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Master Thesis

Logics of n-ary Contact

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Abstract

For an arbitrary connected topological space consider the set of the regular closed sets. With appropriately defined Boolean operations this set forms a (complete) Boolean algebra.

In the case of the m-dimensional Euclidean space we have a special subset of the regular closed sets called polytopes. One of their significant properties is the finite characteristic that they can be represented as a finite union of other polytopes. In addition to that they form a subalgebra of the Boolean algebra of the regular closed sets.

Let us say that n sets are in contact if their set theoretical intersection is non-empty.

The objective of this research is within an appropriate formal system to axiomatise the contact properties in the aforementioned sense of the Boolean algebras of the *polytopes* of the m-dimensional space and the *regular closed* sets of the *connected* topological spaces.

1 Introduction. Formal language and notions

1.1 Introductory notes.

Let us say that n sets of an arbitrary topological space are in *n-ary contact* if their set theoretical intersection is non-empty. Now, consider the *connected* topological spaces. We would like to study the properties of the *n-ary contact* with respect to the *regular closed* sets of such topological spaces.

An approach to this is being examined in [2] by Dimitar Vakarelov by the means of the so called *sequent algebras*. Nevertheless, for this purpose is being used a formal language, which in its essence is of a second-order logic.

As per [1] Philippe Balbiani and Tinko Tinchev study the general notion of the so called "*Boolean logics with relations*". They use a quantifier-free fragment of a first-order logic language to express Boolean notions and their relation properties. Convenient semantic structures are being introduced as well.

In this research we adopt the language and the semantic structures of [1] with an intended interpretation of the relation symbols being the *n-ary contact*. We aim to axiomatise the valid formulas of the language with respect to that semantics in the topological context of the Boolean subalgebras of the *regular closed* sets of some *connected* topological space.

To approach this problem as a representative of a *connected* topological space is used the m-dimensional Euclidean space. It is devised an appropriate axiomatisation of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$. Also, it is considered a special subset of the *regular closed* sets of $\mathbb{R}^m$, namely the *polytopes*. One can think of them in the case of $\mathbb{R}^2$ as the set of the *finite* unions of convex polygons or their (*closed*) complement (with respect to the whole real plane). The *polytopes* form a subalgebra of the Boolean algebra of the *regular closed* sets of the particular m-dimensional topological space. Such an algebra is also axiomatised appropriately. Moreover, the main result of this study shows that the logic of the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ for $m \geq 2$ is the same as the logic of the Boolean algebra of the *regular closed* sets of any *connected* topological space. The last fact trivially is because the axiomatisations of those Boolean algebras are the same.

For the Boolean algebra of the *polytopes* of $\mathbb{R}^1$ is demonstrated to have an axiomatisation that is different than the one of the *polytopes* of $\mathbb{R}^m$ for $m \geq 2$. The difference is the property that distinguishes the *polytopes* of $\mathbb{R}^1$ from the *regular closed* sets of $\mathbb{R}^1$.

To attain the completeness of the corresponding axiomatisations with respect to the aforementioned intended Boolean algebras substantially a common approach is used.
First, the correctness with respect to any Boolean subalgebra of the *regular closed* sets of a *connected* topological space is easy.
With respect to completeness the following steps are made. As a remark, when we talk about satisfiability of a formula in a Boolean algebra we mean the appropriate first-order language semantic structure with a carrier the intended in the particular context Boolean algebra having interpretation of the relation symbols the *n-ary contact*.

- For a formula not deducible in the formal system by appropriate ("external" with respect to the exposition of this research) means is being obtained a finite relational structure (called a Kripke frame) in which the axioms are valid and the intended formula is refutable.

- The relational structure appears of a kind for which a relevant connected graph representation is being associated to. By a particular sequence of modifications that graph is then transformed into an acyclic connected one. Significant is that the associated Kripke structure to the resulting graph is a p-morphic preimage of the originating structure.

- Moreover, it is being elaborated on an appropriate procedure for any of the intended algebras that applied on the acyclic connected graph eventually produces a Kripke structure with particular properties that is isomorphic to the associated to the acyclic graph one.
So far, this means in the resulting Kripke structure the intended formula is refutable.

- With regard to that Kripke structure its carrier has elements of the intended Boolean algebra the structure being obtained for. Furthermore, the set-theoretical Boolean algebra generated by this structure appears isomorphic to the Boolean algebra generated by the elements of its carrier. It follows that the intended formula is refutable in the generated by the carrier Boolean algebra.

- On the other hand, the relations are interpreted as the *n-ary contact*. As a result, the generated by the carrier of the Kripke structure Boolean algebra is a subalgebra of the intended one. Then the formula cannot be valid in the intended Boolean algebra.

These results are developed within the exposition in the following way.

Further in this Section 1 are introduced the necessary notions and adopted results needed throughout the study. It is being clarified the formal language (Section 1.2), the adopted notions and results from [1] (Section 1.3), the notions with respect to graphs (Section 1.4). In Section 1.5 the very basic definitions and results regarding topological spaces needed later are being developed. In Section 1.6 the definitions and properties of *regular closed* sets and *polytopes* and their corresponding Boolean algebras are being highlighted. Finally (Section 1.7), the *n-ary contact* relation is being formally defined.

Section 2 introduces the auxiliary notions of a *contact n-frame* (Section 2.1) and an *n-graph* or *contact n-graph* (Section 2.2). The *contact n-frames* are in some sense generalised relational structures, namely Kripke frames, which impose on the interpretation of the relations properties analogous to those of the *n-ary contact* relation. In short, they are used as the linkage between a general finite Boolean algebra satisfying certain set of axioms and a finite Boolean algebra of particular elements (*regular closed* sets or *polytopes* of $\mathbb{R}^m$) interpreting the relations as the *n-ary contact*. This is achieved by the essential property of the finite *contact n-frames* that is their unique one-to-one correspondence with a special class of graphs, namely the *contact n-graphs* (Section 2.3). This result is established by the pair of Claim 2.3.4 and Claim 2.3.5. Having the aforementioned correspondence, they are demonstrated the connectedness property and a specific condition on the ternary contact of a *contact n-frame* to have an intuitive meaning on the corresponding graph structures (Section 2.4).

Let us consider a finite acyclic contact n-graph and its corresponding contact n-frame. Section 3 elaborates on procedures for obtaining a Kripke frame with interpretation of the relations the *n-ary contact* and specific properties of the carrier that is isomorphic to the given contact n-frame. In short, the properties of the carrier are such that its elements are the atoms of a finite Boolean algebra subalgebra of a particular Boolean algebra of *polytopes* or *regular closed* sets. Section 3.1 treats the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ for $m \geq 2$ and Section 3.2 is for the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ for $m \geq 1$. Section 3.3 within particular conditions for the given acyclic contact n-graph (effectively, those obtained in Section 2.4) demonstrates an approach for the *polytopes* of $\mathbb{R}^1$.

Now, consider an arbitrary *connected* contact n-graph. Section 4 shows a procedure for transforming that contact n-graph into a *connected acyclic* one whose corresponding *contact n-frame* is a *p-morphic preimage* of the corresponding to the originating graph contact n-frame (Claim 4.2.1). Furthermore, they are highlighted the properties of the originating graph being preserved in the resulting one.

The relatively short Section 5 in advance lists the axioms which will be used later in Section 7 to define the Boolean logic of n-ary contact and the appropriate axiomatisation of the considered classes of Boolean algebras of *regular closed* sets. The purpose of Section 5 is to study the implications of the validity of those axioms with respect to the *contact n-frames.* The auxiliary result to be used later is Proposition 5.2.3.

Section 6 considers the semantic structures for representing the Boolean algebras, namely the *Boolean frames*, and demonstrates few results about finite Boolean algebras of *polytopes* and *regular closed* sets being of essential importance later in the exposition The latter is treated in Section 6.2. Furthermore (Section 6.3), they are being demonstrated essential correspondences between the finite Boolean algebras (Boolean frames) and the finite relational structures (Kripke frames). One can check Claim 6.3.1 and Claim 6.3.2.

Section 7 is where the Boolean logic of n-ary contact is formally defined and examined. Section 7.1 introduces the intended semantic structures and defines the relevant formal systems of the axiomatisations to be studied. Section 7.2 demonstrates the correctness of the formal systems. Section 7.3 is where the completeness of the formal systems is being proven as per the aforementioned steps. This is where all the former results find their appropriate use. The key achievements are Proposition 7.3.1, Proposition 7.3.2 and Proposition 7.3.3. Section 7.4 summarises and recaps the aimed outcome of this study. A good illustration are Corollary 7.4.2 and Corollary 7.4.5.

1.2 Formal language

In essence *n-ary contact logic* adopts a reduction of the language of the Boolean logic as introduced in [1], Section 2, *"Syntax"*. That is we have exactly one n-ary relation symbol for each natural $n \geq 1$. A more refined definition is provided as follows.

Recall, for a language $L_{\mathcal{R}}$ of a *Boolean logic* we have countably infinite set $\mathcal{R}$ of relation symbols each being n-ary relation for some natural $n \geq 0$. To $L_{\mathcal{R}}$ we attribute the following logical symbols:

- Parentheses: '(', ')'
- Comma: ','
- Countably many Boolean variables: denoted by lower case Latin letters x, y and so.
- Boolean functions: '0', '$-$' and '$\cup$'
- Connectives: '$\perp$', '$\neg$' and '$\vee$'
- Binary relation symbol: '$\equiv$'

Eventually, we assume that no relation symbol of $\mathcal{R}$ occurs in the set of the logical symbols.

Definition. As a language for the *n-ary contact logic* we consider a Boolean language $L_{\mathcal{R}}$, where $\mathcal{R}$ *consists of* ***exactly one*** *n-ary relation symbol per each positive n.*

Again, ρ denotes the arity function mapping the relation symbols from $\mathcal{R}$ to appropriate natural numbers indicating the intended arity of the respective relation symbol. Hence, by definition of a language for an n-ary contact logic we imply ρ being injective.

Recall the inductive definition of a *term* of $L_{\mathcal{R}}$.

- A Boolean variable is a term.
- The Boolean function symbol 0 is a term.
- If τ is a term then also is $-\tau$.
- If τ_1 and τ_2 are terms then also is $(\tau_1 \cup \tau_2)$

Atomic formulas of $L_{\mathcal{R}}$:

- If P is an n-ary relation symbol and $\tau_1, \ldots, \tau_n$ are terms then $P(\tau_1, \ldots, \tau_n)$ is an atomic formula.
- If τ_1 and τ_2 are terms then $(\tau_1 \equiv \tau_2)$ is atomic formula.

Formulas of $L_{\mathcal{R}}$:

- An atomic formula is a formula.
- $\perp$ is a formula.
- If φ is a formula then also is $\neg\varphi$.
- If φ_1 and φ_2 are formulas then also is $(\varphi_1 \vee \varphi_2)$.

Recall also the abbreviations adopted:

- 1 denotes -0.
- $(\tau_1 \cap \tau_2)$ denotes $-(-\tau_1 \cup -\tau_2)$.
- $\top$ denotes $\neg\perp$

- $(\varphi_1 \wedge \varphi_2)$ denotes $\neg(\neg\varphi_1 \vee \neg\varphi_2)$
- $(\varphi_1 \implies \varphi_2)$ denotes $(\neg\varphi_1 \vee \varphi_2)$
- $(\varphi_1 \iff \varphi_2)$ denotes $((\varphi_1 \implies \varphi_2) \wedge (\varphi_2 \implies \varphi_1))$

For any set of formulas Δ by $BV(\Delta)$ we denote the set of Boolean variables occurring in Δ. Whenever $\Delta = \{\varphi\}$ we simply write $BV(\varphi)$. In a similar way for any term τ by $BV(\tau)$ we denote the set of Boolean variables occurring in τ. By $\varphi[x_1, \ldots, x_n]$ for formula φ we indicate that $BV(\varphi) \subseteq \{x_1, \ldots, x_n\}$.

1.3 Adopted Boolean logic notions

We recall some of the notions adopted from [1], "Boolean logics with relations". Furthermore, will summarise the basic formal understanding when dealing with graphs.

1.3.1 Kripke frames

A *Kripke frame* for $L_{\mathcal{R}}$ is a structure $\mathcal{F} = <S, I>$ where S is a non-empty set and I is an interpretation function mapping the relation symbols of $\mathcal{R}$ to appropriate relations on S. That is for arbitrary P of $\mathcal{R}$ then $I(P)$ is $\rho(P)$-ary relation on S. A *valuation* on $\mathcal{F}$ is function $\mathcal{V}$ mapping the Boolean variables to subsets of S. Recall the recursive extension $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ on the terms of $L_{\mathcal{R}}$:

- $\widetilde{\mathcal{V}}(x) = \mathcal{V}(x)$
- $\widetilde{\mathcal{V}}(0) = \emptyset$
- $\widetilde{\mathcal{V}}(-\tau) = S \setminus \widetilde{\mathcal{V}}(\tau)$
- $\widetilde{\mathcal{V}}(\tau_1 \cup \tau_2) = \widetilde{\mathcal{V}}(\tau_1) \cup \widetilde{\mathcal{V}}(\tau_2)$

A *Kripke model* for $L_{\mathcal{R}}$ is a structure $\mathcal{M} = <\mathcal{F}, \mathcal{V}>$ where $\mathcal{F} = <S, I>$ is a Kripke frame for $L_{\mathcal{R}}$ and $\mathcal{V}$ is a valuation on $\mathcal{F}$. Recall the inductive definition of a formula φ *true in a Kripke model* $\mathcal{M}$ denoted by $\mathcal{M} \Vdash \varphi$.

- $\mathcal{M} \Vdash P(\tau_1, \ldots, \tau_n)$ *iff* there exists $s_1 \in \widetilde{\mathcal{V}}(\tau_1)$, ..., there exists $s_n \in \widetilde{\mathcal{V}}(\tau_n)$ such that $<s_1, \ldots, s_n> \in I(P)$
- $\mathcal{M} \Vdash (\tau_1 \equiv \tau_2)$ *iff* $\widetilde{\mathcal{V}}(\tau_1) = \widetilde{\mathcal{V}}(\tau_2)$
- $\mathcal{M} \nVdash \bot$
- $\mathcal{M} \Vdash \neg\varphi$ *iff* $\mathcal{M} \nVdash \varphi$
- $\mathcal{M} \Vdash (\varphi_1 \vee \varphi_2)$ *iff* $\mathcal{M} \Vdash \varphi_1$ or $\mathcal{M} \Vdash \varphi_2$

Recall that a set of formulas Σ is called *satisfiable* in given Kripke frame should there be a Kripke model based on that frame (equivalently, there is a valuation on that frame) such that all the formulas in Σ are true in that model (respectively, in the model for the Kripke frame and the valuation). Σ is satisfiable in a class of Kripke frames if exists Kripke frame from that class such that Σ is satisfiable in it. A formula φ is *valid* in a Kripke frame $\mathcal{F}$ if φ is *true* in every Kripke model for the frame $\mathcal{F}$. We denote it $\mathcal{F} \Vdash \varphi$. A set of formulas Φ is valid in a Kripke frame $\mathcal{F}$ if every formula in Φ is valid in $\mathcal{F}$. We denote it $\mathcal{F} \Vdash \Phi$. For a set of formulas Φ by C^K_Φ we denote the class of all Kripke frames in which Φ is valid.

1.3.2 Boolean frames

A *Boolean frame* for $L_\mathcal{R}$ is a structure $\mathcal{F} = <A, 0_A, -_A, \cup_A, I>$ for which is satisfied $<A, 0_A, -_A, \cup_A>$ is a non-degenerate Boolean algebra and I is an interpretation function mapping the relation symbols of $\mathcal{R}$ to appropriate relations on A. That is for arbitrary P of $\mathcal{R}$ then $I(P)$ is $\rho(P)$-ary relation on A. Furthermore, they must be satisfied:

- for any $a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n$ in A if $<a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n>$ is in $I(P)$ then $a_i \neq 0_A$
- for all $a_1, \ldots, a_{i-1}, a'_i, a''_i, a_{i+1}, \ldots, a_n$ in A:

$$\begin{array}{l} <a_1, \ldots, a_{i-1}, (a'_i \cup a''_i), a_{i+1}, \ldots, a_n> \in I(P) \\ \qquad \textit{iff} \\ <a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n> \in I(P) \\ \text{or} \\ <a_1, \ldots, a_{i-1}, a''_i, a_{i+1}, \ldots, a_n> \in I(P) \end{array}$$

A *valuation* on $\mathcal{F}$ is a function $\mathcal{V}$ mapping the Boolean variables to elements of A. Recall the recursive extension $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ on the terms of $L_\mathcal{R}$:

- $\widetilde{\mathcal{V}}(x) = \mathcal{V}(x)$
- $\widetilde{\mathcal{V}}(0) = 0_A$
- $\widetilde{\mathcal{V}}(-\tau) = -_A\widetilde{\mathcal{V}}(\tau)$
- $\widetilde{\mathcal{V}}(\tau_1 \cup \tau_2) = \widetilde{\mathcal{V}}(\tau_1) \cup_A \widetilde{\mathcal{V}}(\tau_2)$

A *Boolean model* for $L_\mathcal{R}$ is a structure $\mathcal{M} = <\mathcal{F}, \mathcal{V}>$ where the structure $\mathcal{F} = <A, 0_A, -_A, \cup_A, I>$ is a Boolean frame for $L_\mathcal{R}$ and $\mathcal{V}$ is a valuation on $\mathcal{F}$. Recall the inductive definition of a formula φ *true in a Boolean model* $\mathcal{M}$ denoted by $\mathcal{M} \Vdash \varphi$.

