      end foreach
   end foreach
   foreach p1 in AL.states
      foreach q1 in AR.states
         foreach p2 in AL.states
            foreach q2 in AR.states
               foreach a in alphabet
                  if AR.trans_function(q2, a) = q1 and
                     AL.trans_function(p1, psi(p1, a, q2)) = p2 then
                     AN.add_transition(states[p1][q1], a, states[p2][q2])
                  end if
               end foreach
            end foreach
         end foreach
      end foreach
   end foreach

   let A1 = AN.build_deterministic()

   let new_psi = new Function
   foreach L in A1.states
      foreach a in alphabet
         foreach r in AR.states
            foreach p in AL.states
               foreach q in AR.states
                  if states[p][q] in L and
                     AR.trans_function(r, a) = q then
                     new_psi.define(L, a, r, psi(p, a, r))
                  end if
               end foreach
            end foreach
```

```
      end foreach
    end foreach
  end foreach

  return new Bimachine(A1, AR, new_psi)
end subroutine
```

7.3 Direct Construction of a Bimachine for Context-Sensitive Rewrite Rule

The algorithm `construct_bimachine` directly reflects the construction from section 6 and constructs a bimachine realizing some given rewrite rule.

```
subroutine construct_bimachine(L, E, R, word)
  let AL = new Automaton(".*"+L)
  let AE = new Automaton(E)
  let AR = new Automaton(R+".*")

  let A = rconcat(lconcat(AL, AE), AR)

  let A2 = A.reverse().build_deterministic()
  let AN = A.clone()

  foreach tr in AN.trans
    if tr.target in AR.states then
      AN.remove_transition(tr)
    else
      if tr.target not in AE.start_states then
        AN.add_transition(tr.source, tr.char + alphabet.size, tr.target)
      end if
    end if
  end foreach

  let A1 = AN.build_deterministic()

  let psi = new Function
  let inter = AE.states - AE.start_states - AE.final_states
  foreach L in A1.states
    foreach R in A2.states
      foreach a in alphabet
        if A1.trans_function(L, a).intersect(R).intersect(inter) then
          psi.define(L, a, R, a + alphabet.size)
        else
          psi.define(L, a, R, a)
        end if
      end foreach
```

```
      end foreach
   end foreach

   let B = output_to_input(new Bimachine(A1, A2, psi))

   let final_psi = new Function
   let intermediate  = AE.states - AE.start_states - AE.final_states
   let startnonfinal = AE.start_states - AE.final_states
   let startandfinal = AE.start_states.intersect(AE.final_states)
   foreach L in B.left_automaton.states
      foreach R in B.right_automaton.states
         foreach a in alphabet
            let result = a
            foreach (p,q) in L
               continue unless q = B.right_automaton.trans_function(R, a)
               let common = p.intersect(q)
               if common.intersect(intermediate).length > 0 then
                  result = ""
                  break
               else if common.intersect(startnonfinal).length > 0 then
                  result = word
                  break
               else if common.intersect(startandfinal).length > 0 then
                  result = word + a
                  break
               end if
            end foreach
            if R = B.right_automaton.start_state then
               foreach (p,q) in B.left_automaton.trans_function(L, a)
                  continue unless q = R
                  if p.intersect(q).intersect(AE.start_states).length > 0 then
                     result = result + word
                  end if
               end foreach
            end if
            final_psi.define(L, a, R, result)
         end foreach
      end foreach
   end foreach

   B.output_function = final_psi

   return B
end subroutine
```

8 Complexity

In this section we explore the upper bound of the constructed bimachine's size.

Proposition 8.1. *Let* $\mathcal{B} = \langle \mathcal{A}_L, \mathcal{A}_R, \psi \rangle$ *is an output-driven bimachine and* $\mathcal{B}' = \langle \mathcal{A}'_L, \mathcal{A}'_R, \psi' \rangle$ *is an equivalent input-driven bimachine, constructed by Construction 5.1. Then*

a) $|Q'_R| = |Q_R|$

b) $|Q'_L| \leq (|Q_L| + 1)^{|Q_R|}$

Proof. a) By Construction 5.1 we immediately deduce that $\mathcal{A}_R = \mathcal{A}'_R$, and hence $|Q'_R| = |Q_R|$ is trivially valid.

b) Because of Proposition 5.1 we may look at the states in Q'_L as if they were partial functions from Q_R to Q_L. This means that $|Q'_L|$ is no larger than the number of different partial functions defined from Q_R to Q_L therefore $|Q'_L| \leq (|Q_L| + 1)^{|Q_R|}$.

□

Let's now calculate the upper bound of the constructed bimachine's space complexity. In other words we shall estimate the number of states in the bimachine's automata in terms of the context rule.

Let's fix a context rule $E \to \beta \ / \ L\_R$ and denote $l = |L|$, $e = |E|$ and $r = |R|$. Then the number of states in $\mathcal{A}_L$, $\mathcal{A}_E$ and $\mathcal{A}_R$ will be respectively $\mathcal{O}(l)$, $\mathcal{O}(e)$ and $\mathcal{O}(r)$. Hence it is easily seen that the number of states in $\mathcal{A}$ is bounded by $\mathcal{O}(l + e + r)$.

This gives us an upper bound for the states of $\mathcal{A}_2$, namely $\mathcal{O}(2^{l+e+r})$.

As $\mathcal{A}_1^N$ is obtained from $\mathcal{A}$ through transition relation modifications, its number of states remains as much as $\mathcal{O}(l+e+r)$. This number grows exponentially to $\mathcal{O}(2^{l+e+r})$ after the determinization of $\mathcal{A}_1$. Finally, according to Proposition 8.1, Construction 5.1 constructs $\mathcal{A}'_1$ with $\mathcal{O}(2^{(l+e+r)2^{l+e+r}})$ states.

Thus we obtained the final estimate of the bimachine's space complexity – respectively $\mathcal{O}(2^{n2^n})$ and $\mathcal{O}(2^n)$ for the left and right automata, where $n = l + e + r$ is the total length of the regular expressions in the context-sensitive rewrite rule.

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