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Characterization of uniform sequences of relations and structures

Master thesis

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2018

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1 General concepts

In this chapter we will introduce the notation and remind some common properties of the computable functions, enumeration reducability and $\omega-enumeration$ reducability. More information can be obtained in [7].

1.1 Computable functions and concepts

We will denote with $\mathbb{N}$ the set of all natural numbers which includes 0. If A is a set with χ_A we will denote its characteristic function i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

A function is computable if there is a purely mechanical process to calculate it's values. As a general rule, when we say that a function is computable, we assume that it is total. A function that is not total but can still be shown to have a mechanism for calculating it's values will be called partial computable. With ϕ_e we will denote the partial computable function with index e.
We can encode pairs of natural numbers by a single number using the function $\langle x, y\rangle \mapsto 2^x(2y+1)-1$ or the function $\langle x, y\rangle \mapsto ((x+y)(x+y+1)+y)/2$, which are bijections from $\mathbb{N}^2$ to $\mathbb{N}$ whose inverses are easily computable too. One can then encode triples by using pairs of pairs, and then encode $n-tuples$, and then tuples of arbitrary size, and then tuples of tuples, etc. The same way, we can consider standard effective bijections between $\mathbb{N}$ and various other sets like $\mathbb{Z}$, $\mathbb{Q}$, etc. Given any such finite object a, we use $\ulcorner a \urcorner$ to denote the number coding a. By D_u we denote the finite set D where $u = \Sigma_{y\in D}2^y$. We say that u is a canonical index of the set D.
For $n \in \mathbb{N}$, we sometimes use n to denote the set $\{0, 1, ..., n-1\}$. By $2^{\mathbb{N}}$ we denote the set of all functions from $\mathbb{N}$ to $\{0, 1\}$, which we will sometimes refer to as infinite binary sequences. For any set X, we use $X^{<\mathbb{N}}$ to denote the set of finite tuples of elements from X, which we call strings when $X = 2$ or $X = \mathbb{N}$. For $\sigma \in X^{<\mathbb{N}}$ and $\tau \in X^{<\mathbb{N}}$, we use $\sigma\tau$ to denote the concatenation of these sequences. We use $\sigma \subseteq \tau$ to denote that σ is an initial segment of τ. When X, Y are subsets of $\mathbb{N}$, we use $X \subset Y$ to denote that X is a subset of Y. We will explicitly mention if they are different.
If A and B are sets, with $A \oplus B$ we will denote the following set

$$\{2x : x \in A\} \cup \{2x + 1 : x \in B\}$$

We will say that A is turing reducable to B and write $A \leq_T B$, if there is a natural number e such that $\chi_A = \phi_e^B$, where ϕ_e^B is the function with index e and has as an oracle B. The relation $\leq_T$ is reflexive and transitive. Denote

$$A \equiv_T B \Leftrightarrow A \leq_T B \wedge B \leq_T A.$$

$\equiv_T$ is an equivalence relation. The equivalence classes we call Turing degrees. By $d_T(A)$, we denote the equivalence class containing A. The class of all Turing degrees we denote by D_T which is an upper semi-lattice.
The languages we consider will always be countable and computable. A language $\mathfrak{L}$ consists of three sets of symbols $\{R_i : i \in I_R\}$, $\{f_i : i \in I_F\}$, and $\{c_i : i \in I_C\}$; and two functions a_R and a_F. Each of I_R, I_F, and I_C is an initial segment of $\mathbb{N}$. For $i \in I_R$, $a_R(i)$ is the arity of R_i, the same for the others. A language is computable if the arity functions are computable. This only matters when the language is infinite; finite languages are trivially computable.
Remark: on certain places we will use the notation ω to denote the set $\{0, 1, 2...\}$.

1.2 Enumeration reducibility and ω-enumeration reducibility

Definition 1. *Given sets $A, B \subset \mathbb{N}$ we say that $A \leq_e B$ if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$,i.e.*

$$(\forall x)(x \in A \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_z \wedge D_v \subset B))$$

In the above definition D_v is the finite set with canonical code v and W_z is the computably enumerable (c.e.) set with index z with respect to an effective numbering of all c.e. sets. We can easily see that the relation $\leq_e$ is reflexive and transitive. Let

$$A \equiv_e B \Leftrightarrow A \leq_e B \wedge B \leq_e A.$$

The enumeration degree of a set A is the equivalence class relatively $\equiv_e$.

Definition 2. *Let $A \subset \mathbb{N}$ and let A^+ be defined as $A \oplus (\mathbb{N} \backslash A)$. We say that A is total iff $A \equiv_e A^+$.*

If X is a total set then $A \leq_e X \Leftrightarrow A$ is c.e. in X. An enumeration degree $\mathfrak{a}$ is total if $\mathfrak{a}$ contains a total set. Let $d_e(X)$ be the enumeration degree of a set X. We can define an ordering on the enumeration degrees in the usual way: $d_e(A) \leq d_e(B) \Leftrightarrow A \leq_e B$. Denote by D_e the set of all enumeration degrees.

Definition 3. *Given a set $A \subset \mathbb{N}$, let $K_A^0 = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$. Define A' to be ${K_A^0}^+$. We call A' the enumeration jump of A.*

From now on whenever we use the jump operation, we will mean enumeration jump.
The following properties will be used throughout the paper:

Properties 1. *i) If $n < m$ then $A^{(n)} \leq_e A^{(m)}$ uniformly in n and m.*
ii) If $A \leq_e B$ then $A' \leq_e B'$.
ii') If $A \leq_e B$ then $A^{(n)} \leq_e B^{(n)}$ uniformly in n.
iii) If $n > 0$ then $A^{(n)}$ is a total set.

Denote by $\mathfrak{S}$ the set of all sequences of sets of natural numbers. For each element $\overrightarrow{B} = \{B_n\}_{n<\omega}$ of $\mathfrak{S}$ (such element from now on will be denoted just by $\overrightarrow{B}$), call the jump class of $\overrightarrow{B}$ the set

$$J_{\overrightarrow{B}} = \{d_T(X) : (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in n})\}.$$

For every two sequences $\overrightarrow{A}$ and $\overrightarrow{B}$ let $\overrightarrow{A} \leq_\omega \overrightarrow{B}$ ($\overrightarrow{A}$ is $\omega-enumeration$ reducible to $\overrightarrow{B}$) if $J_{\overrightarrow{B}} \subset J_{\overrightarrow{A}}$. The relation $\leq_\omega$ is reflexive and transitive. Let $\overrightarrow{A} \equiv_\omega \overrightarrow{B}$ if $J_{\overrightarrow{A}} = J_{\overrightarrow{B}}$. Hence $\equiv_\omega$ is an equivalence relation on $\mathfrak{S}$. Let the $\omega - enumeration$ degree of $\overrightarrow{B}$ be $d_\omega(\overrightarrow{B}) = \{\overrightarrow{A} : \overrightarrow{A} \equiv_\omega \overrightarrow{B}\}$ and $D_\omega = \{d_\omega(\overrightarrow{B}) : \overrightarrow{B} \in \mathfrak{S}\}$. If $\mathfrak{a} = d_\omega(\overrightarrow{A})$ and $\mathfrak{b} = d_\omega(\overrightarrow{B})$, then $\mathfrak{a} \leq_\omega \mathfrak{b}$ if $\overrightarrow{A} \leq_\omega \overrightarrow{B}$. Denote by $0_\omega = d_\omega(\emptyset_\omega)$, where $\emptyset_\omega$ is the sequence with all members equal to $\emptyset$. There is a natural embedding of the enumeration degrees into the $\omega - enumeration$ degrees. Given a set A denote by $A \uparrow \omega$ the sequence $\{A_n\}_{n<\omega}$, where $A_0 = A$ and for all $n > 0$ $A_n = \emptyset$. For every $A, B \subset \mathbb{N}$ we have that $A \leq_e B$ iff $A \uparrow \omega \leq_\omega B \uparrow \omega$. So the mapping $\mathfrak{k}(d_e(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of D_e to D_ω. We shall identify the enumeration degree $d_e(A)$ with its representation $d_\omega(A \uparrow \omega)$ in D_ω. So when $\mathfrak{a} = d_e(A)$ and $\mathfrak{b} \in D_\omega$ then by writing $\mathfrak{a} \leq_\omega \mathfrak{b}$ we mean $d_\omega(A \uparrow \omega) \leq_\omega \mathfrak{b}$.
Remark: from now on we will write $\overrightarrow{A} \leq_\omega B$ instead of $\overrightarrow{A} \leq_\omega B \uparrow \omega$.

Definition 4. *Let $\overrightarrow{B}$ be a sequence of sets which are subsets of* $\mathbb{N}$. *We define the respective jump sequence $P(\overrightarrow{B}) = \{P_n(\overrightarrow{B})\}_{n<\omega}$ by induction on n:*
i)$P_0(\overrightarrow{B}) = B_0$;
ii)$P_{n+1}(\overrightarrow{B}) = (P_n(\overrightarrow{B}))' \oplus B_{n+1}$.

We can see that if $X \subset \mathbb{N}$, then $P_n(X \uparrow \omega) \equiv_e X^{(n)}$ uniformly in n.
We will list some simple proprties of the jump sequence which follow easily from the definition.

Properties 2. *i) If $m \leq n$ then $P_m(\overrightarrow{B}) \leq_e P_n(\overrightarrow{B})$ uniformly in n and m.*
ii) If $m \leq n$ then $B_m \leq_e P_n(\overrightarrow{B})$ uniformly in n and m.

The following theorem proven by Soskov links the two reducabilities

Theorem 1. *Let $\overrightarrow{A}_0, ..., \overrightarrow{A}_r, ...$ be a sequence of sets such that for every r, $\overrightarrow{A}_r \not\leq_\omega \overrightarrow{B}$. There is a total set X such that $\overrightarrow{B} \leq_\omega \{X^{(n)}\}_{n<\omega}$ and $\overrightarrow{A}_r \not\leq_\omega \{X^{(n)}\}_{n<\omega}$ for each r.*

Remark: It follows that if $X \subset \mathbb{N}$ then for every sequence $\overrightarrow{A}$ we have: $A_n \leq_e X^{(n)}$ uniformly in n iff $\overrightarrow{A} \leq_\omega \{X^{(n)}\}_{n<\omega}$ iff $\overrightarrow{A} \leq_\omega X \uparrow \omega$. We also have $\overrightarrow{A} \equiv_\omega P(\overrightarrow{A})$.
An important corollary to Theorem 1 is the following:

Lemma 1. *(Soskov [5]) Let $\overrightarrow{A}$ and $\overrightarrow{B}$ be two sequences of sets of natural numbers. The following conditions are equivalent:*
i) $\overrightarrow{A} \leq_\omega \overrightarrow{B}$ i.e. for every total set X, if $B_n \leq_e X^{(n)}$ uniformly in n, then $A_n \leq_e X^{(n)}$ uniformly in n.
ii) $A_n \leq_e P_n(\overrightarrow{B})$ uniformly in n, i.e. there is a computable function g, such tthat $A_n = \Gamma_{g(n)}(P_n(\overrightarrow{B})$ for every n.

2 Introduction to the field of study

We all know that in mathematics there are proofs that are more difficult than others, constructions that are more complicated than others, and objects that are harder to describe than others. The object of *computable mathematics* is to study this complexity, to measure it, and to understand where it comes from.
Here, we will concentrate on the complexity of *structures*. By *structures*,

we mean objects like rings, graphs or linear orderings, which consist of a domain on which we have relations, functions and constants. Also important is to study the interplay betwen complexity and structure. By complexity, we mean descriptional or computational complexity, in the sense of how difficult it is to describe or compute a certain object. By structure, we refer to algebraic or structural properties of mathematical structures. The setting is that of infinite countable structures and thus, within the whole hierarchy of complexity levels, the appropriate tools to measure complexity are those used in computability theory. The motivations for the study come from questions of the following sort: are there syntactical properties that explain why certain objects (like structures, relations or isomorphisms) are easier or harder to compute or to describe?

Given a structure $\mathfrak{A}$, an $\omega - presentation$ of it(or copy) is a structure whose domain is $\mathbb{N}$. What we will need is the $\omega - presentation$ to be isomorphic to $\mathfrak{A}$.The following definition will give a way for representing a structure in order to analyze its computational complexity.

Definition 5. *Let $\mathfrak{L}$ be a first order language. Let $\{\phi_i : i \in \mathbb{N}\}$ be an effective enumeration of all atomic formulas with free variables from the set $\{x_0, x_1, ...\}$. The atomic diagram of an $\omega - presentation$ $\mathfrak{M}$ is the infinite binary string $D(\mathfrak{M}) \in 2^{\mathbb{N}}$ defined by*

$$D(\mathfrak{M})(i) = \begin{cases} 1 & \textit{if } \mathfrak{M} \models \phi_i[x_j \mapsto j : j \in \mathbb{N}] \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 6. *Let $\mathfrak{A}$ be a structure (with domain $\mathbb{N}$). A relation R is relatively intrinsically computably enumerable (r.i.c.e.) if, for every copy $\mathfrak{B}$ of $\mathfrak{A}$, the relation $R^{\mathfrak{B}}$ is c.e. in $D(\mathfrak{B})$.*

Definition 7. *An infinitary Σ_1 forumla is a countable infinite (or finite) disjunction of existential formulas over a finite set of free variables. A computable infinitary Σ_1 formula (denoted Σ_1^c) is an infinite or finite disjunction of a computable list of existential formulas over a finite set of free variables.*

A detailed exposition of infinitary formulas and their properties can be found in [6].