- $\mathcal{M} \Vdash P(\tau_1, \ldots, \tau_n)$ *iff* $<\widetilde{\mathcal{V}}(\tau_1), \ldots, \widetilde{\mathcal{V}}(\tau_n)> \in I(P)$
- $\mathcal{M} \Vdash (\tau_1 \equiv \tau_2)$ *iff* $\widetilde{\mathcal{V}}(\tau_1) = \widetilde{\mathcal{V}}(\tau_2)$
- $\mathcal{M} \nVdash \bot$
- $\mathcal{M} \Vdash \neg\varphi$ *iff* $\mathcal{M} \nVdash \varphi$
- $\mathcal{M} \Vdash (\varphi_1 \vee \varphi_2)$ *iff* $\mathcal{M} \Vdash \varphi_1$ or $\mathcal{M} \Vdash \varphi_2$

Recall that a set of formulas Σ is called *satisfiable* in given Boolean frame should there be a Boolean model based on that frame (equivalently, there is a valuation on that frame) such that all the formulas in Σ are true in that model (respectively, in the model for the Boolean frame and the valuation). Σ

is satisfiable in a class of Boolean frames if exists Boolean frame from that class such that Σ is satisfiable in it. A formula φ is *valid* in a Boolean frame $\mathcal{F}$ if φ is *true* in every Boolean model for the frame $\mathcal{F}$. We denote it $\mathcal{F} \Vdash \varphi$. A set of formulas Φ is valid in a Boolean frame $\mathcal{F}$ if every formula in Φ is valid in $\mathcal{F}$. We denote it $\mathcal{F} \Vdash \Phi$. For a set of formulas Φ by C^B_Φ we denote the class of all Boolean frames in which Φ is valid.

1.3.3 Correspondence

Given $\mathcal{F} = <S, I>$. Recall, by *Boolean frame over* $\mathcal{F}$ we denote the structure $B(\mathcal{F}) = <A', 0_{A'}, -_{A'}, \cup_{A'}, I'>$ such that:

- $<A', 0_{A'}, -_{A'}, \cup_{A'}>$ is the Boolean algebra of all subsets of S
- I' is mapping the relation symbols P of $\mathcal{R}$ to appropriate relations $I'(P)$ on A' such that for any $a_1, \ldots, a_n \in A'$:

  $<a_1, \ldots, a_n> \in I_B(P)$
  iff
  exists $s_1 \in a_1, \ldots,$ exists $s_n \in a_n$ such that $<s_1, \ldots, s_n> \in I(P)$

Remark that, by definition, any valuation $\mathcal{V}$ on a Kripke frame $\mathcal{F}$ is valuation on $B(\mathcal{F})$ and vice versa. Furthermore, the resulting recursive valuations on $\mathcal{F}$ and $B(\mathcal{F})$ are the same. This is trivially inferred due to the simple fact that the zero and the join functions of the Boolean algebra A' are exactly the set theoretical empty set and union for the set of all subsets of S. Really, let $\widetilde{\mathcal{V}}'$ and $\widetilde{\mathcal{V}}''$ be the recursive valuations for $\mathcal{F}$ and $B(\mathcal{F})$ respectively. Then, by induction on the definition of $\widetilde{\mathcal{V}}'$ and $\widetilde{\mathcal{V}}''$ subsequently we have:

- For any Boolean variable x: $\widetilde{\mathcal{V}}'(x) = \mathcal{V}(x) = \widetilde{\mathcal{V}}''(x)$
- $\widetilde{\mathcal{V}}'(0) = \emptyset = 0_{A'} = \widetilde{\mathcal{V}}''(0)$
- By inductive hypothesis $\widetilde{\mathcal{V}}'(\tau) = \widetilde{\mathcal{V}}''(\tau)$ then:

$$\widetilde{\mathcal{V}}'(-\tau) = S \setminus \widetilde{\mathcal{V}}'(\tau) = -_{A'}\widetilde{\mathcal{V}}''(\tau) = \widetilde{\mathcal{V}}''(-\tau)$$

- By inductive hypothesis $\widetilde{\mathcal{V}}'(\tau_1) = \widetilde{\mathcal{V}}''(\tau_1)$ and $\widetilde{\mathcal{V}}'(\tau_2) = \widetilde{\mathcal{V}}''(\tau_2)$ then:

$$\widetilde{\mathcal{V}}'(\tau_1 \cup \tau_2) = \widetilde{\mathcal{V}}'(\tau_1) \cup \widetilde{\mathcal{V}}'(\tau_2) = \widetilde{\mathcal{V}}''(\tau_1) \cup_{A'} \widetilde{\mathcal{V}}''(\tau_2) = \widetilde{\mathcal{V}}''(\tau_1 \cup \tau_2)$$

Next we cite "Proposition 5" from [1], section 4.1 "*From Kripke frames to Boolean frames*".

Proposition 1.3.1. *Let $\mathcal{F} = <S, I>$ be a Kripke frame. Consider the Boolean frame over $\mathcal{F}$ denoted by $B(\mathcal{F}) = <A', 0_{A'}, -_{A'}, \cup_{A'}, I'>$. Let $\mathcal{V}$ be a valuation on $\mathcal{F}$ (and as clarified, equivalently on $B(\mathcal{F})$). Then for every formula φ:*

$$<B(\mathcal{F}), \mathcal{V}> \Vdash \varphi \qquad \text{iff} \qquad <\mathcal{F}, \mathcal{V}> \Vdash \varphi$$

Proof. Trivially by induction on the complexity of the formula φ. □

1.3.4 Formal system

As per [1], Section 7.1, *"Axiomatization"* for any set of formulas Φ the set of axioms of the formal system $\mathcal{L}_\Phi$ is defined and separated in the following groups:

(1) Sentential axioms

(2) Identity axioms: (for τ, τ_1, τ_2 and τ_3 Boolean terms)

- $(\tau \equiv \tau)$
- $(\tau_1 \equiv \tau_2) \implies (\tau_2 \equiv \tau_1)$
- $(\tau_1 \equiv \tau_2) \wedge (\tau_2 \equiv \tau_3) \implies (\tau_1 \equiv \tau_3)$

(3) Congruence axioms (τ_1, τ_2, τ_3 and τ_4 Boolean terms)

- $(\tau_1 \equiv \tau_2) \implies (-\tau_1 \equiv -\tau_2)$
- $(\tau_1 \equiv \tau_3) \wedge (\tau_2 \equiv \tau_4) \implies ((\tau_1 \cup \tau_2) \equiv (\tau_3 \cup \tau_4))$

(4) Boolean axioms: For all Boolean terms τ_1 and τ_2, if τ_1 and τ_2 are equivalent Boolean terms of Boolean logic then the following formula is an axiom of $\mathcal{L}_\Phi$:

- $(\tau_1 \equiv \tau_2)$

(5) Non-degenerate axiom:

- $\neg(0 \equiv 1)$

(6) Proximity axioms: (consider P the n-ary relation symbol, $1 \leq i \leq n$, and $\tau_1, \ldots, \tau_{i-1}, \tau_i, \tau_i', \tau_i'', \tau_{i+1}, \ldots, \tau_n$ arbitrary Boolean terms)

- $P(\tau_1, \ldots, \tau_{i-1}, \tau_i, \tau_{i+1}, \ldots, \tau_n) \implies \neg(\tau_i \equiv 0)$
- $(\tau_i \equiv (\tau_i' \cup \tau_i'')) \implies$

$$(P(\tau_1, \ldots, \tau_{i-1}, \tau_i, \tau_{i+1}, \ldots, \tau_n) \Longleftrightarrow (P(\tau_1, \ldots, \tau_{i-1}, \tau_i', \tau_{i+1}, \ldots, \tau_n) \vee P(\tau_1, \ldots, \tau_{i-1}, \tau_i'', \tau_{i+1}, \ldots, \tau_n)))$$

(7) Φ-axioms: Every formula obtained from a formula of Φ by simultaneously and uniformly substituting Boolean terms for the Boolean variables it contains.

Modus ponens is the only rule of inference for $\mathcal{L}_\Phi$.

$\mathcal{L}_\Phi$*-deduction* of formula φ from given set of formulas Σ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_s$ such that:

- Every φ_i, where $1 \leq i \leq s$, is either:
  - An axiom of $\mathcal{L}_\Phi$
  - Formula from the set Σ
  - Obtained by *Modus ponens* from formulas φ_j and φ_k from the sequence, where $j < i$, $k < i$ and φ_k is the formula $(\varphi_j \implies \varphi_i)$
- φ_s is the formula φ

We say φ is $\mathcal{L}_\Phi$*-deducible* from Σ, denoted as $\Sigma \vdash_{\mathcal{L}_\Phi} \varphi$, should it exist $\mathcal{L}_\Phi$-deduction of φ from Σ.

We say φ is $\mathcal{L}_\Phi$*-deducible* in the particular case when $\Sigma = \emptyset$ and denote it by $\vdash_{\mathcal{L}_\Phi} \varphi$.

1.3.5 Completeness of Boolean logics

Later in the exposition, for proving particular completeness properties of our *n-ary contact logic*, we resort to the more general case results for *Boolean logics* as studied in [1]. In particular, we refer to "Proposition 26", section 7.5 "*Completeness with respect to the Boolean semantics*" and cite it here as the following:

Proposition 1.3.2. *Let Σ be a set of formulas and φ be a formula such that $\Sigma \Vdash_{C^B_\Phi} \varphi$. If $BV(\Sigma)$ is finite then $\Sigma \vdash_{\mathcal{L}_\Phi} \varphi$.*

Effectively, we will be applying Proposition 1.3.2 for empty set Σ. Hence, for convenience sake, we state the form we will use.

Proposition 1.3.3. *For every set of formulas Φ and formula φ of the language $L_{\mathcal{R}}$:*

$$\Vdash_{C^B_\Phi} \varphi \qquad \textit{implies} \qquad \vdash_{\mathcal{L}_\Phi} \varphi$$

1.4 Graphs notions

The definition and notions with respect to graphs are as in [4], "*Graphs: Theory and Algorithms*". Will highlight some of them for the sake of common understanding used in the exposition later.

Denoting a graph by $G = (V, E)$, where V and E are finite sets (unless otherwise explicitly mentioned). The elements of V are called *vertices* and those of E *edges*. Each edge is associated with pair of vertices. We say that any of those vertices is *incident* on the edge. Furthermore, any edge a vertex being *incident* with is also called *incident* on the vertex.

In general we consider *undirected* graphs which means the pairs of vertices associated to the edges are non-ordered. We will tacitly assume *directed* graphs in the case of *rooted trees*. By assumption the direction of all edges is from the root of the tree towards the leafs.

Remark that by definition there is no restriction that every edge is associated with distinct pair of vertices (such graphs may be referred as *multi* graphs). We call two edges associated with a same pair of vertices *parallel* edges or *multi* edges. Should the pair of vertices associated with an edge be singleton then we call that edge *self-loop* at the given vertex or simply a *loop*. Graph having no parallel edges or self-loops is called *simple*.

We call a graph *bipartite* if its set of vertices can be coloured into two colours such that for every edge of the graph its two incident vertices are in different colour.

A graph with no edges is called *empty*.

1.5 Topological spaces and notions. Topological space $\mathbb{R}^m$

Here we briefly introduce the used notions and understanding related to topological spaces. Furthermore, we clarify what is the intended meaning of the topological space $\mathbb{R}^m$.

1.5.1 Topological spaces. Notions

Definition. $\mathbb{T} = <X, \boldsymbol{\tau}>$ is *topological space* if:

- X is a non-empty set and $\boldsymbol{\tau} \subseteq \mathcal{P}(X)$
- $\emptyset \in \boldsymbol{\tau}$ and $X \in \boldsymbol{\tau}$
- If $A_1, A_2 \in \boldsymbol{\tau}$ then $A_1 \cap A_2$ is an element of $\boldsymbol{\tau}$
- If $\{A_i\}_{i \in I}$ is a family of elements of $\boldsymbol{\tau}$ then $\cup_{i \in I} A_i$ is also an element of $\boldsymbol{\tau}$

$\boldsymbol{\tau}$ is called a *topology* on X.

Open set in topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$ is any of the elements of the topology $\boldsymbol{\tau}$.

Closed set is a complement with respect to X of an open set.

We state the definition above in a more convenient form.

Definition. In topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$:

- A is an *open set* *iff* $A \in \boldsymbol{\tau}$
- A is a *closed set* *iff* $X \setminus A \in \boldsymbol{\tau}$

Remark the dual nature of the terms *open* and *closed* sets. It allows to define topological space by $\boldsymbol{\tau}$ being the *closed* sets instead and the *open* sets being their complements with respect to X. Within this exposition though we use the definition already given, namely where $\boldsymbol{\tau}$ is the set of the open sets.

Given topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$ we define *interior* and *closure* of a set. Consider A subset of X.

Definition. *Interior* of a set A, denoted by $Int(A)$, is:

$$Int(A) \ \risingdotseq \ \cup\{B \in \boldsymbol{\tau} \mid B \subseteq A\}$$

Hence, alternatively, the *interior* of a set A is the biggest open set subset of A.

Definition. *Closure* of a set A, denoted by $Cl(A)$, is:

$$Cl(A) \ \risingdotseq \ \cap\{B \mid X \setminus B \in \boldsymbol{\tau} \text{ and } A \subseteq B\}$$

Hence, alternatively, the *closure* of a set A is the smallest closed set for which A is its subset.

Remark the dual nature of *interior* and *closure* of a set. In particular:

- $Cl(A) = X \setminus Int(X \setminus A)$
- $Int(A) = X \setminus Cl(X \setminus A)$

For the definitions of open and closed sets then we have:

- A is *open* *iff* $A \in \boldsymbol{\tau}$ *iff* $A = Int(A)$
- A is *closed* *iff* $X \setminus A \in \boldsymbol{\tau}$ *iff* $A = Cl(A)$

Trivially by definition:

$$Int(A) \subseteq A \subseteq Cl(A)$$

Definition. *Boundary points* of the set A are the elements of the set:

$$Cl(A) \setminus Int(A)$$

Then, in addition to the above, the definitions of *open* and *closed* sets can be restated in the following equivalent way:

- A is *open* *iff* A does not contain any of its boundary points
- A is *closed* *iff* A contains all its boundary points

Again, consider topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$

Definition. The topological space $\mathbb{T}$ is called *connected* if there exist no *open* non-empty A_1 and A_2 subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = X$.

Remark that if $\mathbb{T}$ is *connected* then the only pair A_1 and A_2 of subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = X$ is the sets $\emptyset$ and X.

By definition of a *connected* topological space the following definitions are equivalent:

- There exist no non-empty *open* A_1 and A_2 subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = X$.
- There exist no non-empty *closed* A_1 and A_2 subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = X$.
- The only subsets of X being both *open* and *closed* (*clopen*) are $\emptyset$ and X.
- The only subsets of X having empty set of boundary points are $\emptyset$ and X.

1.5.2 Topological space $\mathbb{R}^m$

Consider the *Euclidean metric* in the Euclidean space $\mathbb{R}^m$.

When we say an *open ball* o for point x of $\mathbb{R}^m$ we mean the set of all points being with *Euclidean distance* to x less than given fixed real positive r. We can think of r as the radius of the "ball". Remark the collision with the term *open* set determined in Section 1.5.1. As a comment, this is not a collision at all as effectively *open ball* eventually is an *open* subset of the topological space $\mathbb{R}^m$ which we are going to define shortly. Nevertheless this result is not going to be explicitly stated as not being essential to the exposition. Confusion should be avoided as the usage of the term *open ball* will be non-ambiguous and clear by the context.

Consider $<\mathbb{R}^m, \boldsymbol{\tau}>$ where $\boldsymbol{\tau}$ is defined as:

- $A \in \boldsymbol{\tau}$ *iff* for every $x \in A$ there exists an open ball $o \ni x$ such that $o \subseteq A$

Remark that $<\mathbb{R}^m, \boldsymbol{\tau}>$ is *topological space*. Really:

- $\emptyset$ and X are in $\boldsymbol{\tau}$ by trivial reasons.

- Let A_1 and A_2 be from $\boldsymbol{\tau}$. Consider arbitrary $x \in A_1 \cap A_2$. By A_1 there is an open ball $o_1 \ni x$ such that $o_1 \subseteq A_1$ and by A_2 there is $o_2 \ni x$ such that $o_2 \subseteq A_2$. Hence $o_1 \cap o_2 \subseteq A_1 \cap A_2$. Trivially, $o_1 \cap o_2$ is an open ball. Furthermore, $o_1 \cap o_2$ is an open ball for x. x was arbitrary element of $A_1 \cap A_2$ therefore $A_1 \cap A_2$ is from $\boldsymbol{\tau}$.

- Consider $\{A_i\}_{i<I}$ family of elements of $\boldsymbol{\tau}$. For an arbitrary $x \in \cup_{i\in I} A_i$ we have that there exists $j \in I$ such that $x \in A_j$. Then there is an open ball $o \ni x$ such that $o \subseteq A_j$. This means $o \subseteq \cup_{i\in I} A_i$. x was arbitrary therefore $\cup_{i\in I} A_i$ is from $\boldsymbol{\tau}$.

For simplicity when we say topological space $\mathbb{R}^m$ we should mean the topological space $<\mathbb{R}^m, \boldsymbol{\tau}>$ as just defined.

Consider the topological space $\mathbb{R}^m$. Having the notions of open, closed set and boundary points of a set in arbitrary topological space then, when in $\mathbb{R}^m$, those can in addition be restated in the following way:

- A is an *open* set *iff* for every $x \in A$ there exists an open ball $o \ni x$ such that $o \subseteq A$

- a is a *boundary point* for A *iff* for every open ball $o \ni a$ there exist in o both points from A and points from the complement of A with respect to $\mathbb{R}^m$ (that is $o \cap A \neq \emptyset$ and $o \cap (\mathbb{R}^m \setminus A) \neq \emptyset$)

- A is a *closed* set *iff* for every $x \in A$ either there exists an open ball $o \ni x$ such that $o \subseteq A$ (in particular, x is from the interior of A) or for every open ball $o \ni x$ there exist in o both points from A and points from the complement of A with respect to $\mathbb{R}^m$ (in particular, x is a boundary point for A)

Finally, remark that:

- The topological space $\mathbb{R}^m$ is *connected.*

Proof notes: Assume the contrary, namely, there exist non-empty open sets A and B subsets of $\mathbb{R}^m$ satisfying $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}^m$. Take arbitrary $a \in A$ and $b \in B$. Apparently $a \neq b$. Consider the segment s being the section between a and b (including a and b) of the straight line connecting a and b in $\mathbb{R}^m$. We build a countable sequence of segments subsets of s in the following way. Let s_0 be s. Consider the Euclidean distance between a and b and take the point in the middle of the segment s_0, denote it by a_1. If $a_1 \in A$ then s_1 is the segment between a_1 and b (including a_1 and b). Otherwise $a_1 \in B$ then s_1 is the segment between a and a_1 (including a and a_1). Remark that the length of the segment s_1 is half the length of s_0. In this way we build the countable sequence of segments $\{s_i\}_{i<\omega}$ each segment being with positive length and half the length of the former one in the sequence. Then the length of the segments s_i converges down to 0 when i tends to infinity. Therefore the sequence $\{s_i\}_{i\to\infty} \to c$, where c is point from s (the latter statement requires some further formal refinement). Remark that, by definition of the sequence $\{s_i\}_{i<\omega}$, for every natural i then $c \in s_i$. Consider arbitrary open ball $o \ni c$. Then by the remark just being made there is a natural N such that for every $i > N$ is satisfied $s_i \subseteq o$. By the choice of the elements of the sequence $\{s_i\}_{i<\omega}$ we imply that there are points from A

and points from B in o. o was arbitrary. Then, by A and B *open* sets it means c is neither in A nor in B. This is a contradiction with $A \cup B = \mathbb{R}^m$.

$\square$

1.6 Regular closed sets and polytopes of $\mathbb{R}^m$. Boolean algebras of the regular closed sets and polytopes

Within the exposition they will extensively be used the notions of *regular closed* subsets and *polytopes* of $\mathbb{R}^m$. Basically for a regular closed subset of $\mathbb{R}^m$ is adopted the very standard notion. The meaning of polytopes though has evolved throughout the years so we will define what being intended here.

The regular closed subsets and polytopes of $\mathbb{R}^m$ determine a class of Boolean algebras. They are of special interest for us because the classes of Boolean frames to be studied later will all be with carriers exactly such Boolean algebras. Furthermore, it will also be given the definition of the standard n-ary contact relation in arbitrary topological space. The n-ary contact relations in $\mathbb{R}^m$ on the other hand will be the interpretation of the relation symbols of the n-ary contact language (defined in Section 1.2) in those same classes of Boolean frames.

1.6.1 Regular closed sets

Consider arbitrary topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$

First, let us observe several additional properties of the *interior* and the *closure*. Following consider arbitrary sets A and B of $\mathbb{T}$.

- (*Idempotency*) The following equations hold:

$$Int(Int(A)) = Int(A) \qquad Cl(Cl(A)) = Cl(A)$$

Proof. Trivially by definition. $\square$

- (*Monotonicity*) If $A \subseteq B$ then:

$$Int(A) \subseteq Int(B) \qquad Cl(A) \subseteq Cl(B)$$

Proof. $Int(A) \subseteq A$ hence $Int(A) \subseteq B$. $Int(A)$ is open set then by the latter and by definition: $Int(A) \subseteq Int(B)$. For the *closure* either in analogy or by the just proven fact for the *interior* and by their duality property. In particular, $A \subseteq B$ then $X \setminus B \subseteq X \setminus A$. Thus $Int(X \setminus B) \subseteq Int(X \setminus A)$. Therefore:

$$Cl(A) = X \setminus Int(X \setminus A) \subseteq X \setminus Int(X \setminus B) = Cl(B)$$

$\square$

- *Linear* property of the *interior* with respect to the set-theoretical intersection and the *closure* with respect to the set-theoretical union:

$$Int(A \cap B) = Int(A) \cap Int(B)$$
$$Cl(A \cup B) = Cl(A) \cup Cl(B)$$

Remark that the *linear* properties of the *interior* and the *closure* are not valid if the set-theoretical intersection and union are exchanged in the above equations.

Proof. We have $Int(A) \subseteq A$ and $Int(B) \subseteq B$. Then, trivially:

$$Int(A) \cap Int(B) \subseteq A \cap B$$

$Int(A)$ and $Int(B)$ are open sets then, by definition, such one also is $(Int(A) \cap Int(B))$. Hence, by *idempotency* and *monotonicity*, it follows:

$$Int(A) \cap Int(B) = Int(Int(A) \cap Int(B)) \subseteq Int(A \cap B)$$

For the opposite direction, trivially, for A we have $A \cap B \subseteq A$ and by *monotonicity*: $Int(A \cap B) \subseteq Int(A)$. The same holds for B, namely $Int(A \cap B) \subseteq Int(B)$. Therefore:

$$Int(A \cap B) \subseteq Int(A) \cap Int(B)$$

The statement for the *closure* is by analogous reasoning to the one just proven for the *interior*. Alternatively, it can simply be attained by the duality of the *interior* and the *closure* using the demonstrated fact for the *interior*. Namely:

$$\begin{aligned}
&Cl(A \cup B) = X \setminus Int(X \setminus (A \cup B)) = X \setminus Int((X \setminus A) \cap (X \setminus B)) = \\
&= X \setminus (Int(X \setminus A) \cap Int(X \setminus B)) = (X \setminus Int(X \setminus A)) \cup (X \setminus Int(X \setminus B)) = \\
&= Cl(A) \cup Cl(B)
\end{aligned}$$

□

We are prepared to define and study the notion of *regular closed* set of the arbitrary topological space $\mathbb{T}$.

Definition. A *regular closed* set of $\mathbb{T}$ is a subset of the space X being equal to the closure of its interior. In particular, A is a *regular closed* set if $A = Cl(Int(A))$.

Obviously:

- A *regular closed* set is a *closed* set.

Furthermore, trivially:

- $\emptyset$ and the space X are *regular closed* sets.

We call the operation $Cl(Int(A))$ on an arbitrary set A *regularisation* and the set $Cl(Int(A))$ *regularised*. Therefore *regular closed* sets are those equal to their *regularised* sets. Furthermore, the *regularised* sets are the *regular closed* sets. In particular:

- For an arbitrary set A of $\mathbb{T}$ then $Cl(Int(A))$ is a *regular closed* set.

This statement is equivalent to:

- For an arbitrary set A of $\mathbb{T}$ it holds:

$$Cl(Int(A)) = Cl(Int(Cl(Int(A))))$$

Proof. By definition of *interior* we have $Int(Cl(Int(A))) \subseteq Cl(Int(A))$. Then, by the *monotonicity*, we obtain $Cl(Int(Cl(Int(A)))) \subseteq Cl(Cl(Int(A)))$. Now by *idempotency* eventually:

$$Cl(Int(Cl(Int(A)))) \subseteq Cl(Int(A))$$

For the opposite direction we have $Int(A) \subseteq Cl(Int(A))$. By *monotonicity* and *idempotency*, consequently, we obtain: $Int(A) \subseteq Int(Cl(Int(A)))$. By *monotonicity* once again:

$$Cl(Int(A)) \subseteq Cl(Int(Cl(Int(A))))$$

□

The following statements hold:

- For arbitrary *regular closed* sets A and B then $(A \cup B)$ also is a *regular closed* set.
- For arbitrary *regular closed* set A then $Cl(X \setminus A)$ also is a *regular closed* set.

Proof.
For the first one we have to demonstrate: $A \cup B = Cl(Int(A \cup B))$.

By *monotonicity* applied on $A \subseteq A \cup B$ we obtain that: $Cl(Int(A)) \subseteq Cl(Int(A \cup B))$ Now, by A being a *regular closed* set, namely $A = Cl(Int(A))$, we have $A \subseteq Cl(Int(A \cup B))$. In analogy $B \subseteq Cl(Int(A \cup B))$. By those:

$$A \cup B \subseteq Cl(Int(A \cup B))$$

For the other direction, we have $Int(A \cup B) \subseteq A \cup B$. By *monotonicity*, *idempotency* and the *linear* properties of the *closure*, subsequently:

$$Cl(Int(A \cup B)) \subseteq Cl(A \cup B) = Cl(A) \cup Cl(B)$$

A and B are *closed* sets hence $A = Cl(A)$ and $B = Cl(B)$. Therefore:

$$Cl(Int(A \cup B)) \subseteq A \cup B$$

For the second one we have to demonstrate: $Cl(X \setminus A) = Cl(Int(Cl(X \setminus A)))$.

This is true by the duality of the *interior* and the *closure*. In particular, subsequently, we obtain:

$$Cl(Int(Cl(X \setminus A))) = Cl(Int(X \setminus Int(A))) = Cl(X \setminus Cl(Int(A)))$$

By this and A being a *regular closed* set, namely $A = Cl(Int(A))$, then the intended equality holds.

□

1.6.2 Boolean algebras of regular closed sets

Again, consider an arbitrary topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$. Denote by $RC(\mathbb{T})$ the set of the *regular closed* sets of $\mathbb{T}$. Consider the structure:

$$RC = <RC(\mathbb{T}), -_{RC}, \cup_{RC}, \cap_{RC}>,$$

where for arbitrary A and B being *regular closed* sets the operations $\cup_{RC}$, $\cap_{RC}$ and $-_{RC}$ are defined as:

- $A \cup_{RC} B = A \cup B$
- $A \cap_{RC} B = Cl(Int(A \cap B))$
- $-_{RC} A = Cl(X \setminus A)$

As per Section 1.6.1 the result of applying $\cup_{RC}$, $\cap_{RC}$ or $-_{RC}$ on arbitrary *regular closed* sets is a *regular closed* set. Our goal then is to see that RC is a Boolean algebra.

First of all, remark that the *de Morgan* laws are satisfied. Namely:

- $A \cap_{RC} B = -_{RC}((-_{RC} A) \cup_{RC} (-_{RC} B))$
- $A \cup_{RC} B = -_{RC}((-_{RC} A) \cap_{RC} (-_{RC} B))$

Proof. By definition and using the properties demonstrated in Section 1.6.1, subsequently:

$$\begin{aligned}
&-_{RC}((-_{RC} A) \cup_{RC} (-_{RC} B)) = \\
&= Cl(X \setminus ((-_{RC} A) \cup_{RC} (-_{RC} B))) = \\
&= Cl(X \setminus (Cl(X \setminus A) \cup Cl(X \setminus B))) = \\
&= Cl(X \setminus Cl((X \setminus A) \cup (X \setminus B))) = \\
&= Cl(X \setminus Cl(X \setminus (A \cap B))) = \\
&= Cl(X \setminus (X \setminus Int(A \cap B))) = \\
&= Cl(Int(A \cap B)) = A \cap_{RC} B
\end{aligned}$$

For the other equation:

$$\begin{aligned}
&-_{RC}((-_{RC} A) \cap_{RC} (-_{RC} B)) = \\
&= Cl(X \setminus ((-_{RC} A) \cap_{RC} (-_{RC} B))) = \\
&= Cl(X \setminus Cl(Int((-_{RC} A) \cap (-_{RC} B)))) = \\
&= Cl(X \setminus Cl(Int(Cl(X \setminus A) \cap Cl(X \setminus B)))) = \\
&= Cl(X \setminus Cl(Int(Cl(X \setminus A)) \cap Int(Cl(X \setminus B))))
\end{aligned}$$

Remark that for arbitrary *regular closed* set A:

$$\begin{aligned}
&Int(Cl(X \setminus A)) = X \setminus Cl(X \setminus Cl(X \setminus A)) = \\
&= X \setminus Cl(Int(A)) = X \setminus A
\end{aligned}$$

Now, by substituting these in the former equation we obtain:

$$\begin{aligned}
&= Cl(X \setminus Cl((X \setminus A) \cap (X \setminus B))) = \\
&= Cl(X \setminus Cl(X \setminus (A \cup B))) = Cl(Int(A \cup B))
\end{aligned}$$

Recall, for *regular closed* sets A and B, as per Section 1.6.1, we have:

$$Cl(Int(A \cup B)) = A \cup B,$$

by which the equation is proven.

□

To show that RC is a Boolean algebra we need to verify a sufficient set of axioms. We adopt one as by [3]. In particular (expectedly, by '$\cup$' and '$\cap$' are denoted the Boolean algebra operations join and meet respectively):

(A1)

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

(A2)

$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

(A3)

$$(A \cap B) \cup B = B, \qquad (A \cup B) \cap B = B$$

(A4)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(A5)

$$(A \cap -A) \cup B = B, \qquad (A \cup -A) \cap B = B$$

As a comment, it is a known fact (also mentioned in [3]) that axiom (A4) could be omitted. Nevertheless we will consider the original set of axioms so (A4) will be demonstrated as well.

(A1)

Trivially by definition and the associativity of set-theoretical union and intersection.

(A2)

$A \cup_{RC} (B \cup_{RC} C) = (A \cup_{RC} B) \cup_{RC} C$ is trivial by definition of '$\cup_{RC}$'. For the other one first remark the following property (a kind of a "dual" form of the *regularisation*).

- For an arbitrary set A of $\mathbb{T}$:

$$Int(Cl(A)) = Int(Cl(Int(Cl(A))))$$

Proof of the property: By definition of *closure*: $Int(Cl(A)) \subseteq Cl(Int(Cl(A)))$. By Section 1.6.1 *monotonicity* and *idempotency*, subsequently:

$$Int(Cl(A)) = Int(Int(Cl(A))) \subseteq Int(Cl(Int(Cl(A))))$$

For the other direction, by definition of *interior*, we have $Int(Cl(A)) \subseteq Cl(A)$. Then by Section 1.6.1 *monotonicity* and *idempotency*, subsequently:

$$Cl(Int(Cl(A))) \subseteq Cl(Cl(A)) = Cl(A)$$

Now, by *monotonicity*:

$$Int(Cl(Int(Cl(A)))) \subseteq Int(Cl(A))$$

$\square$

Remark that whenever A is closed then, by $A = Cl(A)$ and the demonstrated property, we imply:

- For an arbitrary closed set A:

$$Int(A) = Int(Cl(Int(A)))$$

Now:

$$\begin{aligned} &A \cap_{RC} (B \cap_{RC} C) = Cl(Int(A \cap Cl(Int(B \cap C)))) = \\ &= Cl(Int(A) \cap Int(Cl(Int(B \cap C)))) \end{aligned}$$

Recall B and C are *closed* sets then, by the demonstrated property:

$$Int(Cl(Int(B \cap C))) = Int(B \cap C)$$

Using this in the former equation we obtain:

$$= Cl(Int(A) \cap Int(B \cap C)) = Cl(Int(A) \cap Int(B) \cap Int(C)))$$

By similar reasoning we obtain the same result for $(A \cap_{RC} B) \cap_{RC} C$ as well. Hence, the equality holds.

(A3)

First, remark the following property:

- For arbitrary sets A and B of $\mathbb{T}$:

$$A \cup Int(A \cap B) = A$$

Proof of the property: Trivially, by $Int(A \cap B) \subseteq (A \cap B)$:

$$A \subseteq A \cup Int(A \cap B) \subseteq A \cup (A \cap B) = A$$

This proves the equality. □

Now:

$$(A \cap_{RC} B) \cup_{RC} B = Cl(Int(A \cap B)) \cup B$$

B is *regular closed* set then B is closed hence $B = Cl(B)$. Then, by the property just demonstrated and the *linear* property of the *closure* as per Section 1.6.1, subsequently:

$$\begin{aligned} &Cl(Int(A \cap B)) \cup B = Cl(Int(A \cap B)) \cup Cl(B) = \\ &= Cl(Int(A \cap B) \cup B) = Cl(B) = B \end{aligned}$$

For the other equality it is directly by definition and by A and B (in particular B) being *regular closed* sets, namely:

$$\begin{aligned} &(A \cup_{RC} B) \cap_{RC} B = Cl(Int((A \cup B) \cap B)) = \\ &= Cl(Int(B)) = B \end{aligned}$$

(A4)

It is sufficient to demonstrate one of the equations as long as the other is a direct implication of the former by applying *de Morgan* laws. We will prove the equation: $A \cap_{RC} (B \cup_{RC} C) = (A \cap_{RC} B) \cup_{RC} (A \cap_{RC} C)$. For its left side we have:

$$
\begin{aligned}
&A \cap_{RC} (B \cup_{RC} C) = Cl(Int(A \cap (B \cup_{RC} C)))) = \\
&= Cl(Int(A \cap (B \cup C)))) = \\
&= Cl(Int((A \cap B) \cup (A \cap C)))
\end{aligned}
$$

For the right side of the equation:

$$
\begin{aligned}
&(A \cap_{RC} B) \cup_{RC} (A \cap_{RC} C) = (A \cap_{RC} B) \cup (A \cap_{RC} C) = \\
&= Cl(Int(A \cap B)) \cup Cl(Int(A \cap C))
\end{aligned}
$$

A, B and C are *closed* sets then also are $(A \cap B)$ and $(A \cap C)$. Therefore, by directly applying the observation below on the closed sets $(A \cap B)$ and $(A \cap C)$, the equality will hold. It remains to prove then the following observation:

- Let A and B be *closed* sets of $\mathbb{T}$. Then:

$$Cl(Int(A \cup B)) = Cl(Int(A)) \cup Cl(Int(B))$$

Remark that by this equality it is directly implied the fact that for any *regular closed* sets A and B then $(A \cup B)$ is also a *regular closed* set. We have proven the latter explicitly for the sake of simplicity as long as the current property is a slightly less trivial observation.