Definition 8. *A relation R is $\Sigma_1^c - definable$ in $\mathfrak{A}$ with parameters if there is a tuple $\overline{p} \in \mathbb{N}^{<\mathbb{N}}$ and a computable sequence of Σ_1^c formulas $\psi_{i,j}(x_1, ..., x_{|\overline{p}|}, y_1, ..., y_j)$, for $i, j \in \mathbb{N}$ such that*

$$R = \{\langle \overline{b} \rangle \in \mathbb{N}^{<\mathbb{N}} : \mathfrak{A} \models \psi_{i,|\overline{b}|}\langle \overline{p}, \overline{b} \rangle\}.$$

The elements in $\overline{p}$ are the parameters in the definition of R.

The following fundamental theorem was proven by Ash, Knight, Manasse and Slaman [2], and independently by Chisholm [3].

Theorem 2. *Let $\mathfrak{A}$ be a structure and R a relation on it. The following are equivalent*

1. *R is r.i.c.e.*
2. *R is Σ_1^c definable in $\mathfrak{A}$ with parameters.*

Ash, Knight [1] proved further

Theorem 3. *Let $\mathfrak{A}$ be a computable structure, and let R and P be further relations on $\mathfrak{A}$. Then the following are equivalent:*

1. *For all $\mathfrak{B} \cong \mathfrak{A}$, if $R^{\mathfrak{B}}$ is Σ_n^0 relative to $\mathfrak{B}$, then so is $P^{\mathfrak{B}}$.*
2. *P is definable in the structure $\mathfrak{A}$ by a computable infinitary Σ_n formula $\phi(\overline{x}, \overline{c})$, with a finite tuple of parameters $\overline{c}$, in which the relation symbol for R appears only positively.*

Soskov and Baleva [4] introduced the concept of $\alpha - intrinsic$ relations to prove a generalization of the above results.

Definition 9. *For a structure $\mathfrak{A}$, a further relation R and a sequence of relations $\overrightarrow{B}$, we say that R is relatively $\alpha -$ intrinsic on $\mathfrak{A}$ with respect to $\overrightarrow{B}$ if for every $\mathfrak{B} \cong \mathfrak{A}$, if the inverse image of $\overrightarrow{B}$ is enumeration reducible to $\mathfrak{B}$, then $R^{\mathfrak{B}}$ is enumeration reducible to $\mathfrak{B}$.*

Definition 10. *Let $A \subset \mathbb{N}$. The set A is formally $\alpha - definable$ on $\mathfrak{A}$ with respect to the sequence $\overrightarrow{B}$ if there exists a Σ_α^+ formula Φ with free variables among $W_1, ..., W_r, X$ and elements $t_1, ..., t_r$ of $\mathbb{N}$ such that for every element s of $\mathbb{N}$ the following equivalence holds:*

$$s \in A \Leftrightarrow \mathfrak{A} \models \Phi(W_1/t_1, ..., W_r/t_r, X/s).$$

Refer to [4] for the more involved definition of the Σ_α^+ formulas.

Theorem 4. *(Soskov, Baleva [4]). For a structure $\mathfrak{A}$, a further relation R and a sequence of relations $\overrightarrow{B}$, the following are equivalent*

1. *R is relatively $\alpha -$ intrinsic on $\mathfrak{A}$ with respect to $\overrightarrow{B}$.*

2. *R is formally $\alpha - definable$ on $\mathfrak{A}$.*

The work concentrates around the following two problems

1. Find syntactical conditions, on a given structure $\mathfrak{A}$ and sequences of relations $\overrightarrow{A}$ and $\overrightarrow{B}$, guaranteeing that in all copies $\mathfrak{B}$ of $\mathfrak{A}$, if the sequence $\overrightarrow{B}$ with respect to the copy $\mathfrak{B}$ is enumeration reducible to $\mathfrak{B}$, then the sequence $\overrightarrow{A}$ with respect to the copy $\mathfrak{B}$ is enumeration reducible to $\mathfrak{B}$.

2. Find syntactical conditions, on a sequence of structures $\overrightarrow{\mathfrak{A}}$ and further sequences of relations $\overrightarrow{A}$ and $\overrightarrow{B}$, guaranteeing that in all copies $\overrightarrow{\mathfrak{B}}$ of $\overrightarrow{\mathfrak{A}}$, if the sequence $\overrightarrow{B}$ with respect to the copies is $\omega - enumeration$ reducible to $\overrightarrow{\mathfrak{B}}$, then the sequence $\overrightarrow{A}$ with respect to the copies is $\omega - enumeration$ reducible to $\overrightarrow{\mathfrak{B}}$.

3 Relatively intrinsic sequence on a structure

3.1 General concepts, forcing and modelling relations, forcing definability

Suppose we are given the first order relational language $\mathfrak{L} = (T_1, ..., T_k)$. Let $\mathfrak{A} = (\mathbb{N}, R_1, ...R_k)$ be a structure for $\mathfrak{L}$, where the predicates $=$ *and* $\neq$ are among the list $R_1, ..., R_k$ and $\mathbb{N}$ is the set of all natural numbers. We are also given two sequences of subsets of $\mathbb{N}$, i.e. $\overrightarrow{A}$ and $\overrightarrow{B}$.(We assume that each of A_n and B_m is a subset of $\mathbb{N}$ for simplicity. The proofs in the general case are similar)
A total mapping from $\mathbb{N}$ onto $\mathbb{N}$ is called an enumaration of the structure $\mathfrak{A}$. Given an enumeration f and a sequence $\overrightarrow{A}$, by $f^{-1}(\overrightarrow{A})$ we will denote the following sequence $f^{-1}(A_0), f^{-1}(A_1), ...$

Definition 11. *We say that the sequence $\overrightarrow{A}$ of subsets of $\mathbb{N}$ is relatively intrinsic on $\mathfrak{A}$ with respect to the sequence $\overrightarrow{B}$ if for every enumeration f of $\mathfrak{A}$ such that $f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)}$ uniformly in n, the sequence $f^{-1}(\overrightarrow{A})$ is $\omega - enumeration$ reducible to $f^{-1}(\mathfrak{A})$.*

We introduce a more convenient definition of a copy, when the language under consideration is finite. They can be shown to be Turing equivalent.

Definition 12. *Let f be an enumaration of the structure $\mathfrak{A}$ and let B be a subset of $\mathbb{N}^n$. Then, $f^{-1}(B) = \{\langle x_1, ..., x_n\rangle : (f(x_1), ..., f(x_n)) \in B\}$ and $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus f^{-1}(R_2) ... \oplus f^{-1}(R_k)$. The latter set is called the copy of the structure $\mathfrak{A}$.*

In particular, if f is the identity function, we will denote $f^{-1}(\mathfrak{A})$ by $D(\mathfrak{A})$.

Definition 13. *An enumeration f of $\mathfrak{A}$ is called acceptable with repsect to $\overrightarrow{B}$ if*

$$(\forall n)[f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n\ ].$$

Definition 14. *Let f be an acceptable enumeration of $\mathfrak{A}$ with respect to $\overrightarrow{B}$. We denote by $P^f = \{P^f_n\}_{n<\omega}$ the respective jump sequence of the sequence $\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1) ..., f^{-1}(B_n), ...\}$, where*

$$P^f_n = P_n(\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1) ..., f^{-1}(B_n), ...\}).$$

Lemma 2. *An enumeration f on $\mathfrak{A}$ is acceptable with respect to $\overrightarrow{B}$ iff $P^f \leq_\omega f^{-1}(\mathfrak{A})$.*

Proof. Since f is acceptable, we have that $f^{-1}(B_n) = W_{h(n)}(f^{-1}(\mathfrak{A})^{(n)})$, where h is a computable function. We shall prove by induction on n that $P^f_n = W_{g(n)}(f^{-1}(\mathfrak{A})^{(n)})$, where g is a computable function.

1. Let $n = 0$. By definition, $P^f_0 = f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0)$. Let $h(0) = e'_0$. By assumption $f^{-1}(B_0) = W_{e'_0}(f^{-1}(\mathfrak{A}))$ and $f^{-1}(\mathfrak{A}) = W_{e_0}(f^{-1}(\mathfrak{A}))$, where e_0 is obtained effectively. Then we can obtain effectively an index i_0 such that $P^f_0 = W_{i_0}(f^{-1}(\mathfrak{A}))$. Let $g(0) = i_0$.

2. Assume the statement is true for n and we will prove it for $n+1$. Using the definition of the jump sequence $P^f_{n+1} = (P^f_n)' \oplus f^{-1}(B_{n+1})$. By assumption, we can obtain effectively an index e'_{n+1} such that $f^{-1}(B_{n+1}) = W_{e'_{n+1}}(f^{-1}(\mathfrak{A})^{(n)})$. By induction hypothesis we have $P^f_n = W_{i_n}(f^{-1}(\mathfrak{A})^{(n)})$ where $g(n) = i_n$. By the properties of the enumeration jump, we can effectively obtain an index e_{n+1} such that $(P^f_n)' = W_{e_{n+1}}(f^{-1}(\mathfrak{A})^{(n+1)})$. Then we can effectively obtain an index i_{n+1} from e_{n+1} and e'_{n+1}, such that $P^f_{n+1} = W_{i_{n+1}}(f^{-1}(\mathfrak{A})^{(n+1)})$.

We have

$$g(0) = i_0$$

$$g(n+1) = H(g(n), h(n))$$

where H is computable function.

($\leftarrow$) Assume $P_n^f \leq_e f^{-1}(\mathfrak{A})^{(n)}$ uniformly in n via the computable function g i.e. $P_n^f = W_{g(n)}(f^{-1}(\mathfrak{A})^{(n)})$. Let $g(n) = e_n$. Then we can effectively pass from an index e_n such that $P_n^f = W_{e_n}(f^{-1}(\mathfrak{A})^{(n)})$ to an index $h(n) = i_n$ such that $f^{-1}(B_n) = W_{i_n}(f^{-1}(\mathfrak{A})^{(n)})$. □

Definition 15. *Let f be an enumeration of $\mathfrak{A}$. For every n,x and $e \in \mathbb{N}$, we define the relations $f \models_n F_e(x)$ and $f \models_n \neg F_e(x)$ by induction on n:*

i) $f \models_0 F_e(x)$ iff $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
a) $u = \langle 0, \langle i, x_1^u, ..., x_{r_i}^u\rangle\rangle \wedge (f(x_1^u), ..., f(x_{r_i}^u)) \in R_i$ or
b) $u = \langle 2, x_u\rangle \wedge f(x_u) \in B_0]$

ii) $f \models_{n+1} F_e(x)$ iff $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
$((u = \langle 0, e_u, x_u\rangle \wedge f \models_n F_{e_u}(x_u)) \vee$
$(u = \langle 1, e_u, x_u\rangle \wedge f \models_n \neg F_{e_u}(x_u)) \vee$
$(u = \langle 2, x_u\rangle \wedge f(x_u) \in B_{n+1}))]$

iii) $f \models_n \neg F_e(x)$ iff $f \not\models_n F_e(x)$

Remark: We have an arbitrary coding of the tuples of natural numbers. We are not interested in what exactly it looks like, but we can say that there is an effective way to go from this coding to a coding that would resemble the elements of the sets in their entirety.
We will need the following properties of the jump sequence:

Properties 3. *i) $P_n^f \leq_e P_n(P^f)$ uniformly in n.*
ii) $P_n(P^f) \leq_e P_n^f$ uniformly in n.

Lemma 3. *i) Let $C \subset N$, $n \in N$. Then $C \leq_e P_n^f$ iff there is $e \in N$ such that $C = \{x : f \models_n F_e(x)\}$*
ii) Let $\overrightarrow{C}$ be a sequence of sets. $\overrightarrow{C} \leq_\omega P^f$ iff there exists a total computable function g, such that $C_n = \{x : f \models_n F_{g(n)}(x)\}$

Proof. i) We will prove the statement by induction on the definition of the modelling relation. To be more precise, we will prove for every n,

$$C = W_e(P_n^f) \Leftrightarrow C = \{x : f \models_n F_e(x)\}.$$

1. Let $n = 0$
   ($\rightarrow$) Following the definition of enumeration reducability,

   $$x \in C \Leftrightarrow \exists v(\langle v, x\rangle \in W_e \wedge D_v \subset P_0^f).$$

   Recall that $P_0^f = f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0)$. From the definition of the modelling relation, we get $f \models_0 F_e(x)$. Hence $C = \{x : f \models_0 F_e(x)\}$.
   ($\leftarrow$) Fix a natural number e and assume $C = \{x : f \models_0 F_e(x)\}$. Hence,

   $$x \in C \Leftrightarrow f \models_0 F_e(x)$$
   $$\Leftrightarrow \exists v(\langle v, x\rangle \in W_e \wedge D_v \subset P_0^f)$$
   $$\Leftrightarrow x \in W_e(P_0^f).$$

   Thus, we get $C \leq_e P_0^f$.

2. Assume the statement is true for n. We will prove it for $n + 1$.
   ($\rightarrow$) Let $C \leq_e P_{n+1}^f$ i.e. $C = W_e(P_{n+1}^f)$ for some index e. From the definition of enumeration reducability and its jump we have the following equivalences:

   $$x \in C \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subset P_{n+1}^f)$$
   $$\Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge ((\forall u \in D_v)$$
   $$(u = \langle 0, e_u, x_u\rangle \wedge x_u \in W_{e_u}(P_n^f))\vee$$
   $$(u = \langle 1, e_u, x_u\rangle \wedge x_u \notin W_{e_u}(P_n^f))\vee$$
   $$(u = \langle 2, x_u\rangle \wedge x_u \in f^{-1}(B_{n+1})))).$$

   Let $C_u = W_{e_u}(P_n^f)$. By induction hypothesis, $x_u \in C_u \Leftrightarrow f \models_n F_{e_u}(x_u)$ and $x_u \notin C_u \Leftrightarrow f \models_n \neg F_{e_u}(x_u)$. We can rewrite the equivalences as follows:

   $$x \in C \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subset P_{n+1}^f)$$
   $$\Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge ((\forall u \in D_v)$$
   $$(u = \langle 0, e_u, x_u\rangle \wedge x_u \in C_u)\vee$$

$$(u = \langle 1, e_u, x_u \rangle \wedge x_u \notin C_u) \vee$$
$$(u = \langle 2, x_u \rangle \wedge x_u \in f^{-1}(B_{n+1}))))$$
$$\Leftrightarrow (\exists v)(\langle v, x \rangle \in W_e \wedge ((\forall u \in D_v)$$
$$(u = \langle 0, e_u, x_u \rangle \wedge f \models_n F_{e_u}(x_u)) \vee$$
$$(u = \langle 1, e_u, x_u \rangle \wedge f \models_n \neg F_{e_u}(x_u)) \vee$$
$$(u = \langle 2, x_u \rangle \wedge x_u \in f^{-1}(B_{n+1})))).$$

Now by the modelling definition, we get what we needed.

($\leftarrow$) Let $C = \{x : f \models_n F_e(x)\}$. We want to see $C = W_e(P^f_{n+1})$. By assumption and the modelling definition:

$$x \in C \Leftrightarrow f \models_{n+1} F_e(x) \Leftrightarrow (\exists v)(\langle v, x \rangle \in W_e \wedge (\forall u \in D_v)$$
$$((u = \langle 0, e_u, x_u \rangle \wedge f \models_n F_{e_u}(x_u)) \vee$$
$$(u = \langle 1, e_u, x_u \rangle \wedge f \models_n \neg F_{e_u}(x_u)) \vee$$
$$(u = \langle 2, x_u \rangle \wedge f(x_u) \in B_{n+1})))$$

Let $C_u = \{x : f \models_n F_{e_u}(x)\}$. By induction hypothesis $C_u = W_{e_u}(P^f_n)$. Thus we have:

$$f \models_n F_{e_u}(x_u) \Leftrightarrow x_u \in C_u \Leftrightarrow x_u \in W_{e_u}(P^f_n)$$
$$f \models_n \neg F_{e_u}(x_u) \Leftrightarrow x_u \notin C_u \Leftrightarrow x_u \notin W_{e_u}(P^f_n)$$

Hence we can rewrite the equivalences in the following way:

$$x \in C \Leftrightarrow (\exists v)(\langle v, x \rangle \in W_e \wedge (\forall u \in D_v)$$
$$((u = \langle 0, e_u, x_u \rangle \wedge x_u \in W_{e_u}(P^f_n)) \vee$$
$$(u = \langle 1, e_u, x_u \rangle \wedge x_u \notin W_{e_u}(P^f_n)) \vee$$
$$(u = \langle 2, x_u \rangle \wedge f(x_u) \in B_{n+1})))$$

Hence $x \in W_e(P^f_{n+1})$ and thus $C \leq_e P^f_{n+1}$.
ii) ($\leftarrow$) We want to prove $C_n \leq_e P_n(P^f)$ uniformly in n. Since $P^f_n \leq_e P_n(P^f)$ uniformly in n, it will be enough to prove $C_n \leq_e P^f_n$ uniformly in n. By assumption we have for every n, $C_n = \{x : f \models_n F_{g(n)}(x)\}$. By i)

$$C_n = W_{g(n)}(P^f_n),$$

where g is a computable function.

($\rightarrow$) By assumption we have $C_n \leq_e P_n(P^f)$ uniformly in n. Since $P_n(P^f) \leq_e P_n^f$ uniformly in n, we have $C_n \leq_e P_n^f$ uniformly in n. By i)

$$C_n = \{x : f \models_n F_{g(n)}(x)\},$$

where g is the computable function from the last uniformity. $\square$

Remark: To be more precise i) would look like $C \leq_e P_n^f$ iff there is $e \in N$ such that $C = \{x : f \models_n F_{h(e)}(x)\}$, where h is a computable function that we use to pass from our coding to a coding that resembles the sets in the jump sequence P_n^f.

Definition 16. *The forcing conditions, called finite parts, are finite mappings τ of $\mathbb{N}$ to $\mathbb{N}$. We will denote the finite parts by letters δ, τ, ρ. For each $n, e, x \in \mathbb{N}$ and for every finite part τ, define the forcing relations $\tau \Vdash_n F_e(x)$ and $\tau \Vdash_n \neg F_e(x)$ following the definition of the relation "$\models_n$".*

i) $\tau \Vdash_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \wedge (\forall u \in D_v))$ either
a) $u = \langle 0, \langle i, x_1^u, ..., x_{r_i}^u\rangle\rangle \wedge x_1^u, ..., x_{r_i}^u \in dom(\tau) \wedge (\tau(x_1^u), ..., \tau(x_{r_i}^u)) \in R_i$ or
b) $u = \langle 2, x_u\rangle \wedge x_u \in dom(\tau) \wedge \tau(x_u) \in B_0$

ii) $\tau \Vdash_{n+1} F_e(x)$ iff $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
$((u = \langle 0, e_u, x_u\rangle \wedge \tau \Vdash_n F_{e_u}(x_u)) \vee$
$(u = \langle 1, e_u, x_u\rangle \wedge \tau \Vdash_n \neg F_{e_u}(x_u)) \vee$
$(u = \langle 2, x_u\rangle \wedge \tau(x_u) \in B_{n+1}))]$

iii) $\tau \Vdash \neg F_e(x)$ iff $(\forall \rho \supseteq \tau)[\rho \nVdash_n F_e(x)]$

Definition 17. *Let f be an enumeration of $\mathfrak{A}$. We say that f is k-generic with respect to $\overrightarrow{B}$ if for every $j < k$ and $e, x \in \mathbb{N}$:*

$$(\exists \tau \subseteq f)(\tau \Vdash_j F_e(x) \vee \tau \Vdash_j \neg F_e(x))$$

Lemma 4. *i) If $\tau \subseteq \rho$ then $\tau \Vdash_k (\neg)F_e(x)$ implies $\rho \Vdash_k (\neg)F_e(x)$;*
ii) For every $(k+1) -$ generic enumeration f of A, $f \models_k (\neg)F_e(x)$ iff $(\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x))$.

Proof. i) Let $\tau \subseteq \rho$. We will prove the assertion by induction on k.

1. Let $k = 0$. Let $\tau \Vdash_0 F_e(x)$. Then there exists v such that D_v has the properties from the definition. From $\tau \subseteq \rho$ we have $\rho \Vdash_0 F_e(x)$.
   Let $\tau \Vdash_0 \neg F_e(x)$. Assume that $\rho \nVdash_0 \neg F_e(x)$. From the definition of forcing we have that $\exists \delta \supseteq \rho \supseteq \tau$ such that $\delta \Vdash_0 F_e(x)$. From the definition of forcing and $\delta \supseteq \tau$, we get $\delta \nVdash_0 F_e(x)$. Contradiction.

2. Let the assertion be true for $k = n$. We will prove it for $k + 1$.
   Let $\tau \Vdash_{n+1} F_e(x)$. Then exists v such that D_v is a finite set that has the properties form the definition of forcing. Let $u \in D_v$. From the definition of forcing we have the following three cases:
   case 1: $u = \langle 0, e_u, x_u \rangle \wedge \tau \Vdash_n F_{e_u}(x_u)$. By induction hypothesis $\rho \Vdash_n F_{e_u}(x_u)$.
   case 2: $u = \langle 1, e_u, x_u \rangle \wedge \tau \Vdash_n \neg F_{e_u}(x_u)$. By induction hypothesis $\rho \Vdash \neg F_{e_u}(x_u)$.
   case 3: $u = \langle 2, x_u \rangle \wedge \tau(x_u) \in B_{n+1}$. Since $\tau \subseteq \rho$, we have $\rho(x_u) \in B_{n+1}$.
   Combining the three cases and the definition of forcing, we have $\rho \Vdash_{n+1} F_e(x)$.

   Let $\tau \Vdash_{n+1} \neg F_e(x)$. Assume that $\rho \nVdash_{n+1} \neg F_e(x)$. From the definition of forcing we have that $\exists \delta \supseteq \rho \supseteq \tau$ such that $\delta \Vdash_{n+1} F_e(x)$. From the definition of forcing and $\delta \supseteq \tau$, we get $\delta \nVdash_{n+1} F_e(x)$. Contradiction.

ii) We prove the assertion by induction on k.

1. Let $k = 0$. We look at the positive case.
   ($\leftarrow$) We have $(\exists \tau \subseteq f)(\tau \Vdash_0 F_e(x))$. Using the finite set D_v from the definition of forcing and applying it to the definition of the modelling relation, we get $f \models_0 F_e(x)$.
   ($\rightarrow$) We have $f \models_0 F_e(x)$. Using the set D_v from the definition of the modelling relation, we can get a finite part $\tau \subseteq f$, such that it is defined for the elements that are part of the coding for the elements $u \in D_v$.
   Now we turn our attention to the negative part. Let f be $1-generic$.
   ($\rightarrow$) Let $f \models_0 \neg F_e(x)$. Assume that $(\nexists \tau \subseteq f)(\tau \Vdash_0 \neg F_e(x))$. Since f is $1-generic$, we have $(\exists \tau \subseteq f)(\tau \Vdash_0 F_e(x))$. By i), we get $f \models_0 F_e(x)$. Contradiction.
   ($\leftarrow$) Fix a finite part $\tau \subseteq f$ such that $\tau \Vdash_0 \neg F_e(x)$, but assume $f \not\models_0 \neg F_e(x)$, which, by definition, means that $f \models_0 F_e(x)$. By the positive case, there is a finite part $\delta \subseteq f$ such that $\delta \Vdash_0 F_e(x)$. By i), we can

take δ to be such that $\tau \subseteq \delta$. Since $\tau \Vdash_0 \neg F_e(x)$, we get $\delta \nVdash_0 F_e(x)$. Contradiction.

2. Let the assertion be true for k. We will prove it for $k+1$.
Let f be $k+1-$ *generic*. We first consider the positive case.
$(\rightarrow)$ Suppose that $f \models_{n+1} F_e(x)$. Then

$$f \models_{n+1} F_e(x) \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$$
$$((u = \langle 0, e_u, x_u\rangle \wedge f \models_n F_{e_u}(x_u))\vee$$
$$(u = \langle 1, e_u, x_u\rangle \wedge f \models_n \neg F_{e_u}(x_u))\vee$$
$$(u = \langle 2, x_u\rangle \wedge f(x_u) \in B_{n+1})))$$

By induction hypothesis (for the positive and negative case) we can choose appropriate finite parts τ_u and let $\tau = \bigcup_u \tau_u$. By i), since every $\tau_u \subseteq \tau$,

$$\tau_u \Vdash_n F_{e_u}(x_u) \text{ implies } \tau \Vdash_n F_{e_u}(x_u)$$
$$\tau_u \Vdash_n \neg F_{e_u}(x_u) \text{ implies } \tau \Vdash_n \neg F_{e_u}(x_u)$$

and $\tau(x_u) \in B_n$.It follows that $f \models_{n+1} F_e(x)$ implies $\tau \Vdash_{n+1} F_e(x)$. Since $\tau \subseteq f$, the conclusion follows.
$(\leftarrow)$ Suppose there is $\tau \subseteq f$ such that $\tau \Vdash_{n+1} F_e(x)$. By the definition of forcing and the induction hypothesis,

$$\tau \Vdash_{n+1} F_e(x) \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$$
$$((u = \langle 0, e_u, x_u\rangle \wedge \tau \Vdash_n F_{e_u}(x_u))\vee$$
$$(u = \langle 1, e_u, x_u\rangle \wedge \tau \Vdash_n \neg F_{e_u}(x_u))\vee$$
$$(u = \langle 2, x_u\rangle \wedge \tau(x_u) \in B_{n+1})))$$

i.e.

$$(\exists v)(\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$$
$$((u = \langle 0, e_u, x_u\rangle \wedge f \models_n F_{e_u}(x_u))\vee$$
$$(u = \langle 1, e_u, x_u\rangle \wedge f \models_n \neg F_{e_u}(x_u))\vee$$
$$(u = \langle 2, x_u\rangle \wedge f(x_u) \in B_{n+1})))$$

Hence $f \models_{n+1} F_e(x)$.
Now for the negative case:

($\rightarrow$) Let $f \models_{n+1} \neg F_e(x)$. Assume that $(\not\exists \tau \subseteq f)(\tau \Vdash_{n+1} \neg F_e(x))$. Since f is $(k+1) - generic$, we have $(\exists \tau \subseteq f)(\tau \Vdash_{n+1} F_e(x))$. By i), we get $f \models_{n+1} F_e(x)$. Contradiction.
($\leftarrow$) Fix a finite part $\tau \subseteq f$ such that $\tau \Vdash_{n+1} \neg F_e(x)$, but assume $f \not\models_{n+1} \neg F_e(x)$, which, by definition, means that $f \models_{n+1} F_e(x)$. By the positive case, there is a finite part $\delta \subseteq f$ such that $\delta \Vdash_{n+1} F_e(x)$. By i), we can take δ to be such that $\tau \subseteq \delta$. Since $\tau \Vdash_{n+1} \neg F_e(x)$, we get $\delta \nVdash_{n+1} F_e(x)$. Contradiction.