Proof of the observation: By $A \subseteq (A \cup B)$ and by *monotonicity* as per Section 1.6.1, subsequently:

$$Cl(Int(A)) \subseteq Cl(Int(A \cup B))$$

Combining it with the analogous result for B we imply:

$$Cl(Int(A)) \cup Cl(Int(B)) \subseteq Cl(Int(A \cup B))$$

For the other direction it is sufficient to demonstrate that:

$$Int(A \cup B) \subseteq Cl(Int(A)) \cup Cl(Int(B))$$

Having this then, by $Cl(Int(A)) \cup Cl(Int(B))$ being a closed set and the *monotonicity* and *idempotency* as per Section 1.6.1, subsequently we will imply:

$$
\begin{aligned}
&Cl(Int(A \cup B)) \subseteq Cl(Cl(Int(A)) \cup Cl(Int(B))) = \\
&= Cl(Int(A)) \cup Cl(Int(B)),
\end{aligned}
$$

which will prove the equality.

Now, to show $Int(A \cup B) \subseteq Cl(Int(A)) \cup Cl(Int(B))$, for the sake of contradiction, assume the contrary. This means:

$$Int(A \cup B) \cap (X \setminus (Cl(Int(A)) \cup Cl(Int(B)))) \neq \emptyset$$

By the duality of the *interior* and the *closure* we have :

$$\begin{aligned}
&X \setminus (Cl(Int(A)) \cup Cl(Int(B))) = \\
&X \setminus (Cl(Int(A) \cup Int(B))) = \\
&= Int(X \setminus (Int(A) \cup Int(B)))
\end{aligned}$$

Then, by that and the former inequality:

$$\begin{aligned}
&\emptyset \neq Int(A \cup B) \cap Int(X \setminus (Int(A) \cup Int(B))) = \\
&= Int((A \cup B) \cap (X \setminus (Int(A) \cup Int(B)))) = \\
&= Int((A \cup B) \setminus (Int(A) \cup Int(B)))
\end{aligned}$$

Denote:

$$U = Int((A \cup B) \setminus (Int(A) \cup Int(B)))$$

Hence U is an *open* non-empty set and:

$$U \subseteq (A \cup B) \setminus (Int(A) \cup Int(B))$$

Assume $U \subseteq A$. U is *open* then by definition $U \subseteq Int(A)$ which is a contradiction. It follows that both $U \nsubseteq A$ and $U \nsubseteq B$. Then:

$$U \cap (X \setminus A) \neq \emptyset$$

A is *closed* then $(X \setminus A)$ is *open* hence $U \cap (X \setminus A)$ is an *open* non-empty set. By $U \subseteq (A \cup B)$ we have:

$$\begin{aligned}
&U \cap (X \setminus A) \subseteq (A \cup B) \cap (X \setminus A) = \\
&= (A \cup B) \setminus A = B \setminus A \subseteq B
\end{aligned}$$

Nevertheless, $U \nsubseteq B$ hence $(U \cap (X \setminus A)) \nsubseteq B$. We have a contradiction thus our assumption is wrong which proves the observation.

□

(A5)

Again, it is sufficient to prove one of the equations as long as the other is a direct implication of the former by applying the *de Morgan* laws. We will prove $(A \cap_{RC} (-_{RC} A)) \cup_{RC} B = B$. The following is satisfied:

$$\begin{aligned}
&A \cap_{RC} (-_{RC} A) = \\
&= Cl(Int(A \cap (-_{RC} A))) = \\
&= Cl(Int(A \cap Cl(X \setminus A))) = \\
&= Cl(Int(A \cap (X \setminus Int(A)))) = \\
&= Cl(Int(A \setminus Int(A)))
\end{aligned}$$

Remark that:

- For an arbitrary set A is satisfied:

$$Int(A \setminus Int(A)) = \emptyset$$

Proof of the property: Let $U = Int(A \setminus Int(A))$ and assume that $U \neq \emptyset$. $U \subseteq (A \setminus Int(A))$ then $U \cap Int(A) = \emptyset$. Trivially, $U \cup Int(A)$ is an *open* set. Furthermore, $U \cup Int(A) \subseteq A$. Hence, by definition, $U \cup Int(A) \subseteq Int(A)$ thus $U \subseteq Int(A)$, which is a contradiction. □

Applying this observation it follows that:

$$A \cap_{RC} (-_{RC}A) = \emptyset$$

Now, using this, we have:

$$\begin{aligned}&(A \cap_{RC} (-_{RC}A)) \cup_{RC} B =\\&(A \cap_{RC} (-_{RC}A)) \cup B =\\&\emptyset \cup B = B\end{aligned}$$

Eventually we have demonstrated that RC is a Boolean algebra. As a comment, it is a known fact that, furthermore, RC is a *complete* Boolean algebra (intuitively, one allowing infinite meets and joins). Nevertheless the latter will be not needed in our exposition.

Finally, by the equation above, for the *zero* of the Boolean algebra RC we have:

$$0_{RC} = (A \cap_{RC} (-_{RC}A)) = \emptyset$$

For the *unit* of RC, by definition, subsequently:

$$\begin{aligned}&1_{RC} = (A \cup_{RC} (-_{RC}A)) =\\&= A \cup Cl(X \setminus A) \supseteq A \cup (X \setminus A) = X\end{aligned}$$

Therefore:

- The *zero* of the Boolean algebra RC is the empty set.
- The *unit* of the Boolean algebra RC is the space X of the topological space $\mathbb{T}$.

1.6.3 Polytopes of $\mathbb{R}^m$

Consider the topological space $\mathbb{R}^m$ for fixed m, $m \geq 1$.

By a *half-space* of $\mathbb{R}^m$ we intend the standard notion in analogy to half-space of $\mathbb{R}^3$. Then a *half-space* of $\mathbb{R}^2$ is a half-plane, of $\mathbb{R}^1$ is a ray. Formally, a *half-space* of $\mathbb{R}^m$ is the set of points satisfying the inequality:

$$a_1x_1 + \ldots + a_nx_n \geq b,$$

for appropriate real coefficient b and not all zero real coefficients $a_1, \ldots, a_n$.

Definition. Inductive definition of a *polytope* of $\mathbb{R}^m$:

- The empty set is a *polytope* of $\mathbb{R}^m$.
- Any intersection with a *non-empty* interior of finitely many half-spaces is a *polytope* of $\mathbb{R}^m$.
- A union of finitely many *polytopes* of $\mathbb{R}^m$ is a *polytope* of $\mathbb{R}^m$.

Trivially, by definition $\mathbb{R}^m$ itself is also a polytope (as being the set of all points in the empty intersection of half-spaces).

In particular, a polytope of $\mathbb{R}^1$ is a union of finitely many subsets of $\mathbb{R}^1$ each being intersection (with non-empty interior) of finitely many rays. This means a polytope of $\mathbb{R}^1$ effectively is a union of finitely many (genuine) rays or segments of $\mathbb{R}^1$. Polytope in $\mathbb{R}^2$ will be a union of finitely many subsets of $\mathbb{R}^2$ each being intersection (with non-empty interior) of finitely many half-planes.

Remark that any polytope of $\mathbb{R}^m$ can be considered a polytope of $\mathbb{R}^n$ for $m \leq n$. The polytope of $\mathbb{R}^m$ can be considered as obtained by the $\mathbb{R}^m$ projection of $\mathbb{R}^n$. Apparently the subset of $\mathbb{R}^n$ whose projection results in the polytope of $\mathbb{R}^m$ is a polytope by definition.

Consider the equivalent topological notions as highlighted in Section 1.5.2. Then, by the formal definition above of a half-space (the points satisfying a non-strong inequality), it is easy to infer that a half-space contains all its *boundary points*. In particular, any point satisfying the equality will also satisfy the definition of a *boundary point* in $\mathbb{R}^m$. Then a half-space in $\mathbb{R}^m$ is a *closed* set. Furthermore, it is easy to see that the *interior* of a half-space are the points satisfying the strong inequality. By this we imply that the *closure* of the *interior* of a half-space is the half-space itself. Therefore:

- The half-spaces are *regular closed* sets.

Furthermore, this similar reasoning can easily be generalised for a finite intersection of half-spaces (such that the intersection is with a non-empty interior), namely the points satisfying the finite set of the inequalities defining each of the half-spaces. Thus we have:

- An intersection of finitely many half-spaces being with non-empty *interior* is a *regular closed* set.

By Section 1.6.1, union of *regular closed* sets is a *regular closed* set. Now, by applying the inductive definition of a *polytope* it follows that:

- The *polytopes* are *regular closed* sets.

1.6.4 Boolean algebras of regular closed sets and polytopes of $\mathbb{R}^m$

As per Section 1.6.3, consider the topological space $\mathbb{R}^m$. In Section 1.6.2 we have demonstrated that the set of all *regular closed* sets forms a Boolean algebra with the well defined Boolean operations '$\cup_{RC}$', '$\cap_{RC}$' and '$-_{RC}$'. Our purpose now is to demonstrate that the set of all *polytopes* of $\mathbb{R}^m$ forms a Boolean algebra subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$.

Recall that, as per Section 1.6.3, the *polytopes* are *regular closed* sets.

Consider the Boolean algebra complement operation '$-_{RC}$' as defined in Section 1.6.2.

With a similar reasoning as in Section 1.6.3, we also imply that the set-theoretical complement of a half-space is the *interior* of the counter half-space (formally, the one obtained by negating all the coefficients of the given half-space). Then the closure of that set will be the counter half-space of the given half-space. Then:

- The Boolean algebra complement operation '$-_{RC}$' applied on a half-space is a half-space.

Now, consider a finite intersection of half-spaces having a non-empty *interior*. Then, the set-theoretical complement of that set is the finite union of the set theoretical complements of each of the half-spaces participating in the finite intersection. As clarified, the set-theoretical complement of a half-space is the *interior* of the counter half-space. Therefore the set-theoretical complement of an intersection of finitely many half-spaces is the union of the *interiors* of their corresponding counter half-spaces. Now, by the *linearity* of the *closure* as per Section 1.6.1 then the *closure* of that set is the finite union of the *regularisations* of the respective counter half-spaces. As already shown in Section 1.6.3, half-spaces are *regular closed* sets hence they are preserved under *regularisation*. By definition of *politope* we conclude:

- The Boolean algebra complement operation '$-_{RC}$' applied on a finite intersection of half-spaces having a non-empty *interior* results in a *polytope*.

Now, consider the set-theoretical complement of an arbitrary *polytope*. The cases of the polytope being the empty set or $\mathbb{R}^m$ are trivial so, in general, consider the polytope is a finite union of finite intersections of half-spaces (with a non-empty *interior*). Then, the set-theoretical complement is the intersection of the unions of the *interiors* of the corresponding counter half-spaces. Reorganising this properly we obtain a finite union of finite intersections of the interiors of the counter half-spaces. The *closure* of that finite union is then the union of the *closures* of each such finite intersection. As clarified, the closure of a finite intersection of the *interiors* of half-spaces is the intersection of the half-spaces. Therefore we have a finite union of finite intersections of half-spaces, hence, a *polytope*. Finally:

- The Boolean algebra complement operation '$-_{RC}$' applied on a *polytope* results in a *polytope*.

Furthermore, trivially, by definition of *polytopes*, a union of finitely many *polytopes* is a *polytope*. Then:

- The Boolean algebra join operation '$\cup_{RC}$' applied on *polytopes* results in a *polytope*.

Using the *de Morgan* laws and the results for the Boolean algebra complement and join operations we imply:

- The Boolean algebra meet operation '$\cap_{RC}$' applied on *polytopes* results in a *polytope*.

To recapitulate, we have demonstrated that the *polytopes* are *regular closed* sets and they are preserved under the operations of the Boolean algebra of the *regular closed* sets. Finally:

- The *polytopes* of $\mathbb{R}^m$ form a Boolean algebra subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$.

As a subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$, then the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ has the same *zero* and *unit* elements. Therefore, as per Section 1.6.2, we have:

- The *zero* of the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ as well as of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ is the empty set.

- The *unit* of the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ as well as of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ is the set $\mathbb{R}^m$.

Denote by $PRC(\mathbb{R}^m)$ the set of the *polytopes* of $\mathbb{R}^m$. Consider the structure:

$$PRC = <PRC(\mathbb{R}^m), -_{RC}, \cup_{RC}, \cap_{RC}>,$$

where the Boolean operations $-_{RC}$, $\cup_{RC}$ and $\cap_{RC}$ are as per Section 1.6.2. From here on, by RC will be denoted the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$, namely:

$$RC = <RC(\mathbb{R}^m), -_{RC}, \cup_{RC}, \cap_{RC}>$$

Therefore:

- PRC is the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ subalgebra of the Boolean algebra RC of the *regular closed* sets of $\mathbb{R}^m$.

1.7 n-ary contact relation

Consider an arbitrary topological space $\mathbb{T} = <X, \boldsymbol{\tau}>$ and arbitrary $A_1, \ldots, A_n$ subsets of the space X, where $n \geq 1$.

Definition. We say that $A_1, \ldots, A_n$ are in *n-ary contact* if the set-theoretical intersection of $A_1, \ldots, A_n$ is non-empty.

Denote this relation by $\mathcal{C}_n^{\mathbb{T}}$. Then the definition says:

$$<A_1, \ldots, A_n> \in \mathcal{C}_n^{\mathbb{T}} \qquad \textit{iff} \qquad A_1 \cap \ldots \cap A_n \neq \emptyset$$

Apparently $A_1, \ldots, A_n$ are in *n-ary contact* only if $A_1, \ldots, A_n$ are all non-empty.

Consider an arbitrary set $S \subseteq \mathcal{P}(X)$. Whenever we use $\mathcal{C}_n^{\mathbb{T}}$ as a relation on the field S then we naturally mean the restriction of $\mathcal{C}_n^{\mathbb{T}}$ to the field S, namely:

$$\mathcal{C}_n^{\mathbb{T}} \cap (\underbrace{S \times \ldots \times S}_{n})$$

We will use *n-ary contact* always in the context of $\mathbb{R}^m$ hence will omit the topological space superscript and simply write $\mathcal{C}_n$ instead.

2 n-graphs, Contact n-frames and Contact n-graphs

This section studies Kripke frames with specific properties. Non-formally, one can think those intended properties impose in some sense *"finite" contact* behaviour to the considered Kripke frames. *Contact*, from relation symbols interpretation perspective, that is the n-ary relations behave in a way like the standard notion of n-ary contact. Furthermore, those interpretations will have a *finite* character, that is, intuitively, for some $n < \omega$ then every k-ary relation (k-ary relation symbol interpretation) for $k \geq n$ will not add any additional information compared to what already available by the n-ary relation.

An essential commodity of the *finite* frames from the mentioned class of Kripke frames will be that they can be *"encoded"* into convenient graph structure and thus manipulated by graph operations and properties.

2.1 Contact n-frames

Definition 2.1.1. Given Kripke frame $\mathcal{F} = <S, I>$ for $L_{\mathcal{R}}$. Consider $k \geq 1$. Denote by R_k the k-ary relation $I(P)$, where P is the k-ary relation symbol in $\mathcal{R}$.
We call $\mathcal{F}$ a *contact n-frame* for $L_{\mathcal{R}}$ if:

- **(a)** $<s_1, \ldots, s_k> \in R_k$ implies for every $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ is satisfied $<s_{\sigma(1)}, \ldots, s_{\sigma(k)}> \in R_k$
- **(b)** $<s_1, s_1, s_2, \ldots, s_k> \in R_{k+1}$ if and only if $<s_1, s_2, \ldots, s_k> \in R_k$
- **(c)** $<s, s> \in R_2$
- **(d)** For n are satisfied:
  - **(d.1)** They exist distinct $s_1, \ldots, s_n$ such that $<s_1, \ldots, s_n> \in R_n$
  - **(d.2)** For every $k \geq 1$, for every $s_1, \ldots, s_k$ such that $<s_1, \ldots, s_k> \in R_k$ then $\overline{\overline{\{s_1, \ldots, s_k\}}} \leq n$

Remark 2.1.1. By **(c)** and **(b)** we imply that $R_1 = S$.

Now, due to Remark 2.1.1 we adopt the following:

Notation. Contact n-frames will be denoted by:

$$\mathcal{F} = <S, R_2, \ldots, R_n, \ldots>$$

that is contact n-frame $\mathcal{F}$ with carrier S and interpretation of:

- The unary relation symbol as S
- The k-ary relation symbol for $k \geq 2$ as R_k

and explicitly pointing out the n-ary relation R_n.

Consider contact n-frame $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$, $W \neq \emptyset$. By Definition 2.1.1 point **(a)** for every tuple $<w_1, \ldots, w_k>$, such that $<w_1, \ldots, w_k> \in R_k$, an arbitrary permutation of $w_1, \ldots, w_k$ is also in R_k. By this the following definition is correct:

- $R_k^q \leftrightharpoons \Big\{ \{w_1, \ldots, w_k\} \;\Big|\; <w_1, \ldots, w_k> \in R_k \;\&\; \overline{\overline{\{w_1, \ldots, w_k\}}} = q \Big\},\; k \geq 2$

Observation 2.1.1. *The following are satisfied:*

- *Whenever* $q > k$ *then*
$$R_k^q = \emptyset$$
- *Whenever* $1 \leq q \leq k_1, k_2$ *then*
$$R_{k_1}^q = R_{k_2}^q$$

Proof. Should $q > k$ then immediately by the definition of R_k^q.

The case $k_1 = k_2$ is trivially satisfied.