□

Definition 18. *We say that the sequence $\overrightarrow{A}$ is forcing definable on $\mathfrak{A}$ with respect to the sequence $\overrightarrow{B}$ if there exists a finite part δ, and a computable function g,$x \in \mathbb{N}$, such that for every n in $\mathbb{N}$:*

$$s \in A_n \text{ iff } (\exists \tau \supseteq \delta)(\tau(x) = s \wedge \tau \Vdash_n F_{g(n)}(x)).$$

Theorem 5. *Let $\overrightarrow{A}$ be not forcing definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$. Then there exists an enumeration f of $\mathfrak{A}$, such that $f^{-1}(\overrightarrow{A}) \not\leq_\omega P^f$.*

Proof. We will construct the enumeration f on stages via finite parts δ_q. We want $\delta_q \subseteq \delta_{q+1}$ and then we will take $f = \bigcup_q \delta_q$. On stages $q = 3r$ we will ensure that f is total and surjective. On stages $q = 3r + 1$ we ensure that f is $k - generic$ for each $k > 0$. On stages $q = 3r + 2$ we will ensure that f satisfies the omitting condition: $f^{-1}(\overrightarrow{A}) \not\leq_\omega P^f$.
Let $g_0, g_1, ...$ be an enumeration of all 1-ary computable functions. For each $n, x, e \in \mathbb{N}$, we denote $Y^n_{<e,x>}$ to be the set of all finite parts ρ such that $\rho \Vdash_n F_e(x)$.
Let δ_0 be the empty finite part and suppose that δ_q is already defined.

1. case $q = 3r$: Let x_0 be the least natural number which does not belong to $dom(\delta_q)$ and let s_0 be the least natural number which does not belong to $ran(\delta_q)$. Set $\delta_{q+1}(x_0) = s_0$ and $\delta_{q+1}(x) = \delta_q(x)$ for $x \neq x_0$.

2. case $q = 3\langle e, n, x\rangle + 1$: Check whether there exists a finite part $\rho \in Y^n_{<e,x>}$, that extends δ_q. If there is such a part, set δ_{q+1} to be the least extension(regarding the length) of δ_q, that belongs to $Y^n_{<e,x>}$. Otherwise set $\delta_{q+1} = \delta_q$.

3. case $q = 3r + 2$: Consider the computable function g_r. Let x_q be the least natural number s.t. $x_q \notin dom(\delta_q)$. For each n denote by

$$C_n = \{x : (\exists \tau \supseteq \delta_q)(\tau(x_q) = x \wedge \tau \Vdash_n F_{g_r(n)}(x))\}.$$

Obviously the sequence of sets $\overrightarrow{C}$ is forcing definable and hence $\overrightarrow{C} \neq \overrightarrow{A}$ i.e. $C_n \neq A_n$ for some n.
Let $\langle x, n, q\rangle$ be the least triple such that

$$(x \in C_n \wedge x \notin A_n) \vee (x \notin C_n \wedge x \in A_n)$$

i) Suppose $x \in C_n$. Then there is a finite part τ such that

$$\tau \supseteq \delta_q \wedge \tau(x_q) = x \wedge \tau \Vdash_n F_{g_r(n)}(x).$$

Set δ_{q+1} to be the least such τ.
ii) Suppose $x \notin C_n$. Then set $\delta_{q+1}(x_q) = x$ and $\delta_{q+1}(y) = \delta_q(y), y \neq x_q$. (Here we have that $\delta_{q+1} \Vdash_n \neg F_{g_r(n)}(x)$)

The construction is finished. Let $f = \bigcup_q \delta_q$.

The enumeration f is total and surjective due to how it is build in the first case. Let $k \in \mathbb{N}$. In order to prove that f is $(k+1)-generic$, suppose $j \leq k$. Consider the stage $q = 3\langle e, j, x\rangle + 1$. If there is a finite part $\rho \supseteq \delta_q$ such that $\rho \Vdash_j F_e(x)$, then from the construction we have $\delta_{q+1} \Vdash_j F_e(x)$. Otherwise $\delta_{q+1} \Vdash_j \neg F_e(x)$. Hence f is $(k+1) - generic$.
To prove the omitting condition, assume the opposite, i.e. $f^{-1}(\overrightarrow{A}) \leq_\omega P^f$. Then there is a computable function g_s, such that for each n,

$$A_n = \{f(x) : f \models_n F_{g_s(n)}(x)\}.$$

Since the enumeration is $(n+1) - generic$, $f \models_n (\neg)F_{g_s(n)}(x)$ iff $(\exists \tau \subseteq f)(\tau \Vdash_n (\neg)F_{g_s(n)}(x))$ for each x. Consider the stage $q = 3s+2$. From the construction we have x_q and n, such that one of the two cases holds:
i) $\delta_{q+1}(x_q) \notin A_n \wedge \delta_{q+1} \Vdash_n F_{g_s(n)}(x_q)$. By the genericity of f, $f(x_q) \notin A_n$ and $f \models_n F_{g_s(n)}(x_q)$. Contradiction.
ii)$\delta_{q+1}(x_q) \in A_n \wedge \delta_{q+1} \Vdash_n \neg F_{g_s(n)}(x_q)$. Hence $f(x_q) \in A_n$ and $f \models_n \neg F_{g_s(n)}(x_q)$. Contradiction. □

A corollary to the above theorem is the following:

Lemma 5. *Let $\overrightarrow{A}_0, \overrightarrow{A}_1, ...$ be a sequence of sequences of sets, s.t. each $\overrightarrow{A}_i$ is not forcing definable on A with respect to $\overrightarrow{B}$. Then there exists an enumeration f of A, s.t. $f^{-1}(\overrightarrow{A}_i) \not\leq_u P^f$ for each i.*

Proof. The proof is almost the same as in the theorem. The difference is that on stages of the form $q = 3 < r, i > +2$, we consider the computable function g_r and ensure that $\overrightarrow{A}_i \neq \overrightarrow{C}$, where the sequence $\overrightarrow{C}$ is defined in the same way. □

Theorem 6. *Let $\overrightarrow{A}$ be a sequence of sets not forcing definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$. Then there exists an acceptable with respect to $\overrightarrow{B}$ enumeration g, such that $g^{-1}(\overrightarrow{A}) \not\leq_\omega P^g$ and the enumeration degree of $g^{-1}(\mathfrak{A})$ is total.*

Proof. Let $\overrightarrow{A}$ be not forcing definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$. By Theorem 5, there exists an enumeration f such that $f^{-1}(\overrightarrow{A}) \not\leq_\omega P^f$. Hence, by Theorem 1, there exists a total set F, such that $P^f \leq_\omega \{F^{(n)}\}_{n<\omega}$ and $f^{-1}(\overrightarrow{A}) \not\leq_\omega \{F^{(n)}\}_{n<\omega}$. From the definition of P^f and $P^f \leq_\omega \{F^{(n)}\}_{n<\omega}$ we have that $f^{-1}(\mathfrak{A}) \leq_e F$ and $f^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n.
Fix two natural numbers, say s, t such that $s \neq t$ and natural numbers x_s and x_t such that $f(x_s) = s, f(x_t) = t$. We define a function g as follows:

$$g(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F, \\ t & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

Clearly, g thus defined is an enumeration of $\mathfrak{A}$. We want to prove that $g^{-1}(\mathfrak{A}) \equiv_e F$.

i) Let's fix a predicate R_i. Let $x_1, ..., x_{r_i}$ be arbitrary natural numbers. We will define natural numbers $y_1, ..., y_{r_i}$. Let $1 \leq j \leq r_i$. We have the following three cases:
a) x_j is even. Then let $y_j = x_j/2$.
b) $x_j = 2z + 1$ and $z \in F$. Then let $y_j = x_s$.
c) $x_j = 2z + 1$ and $z \notin F$. Then let $y_j = x_t$.
We have the following equivalence:

$$\langle x_1, ..., x_{r_i} \rangle \in g^{-1}(R_i) \Leftrightarrow \langle y_1, ..., y_{r_i} \rangle \in f^{-1}(R_i).$$

Hence $g^{-1} \leq_e f^{-1}(R_i) \oplus F \oplus \overline{F}$. From $f^{-1}(\mathfrak{A}) \leq_e F$, we have $g^{-1}(R_i) \leq_e F$. Since R_i was an arbitrary predicate of the structure, we have that $g^{-1}(\mathfrak{A}) \leq_e$

F.

ii) We have the following equivalences:

$$z \in F \Leftrightarrow 2z+1 \in g^{-1}(s) \Leftrightarrow g(2z+1) = s$$
$$z \notin F \Leftrightarrow 2z+1 \in g^{-1}(t) \Leftrightarrow g(2z+1) = t$$

Since $=, \neq$ are among the predicates of the structure, we have $F \leq_e g^{-1}(\mathfrak{A})$.

Combining i) and ii), we get $g^{-1}(\mathfrak{A}) \equiv_e F$. By the properties of $\leq_e$ we have that $g^{-1}(\mathfrak{A})^{(n)} \equiv_e F^{(n)}$ uniformly in n.
Denote by E_g, E_f the sets $E_g = g^{-1}(=), E_f = f^{-1}(=)$.
We have

$$E_f \leq_e F \Longrightarrow E_g \leq_e F \Longrightarrow E_g \leq_e F^{(n)} \text{ uniformly in n.}$$

Fix n. We have:

$$g^{-1}(B_n) = \{x : (\exists y \in f^{-1}(B_n))(\langle x, 2y\rangle \in E_g)\}.$$

Hence $g^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n. Thus, we have proved that g is an acceptable enumeration of $\mathfrak{A}$ with respect to $\overrightarrow{B}$.
To finish the proof, assume that $g^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$. We have $g^{-1}(A_n) = \{x : 2x \in f^{-1}(A_n)\}$. Hence $f^{-1}(\overrightarrow{A}) \leq_\omega g^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$.
By transitivity of $\leq_\omega$, we have $f^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$. Contradiction. □

Theorem 7. *For every sequence $\overrightarrow{A}$, if*

$$(\forall f)[f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n \Longrightarrow f^{-1}(\overrightarrow{A}) \leq_\omega f^{-1}(\mathfrak{A})]$$

then $\overrightarrow{A}$ is forcing definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$.

Proof. Assume that $\overrightarrow{A}$ is not forcing definable. By the Theorem 6, we have an acceptable enumeration g, such that $g^{-1}(\overrightarrow{A}) \not\leq_\omega P^g$. Contradiction. □

3.2 Formal definability

In this section we will show that the forcing definable sequences on the structure $\mathfrak{A}$ coincide with the sequences which are definable on $\mathfrak{A}$ by means of a

certain kind of positive computable Σ^0_n formulas.
Let $\mathfrak{L} = (T_1, ..., T_k)$ be the first order relational language corresponding to the structure $\mathfrak{A}$ which contains the predicates $=, \neq$. So every T_i is $r_i - ary$ predicate sumbol. Let $\{P_n\}_{n<\omega}$ be a computable sequence of unary predicates intended to represent the sets B_n. We shall also suppose that we have a fixed sequence $X_0, X_1, ..., X_n, ...$ of variables. We will use X,Y,W possibly with subscripts as syntactival variables which vary through the variables.
We will define for each natural number n, the Σ^+_n formulas. The definition is by recursion on n, and goes along the definition of indices for the formulas.

Definition 19. *1. An elementary Σ^+_0 formula with free variables among $W_1, ..., W_r$ is an existentional formula of the form*

$$\exists Y_1 ... \exists Y_m \Phi(W_1, ..., W_r, Y_1, ..., Y_m),$$

where Φ is a finite conjunction of atomic formulas in $\mathfrak{L} \cup \{P_0\}$.

2. A Σ^+_n formula is a c.e. disjunction of elementary Σ^+_n formulas.

3. An elementary Σ^+_{n+1} formula is a formula of the form

$$\exists Y_1 ... \exists Y_m \Phi(W_1, ..., W_r, Y_1, ..., Y_m),$$

where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ or Σ^+_n formulas or negations of Σ^+_n formulas in the language

$$\mathfrak{L} \cup \{P_0\} \cup ... \cup \{P_n\}.$$

Remark: We can see that the Σ^+_n formulas are effectiely closed under existential quantification and c.e. disjunctions.
Let Φ be a Σ^+_n formula with free variables among $W_1, ..., W_n$ and let $t_1, ..., t_n \in \mathbb{N}$. Then by $\mathfrak{A} \models \Phi(W_1/t_1, ..., W_n/t_n)$ we denote that Φ is true on $\mathfrak{A}$ under the variable assignment v such that $v(W_1) = t_1, ..., v(W_n) = t_n$.

Definition 20. *Let $\overrightarrow{A}, \overrightarrow{B}, \mathfrak{A}$ be given. We say that $\overrightarrow{A}$ is formally definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$, if there is a computable function γ and a computable sequence $\{\Phi^{\gamma(n)}\}_{n<\omega}$ of formulas, such that for every n, $\Phi^{\gamma(n)}$ is a Σ^+_n formula with free variables among $W_1, ..., W_r$ and elements $t_1, ..., t_r \in \mathbb{N}$, such that for every $x \in \mathbb{N}$:*

$$x \in A_n \Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \Phi^{\gamma(n)}(W_1/t_1, ..., W_r/t_r, X/x).$$

We shall show that every forcing definable sequence is formally definable. Let var be an effetive mapping of the natural numbers onto the variables. Given a natural number x, by X we shall denote the variable $var(x)$.
Let $y_1 < y_2 < ... < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$ and let Φ be an arbitrary formula. Then by $Q(\overline{y} : \overline{y} \in D)\Phi$ we shall denote the formula $QY_1...QY_k\Phi$.

Lemma 6. *Let $D = \{w_1, ..., w_r\}$ be a finite non-empty set of natural numbers and let x,e be elements of $\mathbb{N}$. There exists an uniform recursive way to construct a Σ_n^+ formula $\Phi_{D,e,x}^{\gamma(n)}$ with free variables among $W_1, ..., W_r$ such that for every finite part δ such that $dom(\delta) = D$, the following equivalences are true:*

$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi_{D,e,x}^n(W_1/\delta(w_1), ..., W_r/\delta(w_r)) \Leftrightarrow \delta \Vdash_n F_e(x)$$

$$(\mathfrak{A}, \overrightarrow{B}) \models \Psi_{D,e,x}^n(W_1/\delta(w_1), ..., W_r/\delta(w_r)) \Leftrightarrow \delta \Vdash_n \neg F_e(x)$$

Proof. We shall construct the formula $\Phi_{D,e,x}^{\gamma(n)}$ by recursion on n, following the definition of forcing.

1. Let $n = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element $v \in V$. For every $u \in D_v$ define an atom Π_u as follows:
   a) $u = \langle 0, \langle i, x_1^u, ..., x_{r_i}^u\rangle\rangle$, where $1 \leq i \leq k$ and all $x_1^u, ..., x_{r_i}^u$ are elements of D. Then let $\Pi_u = T_i(X_1^u, ..., X_{r_i}^u)$.
   b) $u = \langle 2, x_u\rangle$ and $x_u \in D$. Then let $\Pi_u = P_0(X_u)$
   c) $\Pi_u = W_1 \neq W_1$ in all other cases.
   Let $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi_{D,e,x}^0 = \bigvee_{v \in V} \Pi_v$.
   Let $\Psi_{D,e,x}^0 = \neg[\bigvee_{D^* \supseteq D}(\exists \overrightarrow{Y} \in D^* \setminus D)\Phi_{D^*,e,x}^0]$.

2. Assume it is done up till n and we will prove for $n + 1$. Let again $V = \{v : \langle v, x\rangle \in W_e \text{ and } v \in V\}$. For every $u \in D_v$ define a formula Π_u as follows:
   a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi_{D,e_u,x_u}^n$.
   b) If $u = \langle 1, e_u, x_u\rangle$, then let $\Pi_u = \Psi_{D,e_u,x_u}^n$.
   c) If $u = \langle 2, x_u\rangle$ and $x_u \in D$, then let $\Pi_u = P_{n+1}(X_u)$.
   d) $\Pi_u = W_1 \neq W_1$ in all other cases.
   Now let $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi_{D,e,x}^{n+1} = \bigvee_{v \in V} \Pi_v$.
   Let $\Psi_{D,e,x}^{n+1} = \neg[\bigvee_{D^* \supseteq D}(\exists \overrightarrow{Y} \in D^* \setminus D)\Phi_{D^*,e,x}^{n+1}]$.

We constructed the formulas in a uniform recursive way, hence we can find a computable function $\gamma(n, D, e, x)$ which gives the code of the formula $\Phi^n_{D,e,x}$. We prove the statement in the lemma by induction on n.

1. Let $n = 0$. By the definition of the forcing relation:

$$\delta \Vdash_0 F_e(x) \Leftrightarrow (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subset \tau^{-1}(\mathfrak{A}) \oplus \tau^{-1}(B_0))$$

$$\Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \bigvee_{v \in V} \Pi_v$$

which is what we need. On the other hand

$$\delta \Vdash_0 \neg F_e(x) \Leftrightarrow \neg(\exists \rho \supseteq \delta)[\rho \Vdash_0 F_e(x)]$$

$$\Leftrightarrow \neg(\exists \rho \supseteq \delta)[(\mathfrak{A}, \overrightarrow{B}) \models \Phi^0_{D,e,x}(\overrightarrow{W} \setminus \overrightarrow{\rho(w)})]$$

$$\Leftrightarrow (\nexists D^* \supseteq D)[(\mathfrak{A}, \overrightarrow{B}) \models (\exists \overrightarrow{Y} \in D^* \setminus D)\Phi^0_{D^*,e,x}(\overrightarrow{W} \setminus \overrightarrow{\delta(w)}, \overrightarrow{Y})]$$

$$\Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \neg \bigvee_{D^* \supseteq D} (\exists \overrightarrow{Y} \in D^* \setminus D)\Phi^0_{D^*,e,x}(\overrightarrow{W} \setminus \overrightarrow{\delta(w)}, \overrightarrow{Y})$$

$$\Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \Psi^0_{D,e,x}(\overrightarrow{W} \setminus \overrightarrow{\delta(w)}).$$

2. Assume the statement is true for n and we will prove it for $n + 1$. From the definition of the forcing relation we get:

$$\delta \Vdash_{n+1} F_e(x) \Leftrightarrow (\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$$

$$((u = \langle 0, e_u, x_u\rangle \wedge \delta \Vdash_n F_{e_u}(x_u))\vee$$

$$(u = \langle 1, e_u, x_u\rangle \wedge \delta \Vdash_n \neg F_{e_u}(x_u))\vee$$

$$(u = \langle 2, x_u\rangle \wedge \delta(x_u) \in B_{n+1}))]$$

$$\Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \bigvee_{v \in V} \bigwedge_{u \in D_v} \Pi_u$$

$$\Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \Phi^{n+1}_{D,e,x}(\overrightarrow{W} \setminus \overrightarrow{\delta(w)})$$

where Π_u formulas are as in the construction. It is easy to see that the formula $\Psi^{n+1}_{D,e,x} = \neg[\bigvee_{D^* \supseteq D}(\exists \overrightarrow{Y} \in D^* \setminus D)\Phi^{n+1}_{D^*,e,x}]$ defines the relation $\delta \Vdash_{n+1} \neg F_e(x)$, we simply proceed by the definition of the forcing relation.

□

Theorem 8. *Let the sequence $\overrightarrow{A}$ be forcing definable. Then $\overrightarrow{A}$ is formally definable.*

Proof. Suppose for every $s \in \mathbb{N}$ and for every $n \in \mathbb{N}$ we have:

$$s \in A_n \Leftrightarrow (\exists \tau \supseteq \delta)(\tau(x) = s \wedge \tau \Vdash_n F_{g(n)}(x)),$$

where g is a computable function and δ is a finite part. Fix n and x. Let $D = dom(\delta) = \{w_1, ..., w_r\}$ and let $\delta(w_i) = t_i$ for $i = 1, ..., r$. By Lemma 6,

$$(\mathfrak{A}, \overrightarrow{B}) \models \exists(\overline{y} \in D^* \setminus (D \cup \{x\}))\Phi^n_{D^*,g(n),x}(W_1/t_1, ..., W_r/t_r, X/s, \overrightarrow{Y})$$

iff there exists a finite part τ such that $dom(\tau) = D^*$, $\tau \supseteq \delta$, $\tau(x) = s$ and $\tau \Vdash_n F_{g(n)}(x)$. For the set A_n we have,

$$x \in A_n \Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \bigvee_{D^* \supseteq D} \exists(\overline{y} \in D^* \setminus (D \cup \{x\}))\Phi^n_{D^*,g(n),x}(W_1/t_1, ..., W_r/t_r, X/s, \overrightarrow{Y})$$

Let $\gamma(n) = \gamma(n, e, x, D)$ (γ is a function on one variable that also depends on e, x, D), where

$$\Xi^n_{e,x,D} = \exists(\overline{y} \in D^* \setminus (D \cup \{x\}))\Phi^n_{D^*,g(n),x}(W_1/t_1, ..., W_r/t_r, X/s, \overrightarrow{Y})$$

The function γ is computable and defines the code of the formula

$$\Phi^{\gamma(n)} = \exists(\overline{y} \in D^* \setminus (D \cup \{x\}))\Phi^n_{D^*,g(n),x}(W_1/t_1, ..., W_r/t_r, X/s, \overrightarrow{Y})$$

Hence

$$x \in A_n \Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \Phi^{\gamma(n)}(\overrightarrow{W}/\overrightarrow{t}, X/s).$$

Thus we conclude that the sequence $\overrightarrow{A}$ is formally definable. □

We will need the following useful statement:

Lemma 7. *Let g be an arbitrary enumeration of $\mathfrak{A}$. There exists a bijective enumeration f of $\mathfrak{A}$, such that $f^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{A})$.*

Proof. Let's form the set $E_g = \{\langle x, y\rangle : g(x) = g(y)\}$. It is easy to see that $E_g^+ \leq_e g^{-1}(\mathfrak{A})$, because $=, \neq$ are among the predicates of the structure. We will define using the recursion scheme a computable function h as follows:

$$h(0) = 0$$

$$h(n+1) = \mu z[(\forall k \leq n)(\langle h(k), z\rangle \notin E_g)]$$

Define $f(n) = g(h(n))$. Let $n_1 \neq n_2$. Without loss of generality assume $n_1 < n_2$. If $f(n_1) = f(n_2)$ then $g(h(n_1)) = g(h(n_2))$, i.e. $\langle h(n_1), h(n_2)\rangle \in E_g$. From $n_1 < n_2$ and the definition of h, it follows that $\langle h(n_1), h(n_2)\rangle \notin E_g$. We obtain a contradiction, hence $f(n_1) \neq f(n_2)$ and so f is injective. By the definition of h, it is true that $n_1 < n_2$ implies $h(n_1) < h(n_2)$. Assume that f is not surjective, i.e. $(\exists k)(\forall n)(f(n) \neq k)$ and so $(\exists k)(\forall n)(g(h(n)) \neq k)$. g is onto $\mathbb{N}$, so $(\exists l)(g(l) = k)$ and $(\forall n)(\langle h(n), l\rangle \notin E_g)$ and so exists t such that $h(t) < l$ and $h(t+1) > l$. Hence $(\exists s \leq t)(\langle h(s), l\rangle \in E_g)$. We get $f(s) = g(h(s)) = g(l) = k$. This is a contradiction with the assumption and hence $(\forall k)(\exists s)(f(s) = k)$. Thus f is onto $\mathbb{N}$. It is easy to see that $E_g^+ \oplus f^{-1}(\mathfrak{A}) \equiv_e g^{-1}(\mathfrak{A})$. □

As a corollary to Lemma 7, we get:

Lemma 8. *Let g be an arbitrary enumeration of $\mathfrak{A}$. Then there exists a bijective enumeration f such that $P^f \leq_\omega P^g$.*

Proof. Let g be an arbitrary enumeration. By Lemma 7, there is a bijective enumeration f such that $f^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{A})$. Let $\overrightarrow{X} = \{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), ...\}$ and $\overrightarrow{Y} = \{g^{-1}(\mathfrak{A}) \oplus g^{-1}(B_0), g^{-1}(B_1), ...\}$ be two sequences. It is enough to prove that $\overrightarrow{X} \leq_e P_n^g$ uniformly in n. We will prove the assertion by induction on n.

1. Let $n = 0$. We want to prove $f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0) = W_{e_0}(g^{-1}(\mathfrak{A}) \oplus g^{-1}(B_0))$ where the index e_0 is obtained effectively. By assumption, we have $f^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{A})$. Let $x \in f^{-1}(B_0)$. We have the following equivalences:

   $$x \in f^{-1}(B_0) \leftrightarrow f(x) \in B_0 \Leftrightarrow (\exists z)[\langle z, y\rangle \in g^{-1}(=) \wedge y \in B_0]$$

   hence we can effectively find an index e_0 such that $f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0) = W_{e_0}(g^{-1}(\mathfrak{A}) \oplus g^{-1}(B_0))$.

2. Let the assertion be true for n and we will prove it for $n+1$. By induction hypothesis, $f^{-1}(B_n) = W_{e_n}(P_n^g)$ and the index e_n is obtained effectively. By analogous equivalences as in the base case and the properties of the jump sequence, we can effectively find an index e_{n+1} from e_n such that $f^{-1}(B_{n+1}) = W_{e_{n+1}}(P_{n+1}^g)$.

□

For the next lemma we will use the following notation: $P^{D_{id}}$ will be the jump sequence of $\{D(\mathfrak{A}) \oplus B_0, B_1, ...\}$.

Lemma 9. *Let Φ be a Σ_n^+ formula. We can effectively find, from the code of the formula Φ an enumeration operator W_{e_n}, such that for arbitrary $\overrightarrow{t}$, we have*

$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow \langle \overrightarrow{t} \rangle \in W_{e_n}(P_n^{D_{id}}).$$

Proof. We will prove the assertion by induction on n.