Without loss of generality let $k_1 < k_2$. Let $q \leq k_1 < k_2$. Consider $w_1, \ldots, w_q$ distinct. By definition of R_k^q and **(a)**:

$$\{w_1, \ldots, w_q\} \;\in\; R_{k_1}^q$$

iff

$$\{w_1^{'}, \ldots, w_{k_1}^{'}\} = \{w_1, \ldots, w_q\} \quad \text{and} \quad <w_1^{'}, \ldots, w_{k_1}^{'}> \in R_{k_1}$$

By **(b)** $k_2 - k_1$ times we obtain:
iff

$$<\underbrace{w_1^{'}, \ldots, w_1^{'}}_{k_2 - k_1}, w_1^{'}, \ldots, w_{k_1}^{'}> \in R_{k_2} \quad \text{and} \quad \{w_1^{'}, \ldots, w_{k_1}^{'}\} = \{w_1, \ldots, w_q\}$$

which by R_k^q definition and **(a)**:
iff

$$\{w_1, \ldots, w_q\} \in R_{k_2}^q$$

□

2.2 n-graphs. Contact n-graphs

2.2.1 n-graphs

Definition 2.2.1. A graph is called *n-graph* for a positive natural n if its set of vertices can be split into two disjoint sets W and V such that:

- W is a non-empty set.
- Every edge of the graph is incident on one vertex from W and the other from V.
- Every vertex from V is incident on at least 2 edges.
- Every vertex from V is incident on not more than n edges.
- If $n > 1$ then exists vertex from V incident on exactly n edges. Otherwise, in the case when $n = 1$, then V is empty.

Definition. Given an n-graph, let W and V be the split of the vertices of the graph in accordance to Definition 2.2.1. Then:

- We call the elements of W *terminal* vertices.
- We call the elements of V (if any) *conector* vertices.
- We call a connector vertex *k-vertex* if incident on exactly k edges.

Lemma 2.2.1. *Any non-empty n-graph is bipartite. An appropriate partitioning is the disjoint sets of the terminal and the connector vertices.*

Proof. We colour every *terminal* vertex (let's say) in black and every *connector* vertex in white. Then by definition of n-graph every edge naturally satisfies the condition for a bipartite graph, namely, to have its incident vertices in different colour. $\square$

Lemma 2.2.1 allows us to define the following:

Notation. We denote an n-graph by $G = (W, V, E)$ where:

- W is the set of *terminal* vertices
- V is the set of *connector* vertices
- E is the set of edges

Remark. By Definition 2.2.1, 1-graph is empty (that is it has no edges).

Having an n-graph $G = (W, V, E)$ then within the standard notion of a graph it is $G = (W \cup V, E)$.

Furthermore, again by Definition 2.2.1, every k-vertex is incident on exactly k edges, where $k \geq 2$.

2.2.2 Contact n-graphs

Definition. Given graph $G = (V, E)$ and vertex $v \in V$. By $Adj_G(v)$ we denote all the adjacent vertices of vertex v in G:

$$Adj_G(v) \leftrightharpoons \Big\{ \; v' \mid v \in V \;\&\; (v, v') \in E \; \Big\}$$

Definition 2.2.2. *Contact* n-graph G is a graph satisfying the following conditions:

(0) G is n-graph

(1) G is simple

(2) If any connector vertices v' and v'' satisfy $Adj_G(v') \subseteq Adj_G(v'')$ then $v' = v''$

By the definition of contact n-graph we easily make the following observation.

Observation 2.2.1. *Acyclic n-graph is contact n-graph.*

Proof. The graph is n-graph so Definition 2.2.2 **(0)** is satisfied and **(1)** is true by the graph being acyclic.

For Definition 2.2.2 **(2)**, denote the acyclic n-graph by $G = (W, V, E)$. Consider $v', v'' \in V$ such that $Adj_G(v') \subseteq Adj_G(v'')$. By Definition 2.2.1 $\overline{\overline{Adj_G(v')}} \geq 2$ hence there are distinct vertices w_1 and w_2 from W such that $\{w_1, w_2\} \subseteq Adj_G(v') \subseteq Adj_G(v'')$. Assume $v' \neq v''$. Then v'-w_1-v''-w_2-v' is a circuit but the graph is acyclic hence a contradiction. $\square$

2.3 n-frames to n-graphs correspondence

Definition 2.3.1. Let $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$ be a finite contact n-frame. Denote by $G = (W, V, E)$ a graph such that W and V are disjoint sets of vertices the union of which is all vertices of G, V is non-necessarily non-empty and E is the set of edges of G.

The graph $G = (W, V, E)$ is called *induced by* $\mathcal{F}$ (denoted by $\mathcal{F} \longrightarrow G$) if:

- W is the carrier of $\mathcal{F}$
- $V \subseteq \mathcal{P}(W)$ such that:

  (g1) $v \in V$ *iff* exists $k \geq 2$ such that:

  $$v \in R_n^k \quad \text{and} \quad \text{for all } i \text{ and for all } b \text{ if } b \in R_n^i \text{ and } v \subseteq b \text{ then } v = b$$

- For the set of edges E it holds:

  (g2) $E = \Big\{ \{w, v\} \mid w \in W \ \& \ v \in V \ \& \ w \in v \Big\}$

Remark 2.3.1. By **(g2)** we immediately imply:

(g3) For any $v \in V$:

$$Adj_G(v) = v$$

Claim 2.3.1. *$\mathcal{F}$ is finite contact n-frame and G is the graph induced by $\mathcal{F}$. Then G is a contact n-graph.*

Proof. Denote $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$ and $G = (W, V, E)$. Will demonstrate all the conditions of Definition 2.2.2 of contact n-graph one by one.

(0): G is n-graph:

- $W \neq \emptyset$ by definition of Kripke frame.
- $e \in E$ then immediately by Definition 2.3.1 **(g2)** e is incident on a vertex from W and a vertex from V.
- Consider $v \in V$. Then by Definition 2.3.1 **(g1)** there is $k \geq 2$ such that $v \in R_n^k$. Then by Remark 2.3.1 **(g3)** $\overline{\overline{Adj_G(v)}} = k$. Remark that by Definition 2.3.1 **(g2)** E does not contain parallel or self-loop edges. Therefore v is incident on exactly k edges. On one hand, by $k \geq 2$, this means v is incident on at least 2 edges. On the other, by Observation 2.1.1, $k \leq n$ which means v is incident on not more than n edges.
- Consider $n = 1$. Assume it exists $v \in V$. By Definition 2.3.1 **(g1)** then exists $k \geq 2$ such that $v \in R_n^k$. Apparently then $k > n$ and hence, by Observation 2.1.1, R_n^k is empty which is a contradiction. Therefore V is empty.
  Consider $n > 1$. Assume there is no $v \in V$ such that $\overline{\overline{v}} = n$.
  By Definition 2.1.1 **(d.1)** of contact connected frame we imply $R_n^n \neq \emptyset$. By that and **(g1)** for every $a \in R_n^n$ it exists i and exists b such that $b \in R_n^i$ and $a \subseteq b$, and $a \neq b$. For a concrete a take arbitrary witnessess i and b. By $a \in R_n^n$ we imply $\overline{\overline{a}} = n$. $b \in R_n^i$ hence $\overline{\overline{b}} = i$. $a \subseteq b$ thus $n \leq i$. Furthermore, $a \neq b$ so $n < i$. Therefore, by Observation 2.1.1, $R_n^i = \emptyset$,

but this contradicts $b \in R_n^i$. Hence our assumption is wrong. This means there is $v \in V$ such that $\overline{\overline{v}} = n$. By Remark 2.3.1 **(g3)**: $\overline{\overline{Adj_G(v)}} = n$. Recall that by Definition 2.3.1 **(g2)** there are no parallel edges hence v is incident on exactly n edges.

(1): Again, by Definition 2.3.1 **(g2)** no self-loop or parallel edges are possible.

(2): Let $v', v'' \in V$ and $Adj_G(v') \subseteq Adj_G(v'')$. Then by Remark 2.3.1 **(g3)** $v' \subseteq v''$. By Definition 2.3.1 **(g1)** they exist k_1 and k_2, both greater or equal 2, such that $v' \in R_n^{k_1}$ and $v'' \in R_n^{k_2}$. Now by those, $v' \subseteq v''$ and again Definition 2.3.1 **(g1)** we imply $v' = v''$. ☐

Claim 2.3.2. *For any $k \geq 2$ the following statements are equivalent:*

(i) $<w_1, \ldots, w_k> \in R_k$

(ii) Either $w_1 = \ldots = w_k$ *or exists* $v \in V$ *such that* $\{w_1, \ldots, w_k\} \subseteq Adj_G(v)$

Proof.

- From *(i)* to *(ii)*

Let $<w_1, \ldots, w_k> \in R_k$ and let $w_1, \ldots, w_k$ be not all equal. Denote $v = \{w_1, \ldots, w_k\}$. Apparently then for $k' = \overline{\overline{v}}$ we have $k' \geq 2$ and $v \in R_n^{k'}$. Consider the set:

$$I = \left\{ \; i \;\middle|\; i \leq n \quad \& \quad (\exists b \in R_n^i)(v \subseteq b \;\&\; v \neq b) \; \right\}$$

In case I is empty then, by Observation 2.1.1 (namely $R_n^i = \emptyset$ whenever $i > n$), we imply that for every i and for every b then if both $b \in R_n^i$ and $v \subseteq b$ are satisfied then $v = b$. Having this, by Definition 2.3.1 **(g1)**, $k' \geq 2$ and $v \in R_n^{k'}$ we obtain $v \in V$. Then, by Remark 2.3.1 **(g3)**, trivially v is a witness to *(ii)*.
Now let I be non-empty.
By definition I is finite hence it has a maximal element. Denote it by i_0. $i_0 \in I$ then it exists b such that $b \in R_n^{i_0}$, $v \subseteq b$ and $v \neq b$. Consider b_0 a witness to that existence.
Assume $b_0 \notin V$.
$b_0 \in R_n^{i_0}$ and $b_0 \notin V$ then, due to Definition 2.3.1 **(g1)**, they exist i and b such that $b \in R_n^i$, $b_0 \subseteq b$ and $b_0 \neq b$. Take witnesses i_1 and b_1.
Now, by $b_0 \subseteq b_1$ we have $\overline{\overline{b_0}} \leq \overline{\overline{b_1}}$. Furthermore, by $b_0 \neq b_1$, then $\overline{\overline{b_0}} < \overline{\overline{b_1}}$. Trivially, by $b_0 \in R_n^{i_0}$ and $b_1 \in R_n^{i_1}$, we have $\overline{\overline{b_0}} = i_0$ and $\overline{\overline{b_1}} = i_1$. Eventually $i_0 < i_1$.
On the other hand, by $b_0 \subseteq b_1$ we have $v \subseteq b_1$. Furthermore, $v \neq b_0$ hence $v \neq b_1$. Eventually $i_1 \in I$ and by i_0 maximal $i_1 \leq i_0$. This is a contradiction.
Therefore our assumption is wrong thus $b_0 \in V$. By $v \subseteq b_0$ and Remark 2.3.1 **(g3)** b_0 is a witness to *(ii)*.

- From *(ii)* to *(i)*

Let $\{w_1, \ldots, w_k\} \subseteq \mathcal{P}(W)$.
If $w_1 = \ldots = w_k$ then by first applying Definition 2.1.1 **(c)** and then **(b)** $k-1$ times consequently we obtain $<w_1, \ldots, w_1> \in R_k$ and by this satisfying *(i)*.
Now, denote $v = \{w_1, \ldots, w_k\}$ and consider $\overline{\overline{v}} \geq 2$. Let $v' \in V$ such that

$v \subseteq Adj_G(v')$ as per *(ii)*. By Remark 2.3.1 **(g3)** the latter is equivalent with $v \subseteq v'$. Let then $\{w_1, \dots, w_k\} = \{w'_1, \dots, w'_{k_1}\}$, where $w'_1, \dots, w'_{k_1}$ are all distinct and $2 \leq k_1 \leq k$. Thus $\overline{\overline{v}} = k_1$. Let $k_2 = \overline{\overline{v'}}$. Hence $k_1 \leq k_2$. Consider then v' the following way:

$$v' = \{w'_1, \dots, w'_{k_1}, w'_{k_1+1}, \dots, w'_{k_2}\}$$

$v' \in V$ then by Definition 2.3.1 **(g1)** and $k_2 = \overline{\overline{v'}}$ we have $v' \in R_n^{k_2}$. By Observation 2.1.1 $R_n^{k_2} = R_{k_2}^{k_2}$ hence $<w'_1, \dots, w'_{k_2}> \in R_{k_2}$. By appropriately applying Definition 2.1.1 **(a)** we obtain:

$$<\underbrace{w'_1, \dots, w'_1}_{k_2 - k_1}, w'_1, \dots, w'_{k_1}> \in R_{k_2}$$

By applying Definition 2.1.1 **(b)** $k_2 - k_1$ times:

$$<w'_1, \dots, w'_{k_1}> \in R_{k_1}$$

Now by appropriately applying (finitely many times) Definition 2.1.1 points **(a)** and **(b)** we obtain:

$$<w_1, \dots, w_k> \in R_k$$

$\square$

Definition 2.3.2. Given contact n-graph $G = (W, V, E)$.
The structure $\mathcal{F} = <W, R_2, \dots, R_n, \dots>$ is called the Kripke frame *induced* by G (denoted by $G \longrightarrow \mathcal{F}$) if satisfied:

- The carrier of the Kripke frame is the set of the *terminal* vertices W of G
- The unary relation symbol of $L_{\mathcal{R}}$ is interpreted as W
- For every $k \geq 2$ the k-ary relation symbol of $L_{\mathcal{R}}$ is interpreted as the k-ary relation R_k defined as:

  (r) $<w_1, \dots, w_k> \in R_k$
  iff
  either $w_1 = \dots = w_k$ or exists $v \in V$ such that

  $$\{w_1, \dots, w_k\} \subseteq Adj_G(v)$$

Claim 2.3.3. *G is contact n-graph and $\mathcal{F}$ is the Kripke frame induced by G. Then $\mathcal{F}$ is contact n-frame.*

Proof. Denote $G = (W, V, E)$ and $\mathcal{F} = <W, R_2, \dots, R_n, \dots>$. We demonstrate the conditions in Definition 2.1.1 of a contact n-frame.

(a): Let $<w_1, \dots, w_k> \in R_k$. Consider arbitrary $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. Should $w_1 = \dots = w_k$ then trivially $<w_1, \dots, w_k> = <w_{\sigma(1)}, \dots, w_{\sigma(k)}>$ hence $<w_{\sigma(1)}, \dots, w_{\sigma(k)}> \in R_k$.
Otherwise, obviously $\{w_{\sigma(1)}, \dots, w_{\sigma(k)}\} \subseteq \{w_1, \dots, w_k\}$. By Definition 2.3.2 **(r)** exists $v \in V$ such that $\{w_1, \dots, w_k\} \subseteq Adj_G(v)$ thus $\{w_{\sigma(1)}, \dots, w_{\sigma(k)}\} \subseteq Adj_G(v)$. Hence again by Definition 2.3.2 **(r)** we imply $<w_{\sigma(1)}, \dots, w_{\sigma(k)}> \in R_k$.

(b): Should $w_1 = \ldots = w_k$ then by Definition 2.3.2 **(r)** trivially both $<w_1, w_1, w_2, \ldots, w_k> \in R_{k+1}$ and $<w_1, \ldots, w_k> \in R_k$ are satisfied.
Otherwise by Definition 2.3.2 **(r)**: $<w_1, w_1, w_2, \ldots, w_k> \in R_{k+1}$
iff
$\{w_1, \ldots, w_k\} \subseteq Adj_G(v)$ for some $v \in V$, by which and again by Definition 2.3.2 **(r)** we have:
iff
$<w_1, \ldots, w_k> \in R_k$.

(c): Follows immediately by Definition 2.3.2 **(r)**.

(d): The case when $n = 1$ hence, by $R_1 = W$, we have Definition 2.1.1 **(d.1)** trivially satisfied. Furthermore, by Definition 2.2.1, $V = \emptyset$ and thus, by Definition 2.3.2 **(r)**, we obtain $<w_1, \ldots, w_k> \in R_k$ *iff* $w_1 = \ldots = w_k$. The latter trivially satisfies Definition 2.1.1 **(d.2)**.
Now, consider $n > 1$. By Definition 2.2.1 of n-graph there is $v^n \in V$ such that v^n is n-vertex in G. Considering Definition 2.2.2 **(1)** clearly $\overline{\overline{Adj_G(v^n)}} = n$. Let then $Adj_G(v^n) = \{w_1, \ldots, w_n\}$ where $w_1, \ldots, w_n$ are distinct. Hence, by Definition 2.3.2 **(r)**, we obtain $<w_1, \ldots, w_n> \in R_n$ which satisfies Definition 2.1.1 **(d.1)**.
Consider arbitrary $k \geq 2$. Let $<w_1, \ldots, w_k> \in R_k$. By Definition 2.3.2 **(r)** either $\overline{\overline{\{w_1, \ldots, w_k\}}} = 1$ or there is $v' \in V$ such that $\{w_1, \ldots, w_k\} \subseteq Adj_G(v')$. By Definition 2.2.1 of n-graph we have $\overline{\overline{Adj_G(v')}} \leq n$. Eventually $\overline{\overline{\{w_1, \ldots, w_k\}}} \leq n$ which satisfies Definition 2.1.1 **(d.2)**.

□

Remark. Consider finite contact n-frame $\mathcal{F}$.

Let G be the *induced* graph by $\mathcal{F}$. Then by Claim 2.3.1 G is contact n-graph.

Let then $\mathcal{F}'$ be the Kripke frame *induced* by G. By Claim 2.3.3 $\mathcal{F}'$ is contact n-frame.

The relationship between $\mathcal{F}$ and $\mathcal{F}'$ is given by the following claim.

Claim 2.3.4. $\mathcal{F} = \mathcal{F}'$

Proof. By Definition 2.3.1 and then by Definition 2.3.2 the carrier of $\mathcal{F}$ is preserved in $\mathcal{F}'$.

By Claim 2.3.2 and Definition 2.3.2 **(r)** for every $k \geq 2$ the k-ary relations of $\mathcal{F}$ and $\mathcal{F}'$ are equal. □

Remark. Consider contact n-graph G.