1. Let $n = 0$. We have a Σ_0^+ formula $\Phi(\overrightarrow{W})$ which is a c.e. disjunction of elementary Σ_0^+ formulas. Hence there is a c.e. set W_{e_0} such that

$$\ulcorner \alpha \urcorner \in W_{e_0} \Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \alpha(\overrightarrow{W}/\overrightarrow{t})$$

where $\alpha(\overrightarrow{W})$ is a disjunct in the formula $\Phi(\overrightarrow{W})$ and $\alpha(\overrightarrow{W})$ has the form $(\exists \overrightarrow{Y})(P_{l_1}(\overrightarrow{W}, \overrightarrow{Y}) \wedge P_{l_2}(\overrightarrow{W}, \overrightarrow{Y}) ... \wedge P_{l_k}(\overrightarrow{W}, \overrightarrow{Y}))$. Hence

$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow$$

there exists elementary Σ_0^+ formula α such that $\ulcorner \alpha \urcorner \in W_{e_0}$ and $(\mathfrak{A}, \overrightarrow{B}) \models \alpha(\overrightarrow{W}/\overrightarrow{t})$

$\Leftrightarrow$ there exists a formula α and natural numbers $\overrightarrow{u}$: $\ulcorner \alpha \urcorner \in W_{e_0}$ and

$(\mathfrak{A}, \overrightarrow{B}) \models P_{l_1}(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u}) \wedge P_{l_2}(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u}) ... \wedge P_{l_k}(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u})$

For simplicity, let's assume we have chosen a coding such that $\langle l_i, \overrightarrow{x} \rangle \in D(\mathfrak{A}) \oplus B_0 \Leftrightarrow \overrightarrow{x} \in P_{l_i}$. Hence

$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow$$

$\Leftrightarrow$ there exists D_v such that $\ulcorner \alpha \urcorner \in W_{e_0}$ and $D_v \subset D(\mathfrak{A}) \oplus B_0$, where D_v effectively determines α

$\Leftrightarrow$ there exists D_v such that $\langle v, \overrightarrow{t} \rangle \in W_{e_0'}$ and $D_v \subset D(\mathfrak{A}) \oplus B_0$, where the code e_0' is effectively determined by e_0.

$$\Leftrightarrow \langle \overrightarrow{t} \rangle \in W_{e_0'}(P_0^{D_{id}})$$

2. Assume that there is a Σ_n^+ formula $\Phi(\overrightarrow{W}/\overrightarrow{t})$ and a c.e. set $W_{e'_n}$ such that
$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow < \overrightarrow{t} > \in W_{e'_n}(P_n^{D_{id}})$$
We will examine the case $n+1$.

$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow$ there exists an elementary Σ_{n+1}^+ formula α such that
$$\ulcorner \alpha \urcorner \in W_{e_{n+1}} \text{ and } (\mathfrak{A}, \overrightarrow{B}) \models \alpha(\overrightarrow{W}/\overrightarrow{t})$$
where α has the form
$$(\exists \overrightarrow{Y})((\neg)\beta_1(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}) \wedge ... \wedge (\neg)\beta_k(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}))$$
where β_i are Σ_n^+ formulas or the membership predicate P_{n+1}. If β is $P_n + 1$ then it cannot have $\neg$ in front of it.

$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow$ there exists an elementary Σ_{n+1}^+ formula α such that
$$\ulcorner \alpha \urcorner \in W_{e_{n+1}} \text{ and } (\mathfrak{A}, \overrightarrow{B}) \models \alpha(\overrightarrow{W}/\overrightarrow{t})$$
$\Leftrightarrow$ there exist formulas $\beta_1, ..., \beta_k$ which are either the predicate P_{n+1} or are Σ_n^+ formulas and natural numbers $\overrightarrow{u}$ such that
$\ulcorner \alpha \urcorner \in W_{e_{n+1}}$ and $(\mathfrak{A}, \overrightarrow{B}) \models (\neg)\beta_1(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u}) \wedge ... \wedge (\neg)\beta_k(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u})$

If β_i is a Σ_n^+ formula, by induction hypothesis, we can effectively find from it's code an enumeration operator e_n^i and for arbitrary $\overrightarrow{t}, \overrightarrow{u}$:
$$(\mathfrak{A}, \overrightarrow{B}) \models \beta_i(\overrightarrow{W}/\overrightarrow{t}, \overrightarrow{Y}/\overrightarrow{u}) \Leftrightarrow \langle \overrightarrow{t}, \overrightarrow{u} \rangle \in W_{e_n^i}(P_n^{D_i})$$
$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow \text{ there exists } D_v \text{ and } \ulcorner \alpha \urcorner \in W_{e_{n+1}}$$
$$\text{and for } i = 1...k \text{ we have } \langle \overrightarrow{t}, \overrightarrow{u} \rangle \in W_{e_n^i}(P_n^{D_{id}}) \text{ if } l_i = 0 \text{ and}$$
$\langle \overrightarrow{t}, \overrightarrow{u} \rangle \notin W_{e_n^i}(P_n^{D_{id}})$ if $l_1 = 1$, where l_i is part of the coding i.e. D_v consists of elements of the form: $\langle l_i, ... \rangle$
$$\Leftrightarrow (\exists v)(\langle v, \overrightarrow{t} \rangle \in W_{e'_{n+1}} \wedge (\forall u \in D_v))$$
$$((u = \langle 0, e_u, x_u \rangle \wedge x_u \in W_{e_u}(P_n^{D_{id}})) \vee$$
$$(u = \langle 1, e_u, x_u \rangle \wedge x_u \notin W_{e_u}(P_n^{D_{id}})) \vee$$

$$(u = \langle 2, x_u \rangle \wedge x_u \in B_{n+1})))$$

where the code e'_{n+1} is effectively determined from e_{n+1}

$\Leftrightarrow$ there exists c.e. set $W_{e''_{n+1}}$ and $(\exists v')(\langle v', \overrightarrow{t} \rangle \in W_{e''_{n+1}}$ and $D_{v'} \subset (P_n^{D_{id}})' \oplus B_{n+1}$

where the code e''_{n+1} is effectively determined from e'_{n+1}

$$\Leftrightarrow \langle \overrightarrow{t} \rangle \in W_{e''_{n+1}}(P_{n+1}^{D_{id}})$$

$\square$

Theorem 9. *Let $\overrightarrow{A}$ be formally definable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$. Then for every acceptable enumeration f, we have that $f^{-1}(\overrightarrow{A}) \leq_\omega f^{-1}(\mathfrak{A})$.*

Proof. Since $\overrightarrow{A}$ is formally definable, there is a sequence of formulas $\{\Phi^{\gamma(n)}\}_{n<\omega}$ of Σ_n^+ formulas and natural numbers $t_1, ..., t_l$ such that:

$$x \in A_n \Leftrightarrow (\mathfrak{A}, \overrightarrow{B}) \models \Phi^{\gamma(n)}(W_1/t_1, ..., W_r/t_r, X/x).$$

Assume that there exists an enumeration of $\mathfrak{A}$, say g, that is acceptable on $\mathfrak{A}$ with respect to $\overrightarrow{B}$, but $g^{-1}(\overrightarrow{A}) \not\leq_\omega g^{-1}(\mathfrak{A})$. By Lemma 8, there exists a bijective enumeration f, such that $f^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{B})$ and $P^f \leq_\omega P^g$.
Let $\mathfrak{B}$ be the structure with domain $\mathbb{N}$ and predicates $f^{-1}(R_1), ..., f^{-1}(R_k)$. Clearly $\mathfrak{A} \cong \mathfrak{B}$ and $f^{-1}(\mathfrak{A}) \equiv_e D(\mathfrak{B})$. Let $f(u_i) = t_i$ for $i \leq l$. We have

$$(\mathfrak{A}, \overrightarrow{B}) \models \Phi^{\gamma(n)}(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow (\mathfrak{B}, f^{-1}(\overrightarrow{B})) \models \Phi^{\gamma(n)}(\overrightarrow{W}/\overrightarrow{u})$$

Hence $f^{-1}(\overrightarrow{A})$ is formally definable in $\mathfrak{B}$. It follows that $f^{-1}(\overrightarrow{A}) \leq_\omega P^f$. We want to prove $g^{-1}(\overrightarrow{A}) \leq_\omega \{f^{-1}(A_n) \oplus E_g^+\}_{n<\omega}$. We will give an explaination how we can effectively obtain an index, let's say e_n, such that $g^{-1}(A_n) = W_{e_n}(P_n(f^{-1}(\overrightarrow{A}) \oplus E_g^+))$. Fix n and assume without loss of generality that $n > 0$. We have the following equivalences:

$$x \in g^{-1}(A_n) \Leftrightarrow g(x) \in A_n \Leftrightarrow g(x) = y \wedge y \in A_n \Leftrightarrow [\langle x, y \rangle \in E_g \wedge y \in A_n].$$

By definition $P_n(f^{-1}(\overrightarrow{A}) \oplus E_g^+)$ is $(P_{n-1}(f^{-1}(\overrightarrow{A})))' \oplus (f^{-1}(A_n) \oplus E_g^+)$. Hence, by the definition of $\oplus$, we can effectively obtain an index e_n such that $g^{-1}(A_n) = W_{e_n}(P_n(f^{-1}(\overrightarrow{A}) \oplus E_g^+))$. But n was arbitrary, hence we get what

we needed.
Since $f^{-1}(\overrightarrow{A}) \oplus E_g^+ \leq_\omega P^g$ and by transitivity of $\leq_\omega$, we get $g^{-1}(\overrightarrow{A}) \leq_\omega P^g$ and the enumeration g is acceptable. Hence $g^{-1}(\overrightarrow{A}) \leq_\omega g^{-1}(\mathfrak{A})$. Contradiction. □

Putting everything together we arrive at the following:

Theorem 10. *The following statements are equivalent:*
i) $\overrightarrow{A}$ *is relatively intrinsic on* $\mathfrak{A}$ *with respect to* $\overrightarrow{B}$
ii) $\overrightarrow{A}$ *is forcing definable on* $\mathfrak{A}$ *with respect to* $\overrightarrow{B}$
iii) $\overrightarrow{A}$ *is formally definable on* $\mathfrak{A}$ *with respect to* $\overrightarrow{B}$

Proof. i) → ii) is Theorem 7.
ii) → iii) is Theorem 8.
iii) → i) is Theorem 9. □

4 Relatively intrinsic sequence on a sequence of structures

4.1 Forcing definability

We are given a relational language $\mathfrak{L} = (T_1, ..., T_k)$, a list of interpretations (i.e. structures) $\mathfrak{A}_0 = (\mathbb{N}, R_1^0, ..., R_k^0), \mathfrak{A}_1 = (\mathbb{N}, R_1^1, ..., R_k^1), ...$ where $\mathbb{N}$ is the set of natural numbers, $=$ and $\neq$ are present among the predicates. We are also given two sequences of subsets of $\mathbb{N}$, i.e. $\overrightarrow{A}$ and $\overrightarrow{B}$. Here we assume that there is a computable function $\lambda_{x,y}.xy$ that gives the arity of the $y-th$ predicate in the $x-th$ structure.
Remark: We call the total surjective function f is enumeration of $\overrightarrow{\mathfrak{A}}$ if f is enumeration of every single structure.

Definition 21. *We will say that the sequence* $\overrightarrow{A}$ *is relatively intrinsic on* $\overrightarrow{\mathfrak{A}}$ *with respect to the sequence* $\overrightarrow{B}$ *if for every enumertion* f *of* $\overrightarrow{\mathfrak{A}}$*, such that* $f^{-1}(\overrightarrow{B}) \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})$ *then the sequence* $f^{-1}(\overrightarrow{A})$ *is* $\omega-$*enumeration reducible to* $f^{-1}(\overrightarrow{\mathfrak{A}})$.

Definition 22. *We call an enumeration* f *of* $\overrightarrow{\mathfrak{A}}$ *acceptable if* $f^{-1}(\overrightarrow{B}) \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})$.