Let $\mathcal{F}$ be the Kripke frame *induced* by G. By Claim 2.3.3 $\mathcal{F}$ is contact n-frame.

Let then G' be the graph *induced* by $\mathcal{F}$. By Claim 2.3.1 G' is contact n-graph.

The relationship between G and G' is given by the following claim.

Claim 2.3.5. $G \cong G'$

Proof. Denote $G = (W, V, E)$. By Definition 2.3.2 and then by Definition 2.3.1 the *terminal* vertices of G are preserved in G'. Thus denote $G' = (W, V', E')$. Denote $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$ the contact n-frame induced by G.

Adopt the following well defined mappings:

- $f_W(w) = w$
- $f_V(v) = Adj_G(v)$

where $Dom(f_W) = W$ and $Dom(f_V) = V$. Trivially $f_W \cap f_V = \emptyset$. Will demonstrate:

- $f_W : W \rightarrowtail\!\!\!\!\twoheadrightarrow W$
- $f_V : V \rightarrowtail\!\!\!\!\twoheadrightarrow V'$
- $(w, v) \in E$ *iff* $(f_W(w), f_V(v)) \in E'$

Having these satisfied then $f = f_W \cup f_V$ is isomorphism between G and G'.

- $f_W : W \rightarrowtail\!\!\!\!\twoheadrightarrow W$

Trivially satisfied by definition of f_W.

- $f_V : V \rightarrowtail\!\!\!\!\twoheadrightarrow V'$

. f_V is well defined. In particulars if $v \in V$ then $f_V(v) \in V'$:

Let $Adj_G(v) = \{w_1, \ldots, w_k\}$ such that $\overline{\overline{\{w_1, \ldots, w_k\}}} = k, k \geq 2$. By Definition 2.3.2 **(r)** we imply $<w_1, \ldots, w_k> \in R_k$ hence $f_V(v) \in R^k_k$ and by Observation 2.1.1 it means $f_V(v) \in R^k_n$.
Let i and b be such that $b \in R^i_n$ and $f_V(v) \subseteq b$. We have $\overline{\overline{f_V(v)}} = k$. Also $\overline{\overline{b}} = i$ by $b \in R^i_n$. Thus $k \leq i$ by $f_V(v) \subseteq b$. Let then $b = \{w_1, \ldots, w_k, w_{k+1}, \ldots, w_i\}$. By Observation 2.1.1 $R^i_n = R^i_i$ hence $b \in R^i_i$ which gives $<w_1, \ldots, w_i> \in R_i$. Now by Definition 2.3.2 **(r)** (and $i \geq k \geq 2$) exists $v' \in V$ such that $b = \{w_1, \ldots, w_i\} \subseteq Adj_G(v')$ hence $Adj_G(v) \subseteq Adj_G(v')$. By the latter and Definition 2.2.2 **(2)** we obtain $v = v'$. All those give us:

$$f_V(v) = Adj_G(v) \subseteq b \subseteq Adj_G(v') = Adj_G(v) = f_V(v)$$

Eventually $f_V(v) = b$ hence the right side of Definition 2.3.1 **(g1)** is satisfied thus $f_V(v) \in V'$.

. f_V is injection:

For any $v_1, v_2 \in V$ should $f_V(v_1) = f_V(v_2)$ then by Definition 2.2.2 **(2)** we imply $v_1 = v_2$.

. f_V is surjection:

Consider $v' \in V'$. By Definition 2.3.1 **(g1)** for G' we have k, $2 \leq k \leq n$, such that $v' \in R^k_n$. By this on one hand $v' = \{w_1, \ldots, w_k\}$, $\overline{\overline{v'}} = k$, and, on the other, by Observation 2.1.1 $v' \in R^k_k$ thus $<w_1, \ldots, w_k> \in R_k$. Then by Definition 2.3.2 **(r)** for G exists $v \in V$ such that $v' \subseteq Adj_G(v) = f_V(v)$. Furthermore, $f_V(v) \in V'$ as demonstrated in definition correctness of f_V. Now by Definition 2.3.1 **(g1)** for G' there is i such that $f_V(v) \in R^i_n$. Eventually:

$$v' \subseteq f_V(v), \quad v' \in R^k_n, \quad f_V(v) \in R^i_n$$

That result by Definition 2.3.1 **(g1)** gives $v' = f_V(v)$.

- $(w, v) \in E$ *iff* $(f_W(w), f_V(v)) \in E'$

$$(w, v) \in E$$

by G

$$\textit{iff} \quad w \in Adj_G(v)$$

by f_W and f_V

$$\textit{iff} \quad f_W(w) \in f_V(v)$$

by f_W, f_V bijections and by Definition 2.3.1 **(g2)**

$$\textit{iff} \quad (f_W(w), f_V(v)) \in E'$$

□

Remark. By Claim 2.3.1 given a finite contact n-frame we associate to it contact n-graph, namely, the *induced* graph by the contact n-frame. By Claim 2.3.3 given a contact n-graph we associate to it a finite contact n-frame, namely, the *induced* frame by the contact n-graph. Remark that by Claim 2.3.4 and Claim 2.3.5 we can think that there is one-to-one correspondence between the class of finite contact n-frames and the class of contact n-graphs up to isomorphism.

2.4 Few properties of n-graphs and n-frames

Definition 2.4.1. We call a contact n-frame $\mathcal{F}$ *connected* if it holds:

(e) $\mathcal{F} \Vdash (\neg(x \equiv 0) \wedge \neg(-x \equiv 0)) \implies P(x, -x)$

Claim 2.4.1. *A finite contact n-frame is connected* iff *its induced graph is connected.*

Proof. Let $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$ be a finite contact n-frame. Denote $G = (W, V, E)$ the induced contact n-graph by $\mathcal{F}$.

- From left to right:

Assume G is not connected graph.
Let $U_1, \ldots, U_l$ be the partitioning of the vertices $W \cup V$ into the components of the graph G. By G not connected then $l \geq 2$.
Assume $U_i \cap W = \emptyset$ for some i, $1 \leq i \leq l$. Then $U_i \subseteq V$. Let $v \in U_i$, where v is arbitrary element of the non-empty U_i. By Definition 2.3.1 **(g1)** exists $k \geq 2$ such that $v \in R_n^k$ thus $\overline{\overline{v}} = k$. By those, Definition 2.3.1 **(g2)** and Remark 2.3.1 exists $w_0 \in W$ such that $w_0 \in Adj_G(v)$. The induced graph by U_i is component for G and $v \in U_i$ hence $Adj_G(v) \subseteq U_i$. It follows that $w_0 \in U_i$, thus $U_i \cap W \neq \emptyset$ which is a contradiction. Therefore for every i, $1 \leq i \leq l$:

$$U_i \cap W \neq \emptyset$$

Now assume $W \subseteq U_i$ for some i, $1 \leq i \leq l$. Then for every j, $1 \leq j \leq l$ and $i \neq j$ by $U_i \cap U_j = \emptyset$ will have $U_j \cap W = \emptyset$. This is a contradiction with what just demonstrated. Therefore for every i, $1 \leq i \leq l$:

$$U_i \cap W \neq W$$

Consider $W' = U_1 \cap W$. Thus $W' \neq \emptyset$ and $W' \neq W$. Hence $W \setminus W' \neq \emptyset$.
Consider the valuation $\mathcal{V}$ on $\mathcal{F}$ such that $\mathcal{V}(x) = W'$ and is arbitrary for any Boolean variable other than x. Hence for $\mathcal{V}$ is true:

$$\widetilde{\mathcal{V}}(x) = \mathcal{V}(x) = W' \neq \emptyset = \mathcal{V}(0)$$
$$\widetilde{\mathcal{V}}(-x) = W \setminus \mathcal{V}(x) = W \setminus W' \neq \emptyset = \mathcal{V}(0)$$

It follows that:

$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; \neg(x \equiv 0)$$
$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; \neg(-x \equiv 0)$$

Then, by $\mathcal{F}$ connected, hence, satisfying Definition 2.4.1 **(e)** we imply:

$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; P(x, -x)$$

By $\mathcal{V}(x) = W'$ this means exists $w_1 \in W'$ and exists $w_2 \in W \setminus W'$ such that $<w_1, w_2> \in R_2$. By Claim 2.3.2 either $w_1 = w_2$ or exists $v \in V$ such that $\{w_1, w_2\} \subseteq Adj_G(v)$. The former is not possible as long as $W' \cap (W \setminus W') = \emptyset$. Therefore, by the latter, w_1-v-w_2 is a path in G. $w_1 \in W'$ thus $w_1 \in U_1$. U_1 is component for G therefore it follows that $w_2 \in U_1$. Hence $w_2 \in W'$. This is a contradiction with $w_2 \in W \setminus W'$.
As a result our assumption is wrong hence G is connected.

- From right to left:

Consider arbitrary valuation $\mathcal{V}$ on the Kripke frame $\mathcal{F}$. Let the following be satisfied:

$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; \neg(x \equiv 0)$$
$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; \neg(-x \equiv 0)$$

Hence $\mathcal{V}(x) \neq \emptyset$ and $W \setminus \mathcal{V}(x) \neq \emptyset$. Consider then arbitrary $w' \in \mathcal{V}(x)$ and $w'' \in W \setminus \mathcal{V}(x)$. Apparently $w' \neq w''$. By G connected there is a path between w' and w''. By Lemma 2.2.1 and Definition 2.3.1 **(g2)** the path is an alternating sequence of elements between the terminal vertices W and the connector vertices V. Take an arbitrary such path: w'-v_1-w'_1-…-v_r-w'', where $v_1, \ldots, v_r$ are the connector vertices and the others are terminal vertices. By $w' \in \mathcal{V}(x)$, $w'' \in W \setminus \mathcal{V}(x)$ and $\mathcal{V}(x) \cap (W \setminus \mathcal{V}(x)) = \emptyset$ in the path there are subsequent vertices w_1-v-w_2, where $w_1, w_2 \in W$ and $v \in V$, such that $w_1 \in \mathcal{V}(x)$ and $w_2 \in W \setminus \mathcal{V}(x)$. Therefore $\{w_1, w_2\} \subseteq Adj_G(v)$. Hence, by Claim 2.3.2, $<w_1, w_2> \in R_2$, which gives:

$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; P(x, -x)$$

It follows then in all cases:

$$<\mathcal{F}, \mathcal{V}> \;\Vdash\; (\neg(x \equiv 0) \wedge \neg(-x \equiv 0)) \implies P(x, -x)$$

$\mathcal{V}$ was an arbitrary valuation therefore the intended formula **(e)**:

$$(\neg(x \equiv 0) \wedge \neg(-x \equiv 0)) \implies P(x, -x)$$

is valid in $\mathcal{F}$ hence (as per Definition 2.4.1) $\mathcal{F}$ is connected.

□

Claim 2.4.2. *Let $\mathcal{F}$ be a contact n-frame. Then the following statements are equivalent:*

(i) In $\mathcal{F}$ is valid

$$P(x_1, x_2, x_3) \implies (\neg(x_1 \cap x_2 \equiv 0) \vee \neg(x_2 \cap x_3 \equiv 0) \vee \neg(x_1 \cap x_3 \equiv 0))$$

(ii) $\mathcal{F}$ is contact n-frame for $n \leq 2$

Proof.

- From *(i)* to *(ii)*

Consider arbitrary $<s_1, s_2, s_3> \in R_3$ and valuation $\mathcal{V}$ on $\mathcal{F}$ such that:

$$\mathcal{V}(x) = \begin{cases} \{s_i\} & x = x_i \\ \text{arbitrary} & x \notin \{x_1, x_2, x_3\} \end{cases}$$

Then it is true $<\mathcal{F}, \mathcal{V}> \Vdash P(x_1, x_2, x_3)$, by which and *(i)* we imply

$$<\mathcal{F}, \mathcal{V}> \Vdash \neg(x_1 \cap x_2 \equiv 0) \vee \neg(x_2 \cap x_3 \equiv 0) \vee \neg(x_1 \cap x_3 \equiv 0)$$

iff

$$\widetilde{\mathcal{V}}(x_1) \cap \widetilde{\mathcal{V}}(x_2) \neq \emptyset \quad \text{or} \quad \widetilde{\mathcal{V}}(x_2) \cap \widetilde{\mathcal{V}}(x_3) \neq \emptyset \quad \text{or} \quad \widetilde{\mathcal{V}}(x_1) \cap \widetilde{\mathcal{V}}(x_3) \neq \emptyset$$

iff

$$s_1 = s_2 \quad \text{or} \quad s_2 = s_3 \quad \text{or} \quad s_1 = s_3$$

Therefore for arbitrary $<s_1, s_2, s_3> \in R_3$ is satisfied $\overline{\overline{\{s_1, s_2, s_3\}}} < 3$.
Now, assume $\mathcal{F}$ is a contact n-frame for $n \geq 3$. Take a witness $<s'_1, \ldots, s'_n> \in R_n$, with $\overline{\overline{\{s'_1, \ldots, s'_n\}}} = n$. Thus $\overline{\overline{\{s'_1, s'_2, s'_3\}}} = 3$. Then by Definition 2.1.1 **(a)** we imply $<s'_1, \ldots, s'_1, s'_2, s'_3> \in R_n$. Consequently, by Definition 2.1.1 **(b)** applied $n-3$ times, we obtain $<s'_1, s'_2, s'_3> \in R_3$. Nevertheless we've demonstrated the latter implies $\overline{\overline{\{s'_1, s'_2, s'_3\}}} < 3$ hence a contradiction.

- From *(ii)* to *(i)*

Consider $\mathcal{V}$ be an arbitrary valuation on $\mathcal{F}$. Let $<\mathcal{F}, \mathcal{V}> \Vdash P(x_1, x_2, x_3)$. Then there are $s_1 \in \widetilde{\mathcal{V}}(x_1)$, $s_2 \in \widetilde{\mathcal{V}}(x_2)$, $s_3 \in \widetilde{\mathcal{V}}(x_3)$, such that $<s_1, s_2, s_3> \in R_3$. By *(ii)*: $\overline{\overline{\{s_1, s_2, s_3\}}} \leq 2$. Without loss of generality, let then $s_1 = s_2$. It follows that $\widetilde{\mathcal{V}}(x_1) \cap \widetilde{\mathcal{V}}(x_2) \neq \emptyset$, by which $<\mathcal{F}, \mathcal{V}> \Vdash \neg(x_1 \cap x_2 \equiv 0)$ and thus $<\mathcal{F}, \mathcal{V}> \Vdash \neg(x_1 \cap x_2 \equiv 0) \vee \neg(x_2 \cap x_3 \equiv 0) \vee \neg(x_1 \cap x_3 \equiv 0)$. Eventually:

$$<\mathcal{F}, \mathcal{V}> \Vdash P(x_1, x_2, x_3) \implies (\neg(x_1 \cap x_2 \equiv 0) \vee \neg(x_2 \cap x_3 \equiv 0) \vee \neg(x_1 \cap x_3 \equiv 0))$$

The valuation $\mathcal{V}$ was arbitrary therefore *(i)* is satisfied.

$\square$

3 Kripke frames with contact semantics in $\mathbb{R}^m$

In this section for given *finite connected acyclic* n-graph (hence, by Observation 2.2.1, also being a *contact* n-graph) will elaborate on procedures for obtaining a Kripke frame corresponding to the graph for which:

- The carrier of the frame will be a *finite* set such that:
  - All its elements will either be *polytopes* or *regular closed* sets of $\mathbb{R}^m$.
  - Any two distinct elements of the set may have points in common, if any, only some of their boundary points (in short, the intersection of any two distinct elements of the set will have *empty interior*).
  - The union of all elements of the set will be $\mathbb{R}^m$.
- The interpretation of the k-ary relation symbol of $L_{\mathcal{R}}$ will be the standard *k-ary contact relation* (see Section 1.7).
- The Kripke frame will *correspond* to the n-graph, that is, it will be isomorphic to the n-frame induced by that (contact) n-graph.

Non formally, these procedures will "partition" $\mathbb{R}^m$ (that is, any two distinct elements/subsets will have empty *interior* of their intersection) in such a way that the elements/subsets of that partitioning will map one-to-one with the set of the *terminal* vertices of the n-graph. Furthermore, any k elements will have a non-empty intersection if and only if their corresponding *terminal* vertices of the n-graph be adjacent to common *connector* vertex. Having those properties satisfied, we will see such a partitioning used as a carrier of a Kripke frame for which the interpretation of the relation symbols of $L_{\mathcal{R}}$ is the standard *contact relation* (for the corresponding arity) then this frame will be isomorphic to the induced by the n-graph (contact) n-frame. Last but not the least, as mentioned, the elements of this partitioning will all either be *polytopes* or *regular closed* sets of $\mathbb{R}^m$. The important is that they will effectively be the atoms of a finite Boolean subalgebra of either the Boolean algebra of the *polytopes* or the *regular closed sets.* Therefore these procedures will allow us to "build" (finite) Kripke frames with carriers the atoms of Boolean subalgebras of *polytopes* or *regular closed* sets of $\mathbb{R}^m$ and interpretation the standard contact relations which are "corresponding" to given arbitrary (finite acyclic) n-graphs. Such Kripke frames in the later sections will be used as a utility for elaborating on witnesses to particular intended classes of Boolean frames.

The procedures and results to be demonstrated will actually be valid if applied on any *connected regular closed* subset of $\mathbb{R}^m$ instead of $\mathbb{R}^m$ itself.

3.1 Formal approach for polytopes of $\mathbb{R}^m$, $m \geq 2$

3.1.1 Procedure for polytopes of $\mathbb{R}^2$

When saying an *angular region* we mean the *closure* of the section of the real plain $\mathbb{R}^2$ enclosed between two distinct rays having common endpoint. That common endpoint will be called a *corner* point. Remark that a triangle also is an angular region.