We modify the definition of P^f in the following

Definition 23. *Given an enumeration f of $\overrightarrow{\mathfrak{A}}$ denote by $P^f = \{P^f_n\}_{n<\omega}$ the respective jump sequence of the sequence $\{f^{-1}(\mathfrak{A}_0) \oplus f^{-1}(B_0), f^{-1}(\mathfrak{A}_1) \oplus f^{-1}(B_1), ...\}$ where*

$$P^f_n = P_n(\{f^{-1}(\mathfrak{A}_0) \oplus f^{-1}(B_0), f^{-1}(\mathfrak{A}_1) \oplus f^{-1}(B_1), ...\}).$$

Definition 24. *Let f be an enumeration on $\overrightarrow{\mathfrak{A}}$. For every $n, x, e \in \mathbb{N}$, we define the relations $f \models_n F_e(x)$ and $f \models_n \neg F_e(x)$ as follows:*

i) $f \models_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \wedge (\forall u \in D_v))$ either
a) $u = \langle 0, \langle 0, i, x^u_1, ..., x^u_{r_i}\rangle\rangle \wedge (f(x^u_1), ..., f(x^u_{r_i})) \in R^0_i$ or
b) $u = \langle 2, x_u\rangle \wedge f(x_u) \in B_0$

ii) $f \models_{n+1} F_e(x)$ iff $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
$((u = \langle 0, e_u, x_u\rangle \wedge f \models_n F_{e_u}(x_u)) \vee$
$(u = \langle 1, e_u, x_u\rangle \wedge f \models_n \neg F_{e_u}(x_u)) \vee$
$(u = \langle 2, \langle 0, \langle n+1, i, x^u_1, ..., x^u_{r_i}\rangle\rangle\rangle \wedge (f(x^u_1), ..., f(x^u_{r_i})) \in R^{n+1}_i) \vee$
$(u = \langle 2, \langle 2, x_u\rangle\rangle \wedge f(x_u) \in B_{n+1}))]$

iii) $f \models_n \neg F_e(x)$ iff $f \not\models_n F_e(x)$

Lemma 10. *i) Let $C \subset \mathbb{N}$, $n \in \mathbb{N}$. Then $C \leq_e P^f_n$ iff there is an index $e \in \mathbb{N}$ such that $C = \{x : f \models_n F_e(x)\}$*
ii) Let $\overrightarrow{C}$ be a sequence of sets. Then $\overrightarrow{C} \leq_\omega P^f$ iff there exists a computable function g, such that $C_n = \{x : f \models F_{g(n)}(x)\}$

Proof. i) The proof follows the same line as the proof of Lemma 3 i). We proceed by induction on n following the definition of the modelling relation. We have an extra case in the induction step corresponding to the new coded structure.
ii) The proof is the same as the proof of Lemma 3 ii). □

Definition 25. *For each $e, x, n \in \mathbb{N}$ and for every finite part τ, define the forcing relations $\tau \Vdash_n F_e(x)$ and $\tau \Vdash_n \neg F_e(x)$ following the definition of the relation "$\models$".*

i) $\tau \Vdash_0 F_e(x)$ *iff* $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
a) $u = \langle 0, \langle 0, i, x_1^u, ..., x_{r_i}^u\rangle\rangle,$

$$x_1^u, ..., x_{r_i}^u \in dom(\tau) \text{ and } (\tau(x_1^u), ..., \tau(x_{r_i}^u)) \in R_i^0$$

or
b) $u = \langle 2, x_u\rangle \wedge x_u \in dom(\tau) \wedge \tau(x_u) \in B_0$

ii) $\tau \Vdash_{n+1} F_e(x)$ *iff* $(\exists v)[\langle v, x\rangle \in W_e \wedge (\forall u \in D_v)$
$((u = \langle 0, e_u, x_u\rangle \wedge \tau \Vdash_n F_{e_u}(x_u)) \vee$
$(u = \langle 1, e_u, x_u\rangle \wedge \tau \Vdash_n \neg F_{e_u}(x_u)) \vee$
$(u = \langle 2, \langle 0, \langle n+1, i, x_1^u, ..., x_{r_i}^u\rangle\rangle\rangle,$

$$x_1^u, ..., x_{r_i}^u \in dom(\tau) \text{ and } (\tau(x_1^u), ..., \tau(x_{r_i}^u)) \in R_i^{n+1}) \vee$$

$(u = \langle 2, \langle 2, x_u\rangle\rangle \wedge \tau(x_u) \in B_{n+1}))]$

iii) $\tau \Vdash_n \neg F_e(x)$ *iff* $(\forall \rho \supseteq \tau)[\rho \nVdash_n F_e(x)]$

Definition 26. *Let f be an enumeration of* $\overrightarrow{\mathfrak{A}}$. *We say that f is k-generic with respect to* $\overrightarrow{B}$ *if for every* $j < k$ *and* $e, x \in \mathbb{N}$*:*

$$(\exists \tau \subseteq f)(\tau \Vdash_j F_e(x) \vee \tau \Vdash_j \neg F_e(x))$$

Lemma 11. *i) If* $\tau \subseteq \rho$ *then* $\tau \Vdash_k (\neg)F_e(x)$ *implies* $\rho \Vdash_k (\neg)F_e(x)$
ii) For every $(k+1)-$*generic enumeration f of A,* $f \models_k (\neg)F_e(x)$ *iff*
$(\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x))$

Proof. i) The proof is analogous to the proof of Lemma 4 i). In the induction hypothesis we get the extra case $u = \langle 2, \langle 0, \langle n+1, i, x_1^u, ..., x_{r_i}^u\rangle\rangle\rangle \wedge x_1^u, ..., x_{r_i}^u \in dom(\tau) \wedge (\tau(x_1^u), ..., \tau(x_{r_i}^u)) \in R_i^{n+1}$. Since $\tau \subseteq \rho$, this case will be true for ρ
ii) The proof is analogous to the proof of Lemma 4 ii). In the induction hypothesis for the positive case in $(\rightarrow)$, we will chose finite parts τ_u that are also defined for the elements of the domain of f such that $(f(x_1^u), ..., f(x_{r_i}^u)) \in R_i^{n+1}$ and then we proceed as in Lemma 4. □

Lemma 12. *f is an acceptable enumeration on* $\overrightarrow{\mathfrak{A}}$ *with respect to* $\overrightarrow{B}$ *iff* $P^f \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})$.

Proof. ($\rightarrow$) Assume $f^{-1}(B_n) \leq_e P_n(f^{-1}(\overrightarrow{\mathfrak{A}}))$ uniformly in n i .e. there is a computable function h such that $f^{-1}(B_n) = W_{h(n)}(P_n(f^{-1}(\overrightarrow{\mathfrak{A}})))$. We will prove the statement by induction on n.

1. Let $n = 0$. We have $f^{-1}(B_0) = W_{e'_0}(f^{-1}(\mathfrak{A}_0))$, where the $h(0) = e'_0$. Since we can effectively obtain an index e_0 such that $f^{-1}(\mathfrak{A}_0) = W_{e_0}(f_{-1}(\mathfrak{A}_0))$, we can obtain effectively an index i_0 from e_0 and e'_0 such that $f^{-1}(\mathfrak{A}_0) \oplus f^{-1}(B_0) = W_{i_0}(f^{-1}(\mathfrak{A}_0))$.

2. Assume the statement is true for n and we will prove it for $n+1$. We have $P^f_{n+1} = (P^f_n)' \oplus (f^{-1}(\mathfrak{A}_{n+1}) \oplus f^{-1}(B_{n+1}))$. By induction hypothesis $P^f_n = W_{i_n}(P_n(f^{-1}(\overrightarrow{\mathfrak{A}})))$, where i_n is effectively obtained. By assumption, we have $f^{-1}(B_{n+1}) = W_{e'_{n+1}}(P_{n+1}(f^{-1}(\overrightarrow{\mathfrak{A}})))$, where $h(n) = e'_{n+1}$. By the properties of the enumeration jump, we can effectively obtain from i_n an index e_{n+1} such that $(P^f_n)' = W_{e_{n+1}}((P_n(f^{-1}(\overrightarrow{\mathfrak{A}})))')$. Of course, we can effectively obtain an index e''_{n+1} such that $f^{-1}(\mathfrak{A}_{n+1}) = W_{e''_{n+1}}(f^{-1}(\mathfrak{A}_{n+1}))$. Putting everything together, we can effectively obtain an index i_{n+1} such that $P^f_{n+1} = W_{i_{n+1}}(P_{n+1}(f^{-1}(\overrightarrow{\mathfrak{A}})))$.
($\leftarrow$) Let $P^f \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})$. Hence we get $P^f_n \leq_e P_n(f^{-1}(\overrightarrow{\mathfrak{A}}))$ uniformly in n i.e. $P^f_n = W_{h(n)}(P_n(f^{-1}(\overrightarrow{\mathfrak{A}}))$ for a computable function h. Let $n > 0$ and $h(n) = e_n$. By definition $P^f_n = (P^f_{n-1})' \oplus ((f^{-1}(\mathfrak{A}_n) \oplus f^{-1}(B_n))$. Hence from an index e_n such that $P^f_n = W_{e_n}(P_n(f^{-1}(\overrightarrow{\mathfrak{A}})))$, we can effectively find an index i_n such that $f^{-1}(B_n) = W_{i_n}(P_n(f^{-1}(\overrightarrow{\mathfrak{A}})))$.

□

Definition 27. *We say that the sequence $\overrightarrow{A}$ is forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to the sequence $\overrightarrow{B}$ if there exists a finite part δ, and a computable function g, $x \in \mathbb{N}$, such that for every n of $\mathbb{N}$:*

$$s \in A_n \textit{ iff } (\exists \tau \supseteq \delta)(\tau(x) = s \wedge \tau \Vdash_n F_{g(n)}(x)).$$

An analogous

Theorem 11. *Let $\overrightarrow{A}$ be not forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. Then there exists an enumeration f of $\overrightarrow{\mathfrak{A}}$, s.t. $f^{-1}(\overrightarrow{A}) \not\leq_\omega P^f$.*

Proof. The proof is the same as the proof of Theorem 5. We proceed with constructing an enumeration f build up by finite parts δ_q such that $\delta_q \subseteq \delta_{q+1}$ and $f = \bigcup_q \delta_q$. On stages $q = 3r$ we make sure the enumeration is surjective and total, on stages $q = 3r + 1$ we assure f is $k - generic$ for each $k > 0$, and on stages $q = 3r + 2$ we assure f meets the omitting condition. □

Again we can derive the following countable generalization:

Lemma 13. *Let $\overrightarrow{A}_0, \overrightarrow{A}_1, \ldots$ be a sequence of sequences of sets, s.t. each $\overrightarrow{A}_i$ is not forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. Then there exists an enumeration f of $\overrightarrow{\mathfrak{A}}$, s.t. $f^{-1}(\overrightarrow{A}_i) \not\leq_u P^f$ for each i.*

Proof. The same as the case for one structure. □

Theorem 12. *Let $\overrightarrow{A}$ be a sequence of sets not forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. Then there exists an acceptable enumeration g, such that $g^{-1}(\overrightarrow{A}) \not\leq_\omega P^g$ and the enumeration degree of $g^{-1}(\overrightarrow{\mathfrak{A}})$ is total. (The enumeration degree of $g^{-1}(\mathfrak{A}_n)$ is total for each n.)*

Proof. Let $\overrightarrow{A}$ be not forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. From Theorem 11, we find an enumeration f such that $f^{-1}(\overrightarrow{A}) \not\leq_\omega P^f$. Hence there is a total set F such that $P^f \leq_\omega \{F^{(n)}\}_{n<\omega}$ and $f^{-1}(\overrightarrow{A}) \not\leq_\omega \{F^{(n)}\}_{n<\omega}$. From $P^f \leq_\omega \{F^{(n)}\}_{n<\omega}$ we conclude that $f^{-1}(\mathfrak{A}_n) \leq_e F^{(n)}$ uniformly in n and $f^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n.
Fix two natural numbers, say s, t such that $s \neq t$ and natural numbers x_s and x_t s.t. $f(x_s) = s, f(x_t) = t$. We define a function g as follows:

$$g(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z + 1 \text{ and } z \in F, \\ t & \text{if } x = 2z + 1 \text{ and } z \notin F. \end{cases}$$

Thus defined, g is an enumeration of $\overrightarrow{\mathfrak{A}}$. We want to prove $g^{-1}(\overrightarrow{\mathfrak{A}}) \equiv_\omega \{F^{(n)}\}_{n<\omega}$.

i) We have that $g^{-1}(\overrightarrow{\mathfrak{A}}) \leq_\omega \{F^{(n)}\}_{n<\omega} \Leftrightarrow g^{-1}(\mathfrak{A}_n) \leq_e P_n\{F^{(n)}\}$ uniformly in n. (By Lemma 1)
By induction on n and using the definitions of enumeration jump and the fact that F is a total set we can prove that $P_n(\{F^{(n)}\}) \equiv_e F^{(n)}$.
Let's fix a predicate R_i^j of the structure $\mathfrak{A}_n$. Let $x_1, \ldots, x_{r_i}$ be arbitrary natural numbers. We will define natural numbers $y_1, \ldots, y_{r_i}$. Let $1 \leq j \leq r_i$.

a) x_j is even. Then let $y_j = x_j/2$.
b) $x_j = 2z + 1$ and $z \in F$. Then let $y_j = x_s$.
c) $x_j = 2z + 1$ and $z \notin F$. Then let $y_j = x_t$.
We have the following equivalence :

$$\langle x_1, ..., x_{r_i} \rangle \in g^{-1}(R_i^j) \Leftrightarrow \langle y_1, ..., y_{r_i} \rangle \in f^{-1}(R_i^j).$$

From $f^{-1}(\mathfrak{A}_n) \leq_e F^{(n)}$ uniformly in n and the definition of a copy of a structure we have $f^{-1}(R_i^n) \leq_e F^{(n)}$ uniformly in n and hence $g^{-1}(R_i^n) \leq_e F^{(n)}$ uniformly in n. Since this is true for all predicates in the structure, we have that $g^{-1}(\mathfrak{A}_n) \leq_e F^{(n)}$ uniformly in n. Hence by Lemma 1, $g^{-1}(\overrightarrow{\mathfrak{A}}) \leq_\omega \{F^{(n)}\}_{n<\omega}$.

ii) Every structure in the list contains $=, \neq$. Without loss of generality, let's take the structure $\mathfrak{A}_0$. We have the following equivalences:

$$z \in F \Leftrightarrow 2z + 1 \in g^{-1}(s) \Leftrightarrow g(2z + 1) = s$$
$$z \notin F \Leftrightarrow 2z + 1 \in g^{-1}(t) \Leftrightarrow g(2z + 1) = t$$

Hence $F \leq_e g^{-1}(\mathfrak{A}_0)$ (By the same reasoning and proof as in Theorem 6). By a property of $\leq_e$, we have that $F' \leq_e g^{-1}(\mathfrak{A}_0)'$. Again by the properties of $\leq_e$ we obtain $F^{(n)} \leq_e g^{-1}(\mathfrak{A}_0)^{(n)}$ uniformly in n i.e. $F^{(n)} = W_{h(n)}(g^{-1}(\mathfrak{A}_0)^{(n)})$ via the computable function h. By the properties of the jump sequence, we have $P_m(g^{-1}(\overrightarrow{\mathfrak{A}})) \leq_e P_n(g^{-1}(\overrightarrow{\mathfrak{A}}))$ uniformly in n and m and hence $P_0(g^{-1}(\overrightarrow{\mathfrak{A}})) \leq_e P_n(g^{-1}(\overrightarrow{\mathfrak{A}}))$ uniformly in n. From here we get $g^{-1}(\mathfrak{A}_0)^{(n)} = W_{l(n)}(P_n(g^{-1}(\overrightarrow{\mathfrak{A}})))$ via the computable function l. Using the computable functions h and l we can conclude that $F^{(n)} \leq_e P_n(g^{-1}(\overrightarrow{\mathfrak{A}}))$ uniformly in n.