Given a subset of $\mathbb{R}^2$ and a *terminal* vertex w. We say this set is *coloured* in w if the elements of the set are marked as w in an appropriate way. As an example, one can consider the Cartesian product of the given set and the singleton of w. At any stage of the procedure any element of $\mathbb{R}^2$ will be coloured in some of the *terminal* vertices.

Following we define a procedure which for given *finite connected acyclic n*-graph will produce a *finite* set of *polytopes* of $\mathbb{R}^2$ such that:

- Any two distinct elements of the set will have empty *interior* of their intersection.
- The union of the elements of the set will be $\mathbb{R}^2$.
- The set will be one-to-one mapped to the set of the *terminal* vertices of the n-graph.
- Any k elements of the set will have a non-empty intersection if and only if their corresponding *terminal* vertices of the n-graph have common adjacent *connector* vertex.

The procedure considers the *acyclic* n-graph as a *rooted* tree with a root some of the *terminal* vertices. Let us call this tree the n-graph rooted tree. Each (recursive) step of the procedure works on particular distinct *terminal* vertex and examines the sub-tree of the n-graph rooted tree with a root of the sub-tree that *terminal* vertex.

The procedure takes as an input a *terminal* vertex and an *angular region.* The angular region is assumed having already been coloured in the *terminal* vertex. First, the procedure associates distinct points of the *interior* of the angular region with each of the direct *connector* descendants of the input *terminal* vertex. Let us call each such point a *connector* point. Then each *connector* point has corresponding to it *connector* vertex. For each of those *connector* points and their associated *connector* vertices:

- The procedure takes the direct *terminal* descendants of the *connector* vertex.
- For each such *terminal* vertex it associates an angular region having as a *corner* point the *connector* point.
- Every such angular region is a subset of the *interior* of the input angular region.
- Every two such regions have no point in common but their *corner* points, namely, the *connector* point.
- Then the procedure *colours* each of those angular regions in their associated *terminal* vertex.

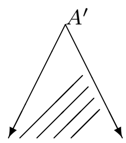


Figure 1: An *angular region* with a *corner* point A'

- Eventually, for each of those angular regions the procedure is applied recursively on the angular region and the *terminal* vertex associated to it.

Remark that, as a result, every two such new angular regions have a point in common only *connector* point being their *corner* point only if their associated *terminal* vertices are direct *terminal* descendants of the *connector* vertex associated to the *connector* point. Otherwise any two new angular regions have no point in common.

Before applying the procedure the entire $\mathbb{R}^2$ is coloured in the root of the n-graph rooted tree. As an initial input then the procedure takes the chosen root of the n-graph rooted tree and an (arbitrary) angular region subset of the coloured already $\mathbb{R}^2$.

Eventually the procedure traverses every vertex of the n-graph rooted tree. Upon completion, to each *terminal* vertex is mapped the polytope defined as the *closure* of the subset of $\mathbb{R}^2$ *coloured* in the *terminal* vertex. Intuitively, the result of the procedure is such that $\mathbb{R}^2$ is "partitioned" into *polytopes* in a way that in some sense being isomorphic to the n-graph. That is, the *connector* points are one to one to the *connector* vertices of the n-graph and the *polytopes* of the "partitioning" are one to one to their corresponding *terminal* vertices. Furthermore, any k *polytopes* have common point some of the *connector* points if and only if their corresponding *terminal* vertices are adjacent to the *connector* vertex associated with the *connector* point. Additionally, it will also become clear that if any k *terminal* vertices are not adjacent to any common *connector* vertex then their corresponding k *polytopes* have empty intersection.

Following is a detailed definition of the procedure.

Assumptions:

- Given $G = (W, V, E)$ finite connected acyclic n-graph, W, V and E non-empty.
- $W = \{w_1, \ldots, w_s\}$ is enumerated, where $\overline{\overline{W}} = s$.
- The finite set of connector vertices V is enumerated properly. By A_v for $v \in V$ we denote A_i for the appropriate index i of v in V such that $1 \leq i \leq \overline{\overline{V}}$ and $v = v_i$.
- We consider G as a *rooted tree* for particularly chosen *root vertex.* Any sub-tree of G will also be considered rooted tree where the root will be clear by the context. All terms then like *predecessor*, *descendant* etc. will be with respect to the current (*rooted*) sub-tree under consideration.

Procedure 3.1. Polytopes of $\mathbb{R}^2$

Input:

- C : *angular region*
- A' : the *corner* point of the *angular region* C
- w' : the *root* of the sub-tree of G being currently traversed, $w' \in W$

Procedure Steps:

1

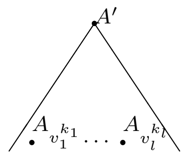
- Consider C being coloured in w'.

②

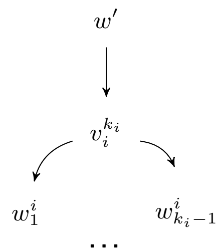
- If w' has no descendants then the current procedure recursive call finishes.
- Otherwise:

Let the direct descendants of w' be $v_1^{k_1}, \ldots, v_l^{k_l}$. By definition of n-graph they are connector vertices. By $v_i^{k_i}$ we denote k_i-vertex.

- For each $v_i^{k_i}$ designate distinct point from C from the *interior* of C. Call it $A_{v_i^{k_i}}$.



Let $w_1^i, \ldots, w_{k_i-1}^i$ be the terminal vertices direct descendants of $v_i^{k_i}$.



- For every w_j^i cut a (non-empty) *angular region* from C and take its closure, denote it by U_j^i, such that:

(i) U_j^i has *corner* vertex $A_{v_i}^{k_i}$

(ii) None of the boundary points of C is in any of U_j^i

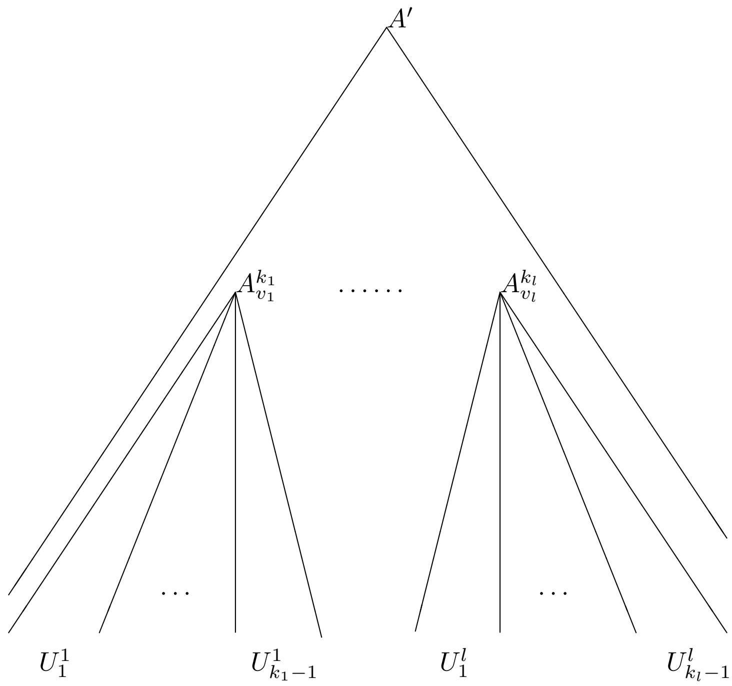
(iii) $A' \notin U_j^i$

(iv) U_j^i is a polytope of $\mathbb{R}^2$ a subset of C

(v) $U_{j_1}^i \cap U_{j_2}^i = \{A_{v_i}^{k_i}\}, \quad j_1 \neq j_2$

(vi) $U_{j_1}^{i_1} \cap U_{j_2}^{i_2} = \emptyset, \quad i_1 \neq i_2$

Remark that condition *(iii)* is direct inference of *(ii)* and *(iv)* follows immediately from what agreed as *angular region.* Both explicitly stated for convenience only.



③

- Colour U_j^i in w_j^i

④

- For every w_j^i apply the procedure (recursively), that is for every i, $1 \leq i \leq l$, and every j, $1 \leq j \leq k_i - 1$:
  - Apply the procedure recursively from ① with the following input:
    * $C := U_j^i$
    * $A' := A_{v_i}^{k_i}$
    * $w' := w_j^i$

Application:
Consider C_0 be a *connected polytope* of $\mathbb{R}^2$. Then:

- Choose any w' from the terminal vertices W of G. Consider this point as the root of the tree G.

⓪

- Colour C_0 in w'.
- Choose an arbitrary point A_0 from the *interior* of C_0.

- Choose an *angular region* being from the *interior* of C_0 with a *corner point* A_0. Consider that *angular region* as C.
- Apply Procedure 3.1 on C, A_0 and w'.

Completion:
Upon completion of the procedure define W_i for i such that $1 \leq i \leq s$:

- $W_i \leftrightharpoons$ the closure of the union of the closures of all regions coloured in w_i

As a note, the set $\{W_1, \ldots, W_s\}$ will be the finite set of polytopes promised. Apparently it will be one-to-one to the set of the vertices $\{w_1, \ldots, w_s\}$ (trivially, the colour defining each of the elements). Furthermore, the union of the elements will clearly be $\mathbb{R}^m$ and each two elements will have empty intersection of their *interiors*. In addition to that every k elements will be in *k-ary contact* if and only if the corresponding terminal vertices have common adjacent *connector* vertex.

Remark 3.1.1. Procedure 3.1 is valid for $\mathbb{R}^m$ for any $m \geq 2$. It is simply that the procedure should be applied on the $\mathbb{R}^2$ projection of $\mathbb{R}^m$ (or its considered connected regular closed subset).

3.1.2 Observations

Following, the intended properties of the set $\{W_1, \ldots, W_s\}$ and the promised results are demonstrated.

Observation 3.1.1. *The following statements are immediately from the definition of Procedure 3.1:*

- *1 is correctly required as being sound with both 0 (the initial input) and 4 (the recursive step).*
- *By definition every U^i_j is non-empty hence every recursive call of the procedure, as per 4, is on input non-empty (current)* angular region C.
- *The procedure eventually completes.*

Proof note: Every recursive step of Procedure 3.1 runs on particular terminal vertex of G considered as a root of the related sub-tree of G. Furthermore, the procedure does never backtrack hence each terminal vertex is traversed only once. G is finite hence the procedure always completes after finite number of steps (in particular exactly s).

□

Observation 3.1.2. *For every i, $1 \leq i \leq s$:*

- *W_i is defined and is completely determined at the step when terminal vertex w_i is being the current root vertex of the traversed by the procedure sub-tree of G*
- *$W_i \neq \emptyset$*

Proof. Once again, Procedure 3.1 does never backtrack. Furthermore, by ② and ④, the procedure traverses every terminal vertex of G and only once. Then, by definition of W_i, ⓪, ① and, either by ② or, by ③, W_i is being completely determined at completion of the step (i.e. recursive call) of traversing the particular terminal vertex w_i.

By the former, the definition of W_i, Observation 3.1.1 (the input *angular region* C non-empty) and, either by both ① and ② or by ③, we imply that W_i is a non-empty set.

□

Observation 3.1.3. *All elements in* $\{W_1, \ldots, W_s\}$ *are polytopes of* $\mathbb{R}^2$.

Proof. By definition of Procedure 3.1 the initial C is a polytope. Furthermore, *(iv)* and ④ guarantee at every recursive step C being a polytope. Let the initial input terminal vertex be w_{i_0}.
Whenever a recursive step terminates at ② then by definition of W_i and Observation 3.1.2, trivially, $W_i = C$ considering the current root vertex being w_i. Hence the set W_i is a polytope. (Remark that if $w_i = w_{i_0}$ then $W_{i_0} = C_0$ which means W_{i_0} is a polytope. In such a case G should have been degenerate graph containing only one terminal vertex and no connector verices or edges. This is not an n-graph for $n \geq 2$. Nevertheless for such a graph the procedure gives trivial solution, namely, $C_0 = W_1$, where w_1 is the only vertex of G. Think of such graph/tree as a particular/exceptional case).
Otherwise, again by Observation 3.1.2, W_i is determined after ③. By *(iv)* the corresponding regions $U_j^{i'}$ are polytopes.
If the current root terminal vertex $w_i \neq w_{i_0}$ then by definition of W_i and that C and all $U_j^{i'}$ are polytopes we imply W_i is a polytope.
Should $w_i = w_{i_0}$ then by definition of W_i we have that W_{i_0} is the closure of the union of $C_0 \setminus C$ on one hand and C minus the union of the appropriate $U_j^{i'}$ regions on the other. Hence W_{i_0} is the closure of C_0 minus the union of the appropriate $U_j^{i'}$ and by all those being polytopes we imply W_{i_0} is a polytope.

□

Observation 3.1.4.

$$C_0 = \bigcup_{i=1}^{s} W_i$$

Proof. By ⓪ at the initial step C_0 is coloured in one colour (the initial root vertex w'). Then, either by both ① and ② or by ③, at any step of the procedure the whole C_0 is kept coloured. As per Observation 3.1.2 and its reasoning each W_i is being determined at the step of traversing terminal vertex w_i and is non-empty. Furthermore, by C_0 being a *regular closed* set and the definition of W_i we imply $W_i \subseteq C_0$ because all elements of W_i are elements of C_0. Then by the former, after traversing all terminal vertices, hence upon completion of the procedure, the equality is satisfied by definition of W_i because eventually every element of C is in at least one W_i.

□

Observation 3.1.5.

$$Int(W_i) \cap Int(W_j) = \emptyset \qquad 1 \leq i < j \leq s$$

Proof. By Observation 3.1.2 $W_{i'}$ is completely determined either by ①, should the procedure terminate at ②, or by ③, should the procedure continue recursively by ④. By definition of $W_{i'}$ and by ②, ③ and ④ W_i may have common point with W_j only if the point is boundary for both W_i and W_j. It follows then $Int(W_i) \cap Int(W_j) = \emptyset$.

□

Observation 3.1.6. *w_i is a terminal vertex for G, $1 \leq i \leq s$, and v is a connector vertex. Then:*
$w_i \in Adj_G(v)$ implies $A_v \in W_i$.

Proof.
Case 1): v is direct descendent of w_i.
Then ② fails. By ① the current C is coloured in w_i. $A_v \in C$ by definition of A_v. Then, by *(i)*, A_v is a boundary point for $U^{i'}_{j'}$ where $w_i = w^{i'}_{j'}$. Then by definition of W_i and ③ we imply $A_v \in W_i$.
Case 2): w_i is direct descendent of v.
In general, consider current root vertex w and corner point A' for C. $A' \in C$ and by ① C is coloured in w. In the case ② and by definition of W_i it trivially follows $W_i = C$ thus $A' \in W_i$. Otherwise by definition of W_i, ③ and *(iii)* we imply $A' \in W_i$. Now by this observation applied on ④ when being called for the vertex w_i in which case $A' = A_v$ we obtain $A_v \in W_i$. □

Observation 3.1.7. *v is a connector vertex, $w_{i_1}, \ldots, w_{i_k}$ are terminal vertices (not necessarily distinct), $1 \leq i_j \leq s$.*
If $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$ then $W_{i_1}, \ldots, W_{i_k}$ are in k-ary contact.

Proof. By Observation 3.1.6 A_v is a witness to the k-ary contact of $W_{i_1}, \ldots, W_{i_k}$.

□

Observation 3.1.8. *Let w_{i_1} and w_{i_2} be distinct terminal vertices.*
If there is no connector vertex v such that $\{w_{i_1}, w_{i_2}\} \subseteq Adj_G(v)$ then W_{i_1} and W_{i_2} are not in binary contact (that is $W_{i_1} \cap W_{i_2} = \emptyset$).

Proof.

1) Without loss of generality let w_{i_2} be descendent of w_{i_1}.

Let v be the direct successor of w_{i_1} in the (single) path towards w_{i_2}, thus v connector vertex. Let w_{i_3} be the direct successor of v towards w_{i_2}, w_{i_3} terminal vertex. Then $w_{i_3} \neq w_{i_2}$ otherwise v is their common adjacent connector vertex which is a contradiction.

$$
\begin{tikzcd}
w_{i_1} \arrow[d] \\
v \arrow[d] \\
w_{i_3} \arrow[d] \\
\cdots \arrow[d] \\
w_{i_2}
\end{tikzcd}
$$

As per Procedure 3.1 consider the region U^i_j where $w^i_j = w_{i_3}$. Continuing this way towards w_{i_2} eventually we obtain a sequence $U_1, \dots, U_{r-1}$, where r is the number of terminal vertices in the path w_{i_1}-…-w_{i_2} (unique by G acyclic and connected) and thus $r \geq 3$ (by definition) such that:

- $U_1 = U^i_j$ where $w^i_j = w_{i_3}$
- $U_{r-1} = U^{i'}_{j'}$ where $w^{i'}_{j'} = w_{i_2}$

By 4 and *(iv)*:

- $U_1 \supseteq \ldots \supseteq U_{r-1}$

By $U_{r-1} = U^{i'}_{j'}$ and 4, 2, 3 and definition of W_i we obtain $W_{i_2} \subseteq U_{r-1}$. By $r \geq 3$ and *(ii)* U_2 has no common point with any of the boundary points of U_1. Hence by definition of W_i and 3 we imply $W_{i_1} \cap U_2 = \emptyset$. By this and $W_{i_2} \subseteq U_{r-1} \subseteq U_2$ it follows that $W_{i_1} \cap W_{i_2} = \emptyset$.

2) None of w_{i_1} and w_{i_2} is descendent of the other.

   2.1) Their closest common ancestor is connector vertex.