Combining i) and ii), we have $g^{-1}(\overrightarrow{\mathfrak{A}}) \equiv_\omega \{F^{(n)}\}_{n<\omega}$.

Denote E_g, E_f to be the sets $E_g = g^{-1}(=), E_f = f^{-1}(=)$ (It doesn't matter from which structure we take $=, \neq$, since they are the same sets.). From $f^{-1}(\overrightarrow{A}) \not\leq_\omega \{F^{(n)}\}_{n<\omega}$ we conclude

$$E_f \leq_e F \Longrightarrow E_g \leq_e F \Longrightarrow E_g \leq_e F^{(n)} \text{ uniformly in n.}$$

Fix n. We have:

$$g^{-1}(B_n) = \{x : (\exists y \in f^{-1}(B_n))(\langle x, 2y \rangle \in E_g)\}.$$

Hence $g^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n i.e. $g^{-1}(\overrightarrow{B}) \leq_\omega g^{-1}(\overrightarrow{\mathfrak{A}})$. Thus, we have proved that g is an acceptable enumeration.
To prove the ommitting condition, assume the opposite i.e. $g^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$. We have

$$g^{-1}(A_n) = \{x : 2x \in f^{-1}(A_n)\}.$$

Hence $f^{-1}(\overrightarrow{A}) \leq_\omega g^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$. By transitivity of $\leq_\omega$, we have $f^{-1}(\overrightarrow{A}) \leq_\omega \{F^{(n)}\}_{n<\omega}$. Contradiction. □

Theorem 13. *For every sequence $\overrightarrow{A}$, if*

$$(\forall f)[f^{-1}(\overrightarrow{B}) \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}}) \Longrightarrow f^{-1}(\overrightarrow{A}) \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})]$$

then $\overrightarrow{A}$ is forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$.

Proof. Assume that $\overrightarrow{A}$ is not forcing definable. By the previous theorem, we find an acceptable enumeration g, s.t. $g^{-1}(\overrightarrow{A}) \not\leq_\omega P^g$. Contradiction. □

4.2 Formal definability

Again we are given a relational language $\mathfrak{L} = (T_1, T_2, ..., T_k)$. The predicates $=, \neq$ are present. In order to prove that every forcing definable sequence is formally definable, we will use the formulas we introduced in the previous section with a slight difference. On each level of the elementary Σ_n^+ formulas, we will add all the predicates of the $\mathfrak{A}_n$ structure. Again we assume that we have a computable sequence $\{P_n\}_{n<\omega}$ of predicates that represents the sequence $\overrightarrow{B}$ i.e. $P_n(X)$ is true if $X \in B_n$. We will make the following abbreviation:

$$\mathfrak{L}_0 = \{T_1^0, ..., T_k^0\},$$
$$\mathfrak{L}_{n+1} = \mathfrak{L}_n \cup \{T_1^{n+1}, ..., T_k^{n+1}\}.$$

Definition 28. *1. An elementary Σ_0^+ formula with free variables among $W_1, ..., W_r$ is an existentional formula of the form*

$$\exists Y_1 ... \exists Y_m \Phi(W_1, ..., W_r, Y_1, ..., Y_m),$$

where Φ is a finite conjunction of atomic formulas in $\mathfrak{L}_0 \cup \{P_0\}$.

2. *A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulas.*

3. *An elementary Σ_{n+1}^+ formula is a formula of the form*

$$\exists Y_1 ... \exists Y_m \Phi(W_1, ..., W_r, Y_1, ..., Y_m),$$

where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ or atoms from $\{T_1^{n+1}, ..., T_k^{n+1}\}$ or Σ_n^+ formulas or negations of Σ_n^+ formulas in the language $\mathfrak{L}_{n+1} \cup \{P_0\} \cup ... \cup \{P_{n+1}\}$.

With a slight modification we arrive at the following:

Definition 29. *Let $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{\mathfrak{A}}$ be given. We say that $\overrightarrow{A}$ is formally definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$, if there is a computable sequence $\{\Phi^{\gamma(n)}\}_{n<\omega}$ of formulas, such that for every n, $\Phi^{\gamma(n)}$ is a Σ_n^+ formula with free variables among $W_1, ..., W_r$ and elements $t_1, ..., t_r \in \mathbb{N}$, such that for every $x \in \mathbb{N}$:*

$$x \in A_n \Leftrightarrow (\overrightarrow{\mathfrak{A}}, \overrightarrow{B}) \models \Phi^{\gamma(n)}(W_1/t_1, ..., W_r/t_r, X/x).$$

Lemma 14. *Let $D = \{w_1, ..., w_r\}$ be a finite non-empty set of natural numbers and let x,e be elements of $\mathbb{N}$. There exists an uniform recursive way to construct a Σ_n^+ formula $\Phi_{D,e,x}^n$ with free variables among $W_1, ..., W_r$ such that for every finite part δ such that $dom(\delta) = D$, the following equivalences are true:*

$$(\overrightarrow{\mathfrak{A}}, \overrightarrow{B}) \models \Phi_{D,e,x}^n(W_1/\delta(w_1), ..., W_r/\delta(w_r)) \Leftrightarrow \delta \Vdash_n F_e(x)$$

$$(\overrightarrow{\mathfrak{A}}, \overrightarrow{B}) \models \Psi_{D,e,x}^n(W_1/\delta(w_1), ..., W_r/\delta(w_r)) \Leftrightarrow \delta \Vdash_n \neg F_e(x)$$

Proof. We shall construct the formula $\Phi_{D,e,x}^n$ by recursion on n, following the definition of forcing.

1. Let $n = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element $v \in V$. For every $u \in D_v$ define an atom Π_u as follows:
   a) $u = \langle 0, \langle 0, i, x_1^u, ..., x_{r_i}^u\rangle\rangle$, where $1 \leq i \leq k$ and all $x_1^u, ..., x_{r_i}^u$ are elements of D. Then let $\Pi_u = T_i^0(X_1^u, ..., X_{r_i}^u)$.
   b) $u = \langle 2, x_u\rangle$ and $x_u \in D$. Then let $\Pi_u = P_0(X_u)$
   c) $\Pi_u = W_1 \neq W_1$ in all other cases.
   Let $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi_{D,e,x}^0 = \bigvee_{v \in V} \Pi_v$.
   Let $\Psi_{D,e,x}^0 = \neg[\bigvee_{D^* \supseteq D}(\exists \overrightarrow{Y} \in D^* \setminus D)\Phi_{D^*,e,x}^0]$.

2. Assume it is done up till n and we will prove for $n+1$. Let again $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$. For every $u \in D_v$ define a formula Π_u as follows:
a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi^n_{D,e_u,x_u}$.
b) If $u = \langle 1, e_u, x_u\rangle$, then let $\Pi_u = \Psi^n_{D,e_u,x_u}$.
c) If $u = \langle 2, \langle 0, \langle n+1, i, x^u_1, ..., x^u_{r_i}\rangle\rangle\rangle$ where $1 \leq i \leq k$ and all $x^u_1, ..., x^u_{r_i}$ are elements of D. Then let $\Pi_u = T^{n+1}_i(X^u_1, ..., X^u_{r_i})$.
d) If $u = \langle 2, \langle 2, x_u\rangle\rangle$, then let $\Pi_u = P_{n+1}(X_u)$
e) $\Pi_u = W_1 \neq W_1$ in all other cases.
Now let $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^{n+1}_{D,e,x} = \bigvee_{v\in V} \Pi_v$.
Let $\Psi^{n+1}_{D,e,x} = \neg[\bigvee_{D^*\supseteq D}(\exists \overrightarrow{Y} \in D^* \setminus D)\Phi^{n+1}_{D^*,e,x}]$.

The proof that the statement in the lemma is acomplished is analogous to the similiar lemma in chapter 3. We proceed by induction on n. The base case remains the same and in the induction step we will have extra atomic formulas from the predicates of $\mathfrak{A}_{n+1}$. Again we note that there is a computable way to recover the index of the formula. □

We tie the forcing definability to the formal definability in the following:

Theorem 14. *Let the sequence $\overrightarrow{A}$ be forcing definable. Then $\overrightarrow{A}$ is formally definable.*

Proof. The proof of this theorem is the same as the proof of the analogous theorem in section 3. □

With a slight modification of the proof of Lemma 7, we get the following useful

Lemma 15. *Let g be an arbitrary enumeration of $\overrightarrow{\mathfrak{A}}$. There exists a bijective enumeration f of $\overrightarrow{\mathfrak{A}}$, such that $f^{-1}(\mathfrak{A}_n) \leq_e g^{-1}(\mathfrak{A}_n)$ for every n.*

A modification to Lemma 8 with the concepts of chapter 4 gives us the following

Lemma 16. *Let g be an arbitrary enumeration of $\overrightarrow{\mathfrak{A}}$. Then there exists a bijective enumeration f such that $P^f \leq_\omega P^g$.*

By $P^{D_{id}}$ we denote the jump sequence of the following sequence

$$\{D(\mathfrak{A}_0) \oplus B_0, D(\mathfrak{A}_1) \oplus B_1, ...\}$$

Lemma 17. *Let Φ be a Σ_n^+ formula. We can effectively find, from the code of the formula Φ an enumeration operator W_{e_n}, such that for arbitrary natural numbers $\overrightarrow{t}$, we have*

$$(\overrightarrow{\mathfrak{A}}, \overrightarrow{B}) \models \Phi(\overrightarrow{W}/\overrightarrow{t}) \Leftrightarrow \langle \overrightarrow{t} \rangle \in W_{e_n}(P_n^{D_{id}}).$$

Proof. The proof follows the same line as the proof of the analogous lemma in section 3. We proceed by induction on n. The base case remains the same and in the induction step, we have the predicates of the structure $\mathfrak{A}_{n+1}$ which occur in the elementary Σ_{n+1}^+ formulas. They are treated the same way as in the base case $n = 0$. □

Theorem 15. *Let $\overrightarrow{A}$ be formally definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. Then for every acceptable enumeration f, we have that $f^{-1}(\overrightarrow{A}) \leq_\omega f^{-1}(\overrightarrow{\mathfrak{A}})$.*

Proof. Let $\overrightarrow{A}$ be formally definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$. Suppose that there is an acceptable enumeration g for which $g^{-1}(\overrightarrow{A}) \not\leq_\omega g^{-1}(\overrightarrow{\mathfrak{A}})$. There exists a bijective enumeration f, such that $f^{-1}(\mathfrak{A}_n) \leq_e g^{-1}(\mathfrak{A}_n)$ for every n and $P^f \leq_\omega P^g$.
Let $\overrightarrow{\mathfrak{B}}$ be the sequence of structures that that is build up by f i.e $\mathfrak{B}_0 = (\mathbb{N}, f^{-1}(R_1^0), ..., f^{-1}(R_k^0))$ etc. We have $\mathfrak{A}_n \cong \mathfrak{B}_n$ and $f^{-1}(\overrightarrow{\mathfrak{A}}) \equiv_\omega D(\overrightarrow{\mathfrak{B}})$. As in Theorem 9, we see that $f^{-1}(\overrightarrow{A})$ is formally definable in $\overrightarrow{\mathfrak{B}}$ and hence $f^{-1}(\overrightarrow{A}) \leq_\omega P^f$. Reasoning as in Theorem 9 , it follows that $g^{-1}(\overrightarrow{A}) \leq_\omega P^g$ with g acceptable enumeration. Contradiction. □

Theorem 16. *The following statements are equivalent:*
i) $\overrightarrow{A}$ is relatively intrinsic on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$
ii) $\overrightarrow{A}$ is forcing definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$
iii) $\overrightarrow{A}$ is formally definable on $\overrightarrow{\mathfrak{A}}$ with respect to $\overrightarrow{B}$

Proof. i) $\rightarrow$ ii) is Theorem 13
ii) $\rightarrow$ iii) is Theorem 14
iii) $\rightarrow$ i) is Theorem 15 □

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