Call it v. In such a case there are direct successors of v, w_{i_3} and w_{i_4}, such that:

- Either w_{i_1} is descendant of w_{i_3} or $w_{i_1} = w_{i_3}$
- Either w_{i_2} is descendant of w_{i_4} or $w_{i_2} = w_{i_4}$
- $w_{i_1} \neq w_{i_3}$ or $w_{i_2} \neq w_{i_4}$

Let w be the direct terminal predecessor of v. Such exists because as an initial root vertex of the tree G we choose terminal vertex (and the graph is bipartite by Lemma 2.2.1).

$$\begin{tikzcd}
 & & & w \arrow[d] & & & \\
 & & & v \arrow[dl, bend right] \arrow[dr, bend left] & & & \\
 & & w_{i_3} \arrow[dl, bend right] & & w_{i_4} \arrow[dr, bend left] & & \\
 & \cdots \arrow[dl, bend right] & & & & \cdots \arrow[dr, bend left] & \\
w_{i_1} & & & & & & w_{i_2}
\end{tikzcd}$$

Then, as per Procedure 3.1, with respect to v there are $U^i_{j_1}$ and $U^i_{j_2}$ such that $w^i_{j_1} = w_{i_3}$ and $w^i_{j_2} = w_{i_4}$. Continuing this way towards w_{i_1} and w_{i_2} we obtain sequences $U'_1, \ldots, U'_{r_1}$ and $U''_1, \ldots, U''_{r_2}$ such that:

- $U'_1 = U^i_{j_1}$
- $U''_1 = U^i_{j_2}$
- $U'_{r_1} = U^{i'}_{j'_1}$ such that $w^{i'}_{j'_1} = w_{i_1}$
- $U''_{r_2} = U^{i''}_{j'_2}$ such that $w^{i''}_{j'_2} = w_{i_2}$

By 4 and *(iv)*:

- $U'_1 \supseteq \ldots \supseteq U'_{r_1}$
- $U''_1 \supseteq \ldots \supseteq U''_{r_2}$

As a simple remark if $w_{i_1} = w_{i_3}$ then $r_1 = 1$ and if $w_{i_2} = w_{i_4}$ then $r_2 = 1$.

By those and by 4, 2, 3 and definition of W_i we imply:

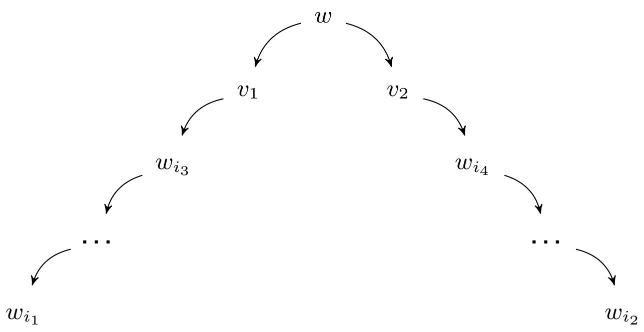
- $W_{i_1} \subseteq U'_{r_1}$
- $W_{i_2} \subseteq U''_{r_2}$

By *(v)* $U'_1 \cap U''_1 = \{A_v\}$ hence $W_{i_1} \cap W_{i_2} \subseteq \{A_v\}$. By $w_{i_1} \neq w_{i_3}$ or $w_{i_2} \neq w_{i_4}$ at least one of r_1 and r_2 is greater than 1. Without loss of generality assume $r_1 > 1$. Then we have $W_{i_1} \subseteq U'_{r_1} \subseteq U'_2 \subseteq U'_1$. By *(iii)* $A_v \notin U'_2$ therefore $A_v \notin W_{i_1}$. It follows then $W_{i_1} \cap W_{i_2} = \emptyset$.

2.2) The closest common ancestor of w_{i_1} and w_{i_2} is terminal vertex.

Call it w. Then there are distinct connector vertices v_1 and v_2 such that v_1 and v_2 are direct successors of w and w_{i_1}, w_{i_2} are descendants of v_1 and v_2 respectively. Let w_{i_3} and w_{i_4} be the terminal direct successors of v_1 and v_2 respectively such that:

- w_{i_1} is descendant of w_{i_3} or $w_{i_1} = w_{i_3}$
- w_{i_2} is descendant of w_{i_4} or $w_{i_2} = w_{i_4}$



Then, as per Procedure 3.1, in a similar manner as per now, we associate $U^{i'_3}_{j_3}$ and $U^{i'_4}_{j_4}$ corresponding to w_{i_3} and w_{i_4} respectively. Remark that $i'_3 \neq i'_4$ because $v_1 \neq v_2$. In similar way as per now we obtain sequences:

- $U'_1, \dots, U'_{r_1}$
- $U''_1, \dots, U''_{r_2}$

such that:

- $U'_1 = U^{i'_3}_{j_3}$ where $w^{i'_3}_{j_3} = w_{i_3}$
- $U''_1 = U^{i'_4}_{j_4}$ where $w^{i'_4}_{j_4} = w_{i_4}$
- $U'_{r_1} = U^{i'_1}_{j_1}$ where $w^{i'_1}_{j_1} = w_{i_1}$
- $U''_{r_2} = U^{i'_2}_{j_2}$ where $w^{i'_2}_{j_2} = w_{i_2}$

By 4 and *(iv)*:

- $U'_1 \supseteq \dots \supseteq U'_{r_1}$
- $U''_1 \supseteq \dots \supseteq U''_{r_2}$

By those, by 4, 2, 3 and by definition of W_i it follows that:

- $W_{i_1} \subseteq U'_{r_1}$
- $W_{i_2} \subseteq U''_{r_2}$

Furthermore, applying *(vi)* we have $U'_1 \cap U''_1 = \emptyset$. Now by the latter and those above we obtain $W_{i_1} \cap W_{i_2} = \emptyset$

□

Observation 3.1.9. *If $W_{i_1}, \dots, W_{i_k}$ are in k-ary contact then exists connector vertex v such that $\{w_{i_1}, \dots, w_{i_k}\} \subseteq Adj_G(v)$.*

Proof. Induction on k.

The case $k = 2$ follows directly by Observation 3.1.8.

Let the claim be true for $k \geq 2$. Consider $k + 1$.

Should $i_1, \dots, i_{k+1}$ be not distinct then the claim follows directly by inductive hypothesis. Consider then $i_1, \dots, i_{k+1}$ distinct.

$W_{i_1}, \ldots, W_{i_{k+1}}$ are in contact. Then $W_{i_1}, \ldots, W_{i_k}$ are in contact. By inductive hypothesis there is a connector vertex v_1 such that $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v_1)$. Furthermore, $W_{i_1}, \ldots, W_{i_{k-1}}, W_{i_{k+1}}$ are in contact. Again by inductive hypothesis exists connector vertex v_2 such that $\{w_{i_1}, \ldots, w_{i_{k-1}}, w_{i_{k+1}}\} \subseteq Adj_G(v_2)$.

Assume $v_1 \neq v_2$. We have $k \geq 2$. If $k > 2$, hence, consider the trail w_{i_1}-v_1-w_{i_2}-v_2-w_{i_1}. This is a circuit, which contradicts G being acyclic. It remains the case when $k = 3$. Then we have $\{w_{i_1}, w_{i_2}\} \subseteq Adj_G(v_1)$ and $\{w_{i_1}, w_{i_3}\} \subseteq Adj_G(v_2)$. Furthermore, again by inductive hypothesis, consider the *connector* vertex v_3 such that $\{w_{i_2}, w_{i_3}\} \subseteq Adj_G(v_3)$. Assume v_3 is equal neither to v_1 nor to v_2. Then consider the trail w_{i_1}-v_1-w_{i_2}-v_3-w_{i_3}-v_2-w_{i_1}. This is a circuit which is a contradiction. Then, without loss of generality, let $v_1 = v_3$. Then the trail w_{i_1}-v_1-w_{i_3}-v_2-w_{i_1} is a circuit. Therefore $v_1 = v_2$ by which we obtain $\{w_{i_1}, \ldots, w_{i_{k+1}}\} \subseteq Adj_G(v_1)$. v_1 is a witness to the existence.

□

Observation 3.1.10. *$W_{i_1}, \ldots, W_{i_k}$ are in k-ary contact*
iff
there exists connector vertex v such that $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$

Proof. The statement is the combined result of Observation 3.1.7 and Observation 3.1.9.

□

Remark. By Observation 2.2.1 the acyclic n-graph G is contact n-graph.

Claim 3.1.1. *Let $\mathcal{F}^c$ be the Kripke frame with carrier $\{W_1, \ldots, W_s\}$ and interpretation of the relation symbols of $L_{\mathcal{R}}$ the standard contact relation for the corresponding arity of the symbols. Let $\mathcal{F}$ be the contact n-frame induced by the contact n-graph G.*
Then $\mathcal{F}^c \cong \mathcal{F}$.

Proof. Consider the mapping $f(w_i) = W_i$. By Observation 3.1.2 f is bijection between W and $\{W_1, \ldots, W_s\}$.

Denote, as per normal, the contact n-frame $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$. Denote $\mathcal{F}^c = <\{W_1, \ldots, W_s\}, I^c>$, where for the k-ary relation symbol P of $L_{\mathcal{R}}$ it is satisfied $I^c(P) = \mathcal{C}_k$. Remark that $\mathcal{F}^c$ is a Kripke frame. Then we have:

$<w_{i_1}, \ldots, w_{i_k}> \in R_k$
(by Claim 2.3.2) *iff*
there exists connector vertex v such that $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$
(by Observation 3.1.10) *iff*
$W_{i_1}, \ldots, W_{i_k}$ are in *k-ary contact*
iff
$<f(w_{i_1}), \ldots, f(w_{i_k})> \in \mathcal{C}_k$

Therefore f is an isomorphism between $\mathcal{F}$ and $\mathcal{F}^c$.

□

3.2 Formal approach for regular closed sets of $\mathbb{R}^m$, $m \geq 1$

In Section 3.1 was demonstrated how to "partition" $\mathbb{R}^2$ into polytopes such that their union is $\mathbb{R}^2$ itself or its regular closed connected subset (Observation 3.1.4) and that "partitioning" is a carrier of a Kripke frame with standard k-ary contact semantics that is isomorphic to the induced by the graph G contact (n-) frame (Claim 3.1.1). An essential property of that "partitioning" is the intersection of the *interiors* of any two distinct polytopes of it is empty (Observation 3.1.3).

Considering k-ary contact for $k > 2$ such a result is not possible for polytopes of $\mathbb{R}^1$ though. For any three polytopes in contact it is easy to see that some two of them will have intersection with a non-empty *interior* (this will be shown later in the exposition). Informally, this limitation is due to the fact a *polytope* is a finite union of segments or rays of $\mathbb{R}^1$. Should we allow infinite such unions then the restriction is alleviated.

In this section will demonstrate how analogous result to the one in Section 3.1 can be obtained when instead of *polytopes* we allow *regular closed* sets of $\mathbb{R}^1$ being possibly infinite unions of segments and rays in $\mathbb{R}^1$.

3.2.1 Procedure for regular closed sets of $\mathbb{R}^1$

In its essence the approach for *regular closed* sets of $\mathbb{R}^1$ will be a decent modification of the ideas in Procedure 3.1. The new procedure will avail of the fact that a union of infinitely many disjoint segments tending to particular point not being in any of those segments eventually is a *regular closed* set and that is the union of those segments and the point they tend to.

Roughly, again, we will first colour the whole $\mathbb{R}^1$ in the root vertex of the n-graph rooted tree and then will use an arbitrary segment of the already coloured set as an input together with the root vertex. Then, in analogy to Procedure 3.1:

- The input segment is sliced into as many segments as the number of the direct *connector* descendants of the input root *terminal* vertex.
- In the interior of each of those segments is dedicated point to the respective *connector* vertex the segment is created for. We call it a *connector* point.
- It is taken monotonic sequence in each of the segments tending to the *connector* point of the segment.
- Such a sequence forms an infinite family of segments (each defined by two subsequent points of the sequence) tending to the *connector* point. Each element of such a family is then separated in as many segments as the direct *terminal* descendant vertices of the corresponding *connector* vertex are. Finally, the left half of each of those finitely many sub-segments is coloured in their corresponding *terminal* vertex.
- Then the procedure is applied recursively but considering only the first element of every one of the infinite families of segments. That is, for each of those first elements for every their new sub-segment coloured in related *terminal* descendant. The input is such a segment and the *terminal* vertex the segment is coloured in.

As a result of this procedure if we take the *closure* of the union of all segments coloured in particular *terminal* vertex then each two such sets have *connector* point in common should their corresponding *terminal* vertices be adjacent to the corresponding *connector* vertex. This is because the *connector* points are in the *closure* of both sets, despite the fact that each two segments from the corresponding unions coloured differently have no point in common from their *interiors*. Furthermore, it will be satisfied that k such sets have point in common if and only if their corresponding *terminal* vertices are adjacent to common *connector* vertex. Finally, they will also hold the rest of the properties as for the result of Procedure 3.1.

Assumptions:
We adopt the same *assumptions* as in Section 3.1.1.

In addition from here on consider only closed segments of $\mathbb{R}^1$ unless stated otherwise. Hence, by *segment* we mean closed interval $[S_0, S_1] \subseteq \mathbb{R}^1$ where $S_0 < S_1$.

Segment in $\mathbb{R}^1$

Procedure 3.2. Regular closed subsets of $\mathbb{R}^1$
Input: $[S_0, S_1]$: the current segment, w' : the terminal vertex as the *root* of the sub-tree of the tree G being currently traversed.
Procedure steps:
1

- Consider $[S_0, S_1]$ as coloured in w'.

2

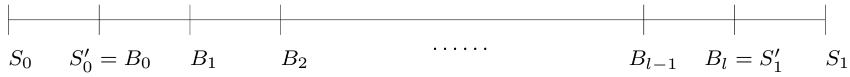
- If w' has no descendants in G then the current procedure recursive call finishes.
- Otherwise:

3

- Take an internal segment $[S'_0, S'_1] \subseteq [S_0, S_1]$ such that $S_0 < S'_0 < S'_1 < S_1$ (*padding left and right*)

Let the direct descendants of w' be $v_1^{k_1}, \dots, v_l^{k_l}$, where $v_i^{k_i}$ is a connector k_i-vertex. Let their direct descendants be $w_1^i, \dots, w_{k_i-1}^i$ for $v_i^{k_i}$ respectively, where w_j^i are terminal vertices.
4

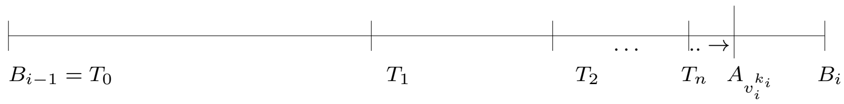
- Slice $[S'_0, S'_1]$ into l segments $[B_0, B_1], \dots, [B_{l-1}, B_l]$ such that $B_0 = S'_0$, $B_l = S'_1$ and $B_i < B_{i+1}$.



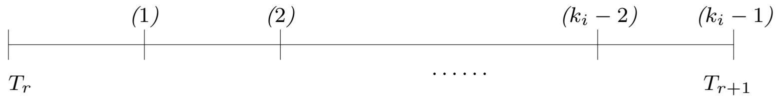
5

- For each of those segments $[B_{i-1}, B_i]$ take (*a witness to* $v_i^{k_i}$) $A_{v_i^{k_i}}$ different from the boundary points, namely: $B_{i-1} < A_{v_i^{k_i}} < B_i$.

- In each of those l segments $[B_{i-1}, A_{v_i^{k_i}}]$ take a strictly increasing sequence of distinct points all different from $A_{v_i^{k_i}}$ with first point in the sequence B_{i-1} and tending to $A_{v_i^{k_i}}$.



- For each of those increasing sequences, for each segment formed by two subsequent points in the sequence $[T_r, T_{r+1}]$:
  - divide the segment into $k_i - 1$ segments



6

  - The j-th for all $1 \le j \le k_i - 1$ of those $k_i - 1$ segments divide into two halves. The left one colour in w_j^i.

6.1

Remark: In this way we form countably many segments coloured in w_j^i. Each such segment is surrounded by coloured in w' (the parrent terminal vertex) segments (or rays) thus no two such segments have a point in common. By the choice of the sequences $\{T_r\}_{r\to\infty}$ the union of all those countably many segments coloured in w_j^i has $A_{v_i^{k_i}}$ in its closure.

7

- For all the intended vertices w_j^i take one segment from the countably many coloured in w_j^i, say the first one with respect to the infinite sequence of points tending to $A_{v_i^{k_i}}$, and on it apply the procedure (recursively) that is for every i, $1 \le i \le l$, for every j, $1 \le j \le k_i - 1$:
  - the input segment $[S_0, S_1]$ be the chosen segment coloured in w_j^i
  - the root terminal vertex w' be w_j^i

Application:

By C let us denote $\mathbb{R}^1$. As a remark, it is sufficient C to be a non-empty *connected regular closed* subset of $\mathbb{R}^1$. Thus we will call C the *initial connected regular closed set* and in this way abstracting from which set we have exactly chosen it to be.

- Choose terminal vertex w' as the root of the tree G.

0

- Colour C in w'.
- Choose an arbitrary segment $[S_0, S_1] \subseteq C$.
- Apply the procedure with input $[S_0, S_1]$ as an initial segment and w' as the root of the tree G.

Completion:

Upon completion of the procedure define W_i for every i, $1 \leq i \leq s$:

- $W_i \risingdotseq$ the closure of the union of the closures of all regions being coloured in w_i

Remark 3.2.1. Procedure 3.2 is valid for $\mathbb{R}^m$ for any $m \geq 1$. It is simply that the procedure should be applied on the $\mathbb{R}^1$ projection of $\mathbb{R}^m$ (or its considered connected regular closed subset).

3.2.2 Observations

Observation 3.2.1. *The following statements are immediately from the definition of Procedure 3.2:*

- *1 is correctly required as being consistent with both 0 (the initial input) and 7 (the recursive step).*
- *The procedure eventually completes.*

Proof note: The reasoning with respect to the completion of the procedure repeats the one in Observation 3.1.1. Informally, it is because both Procedure 3.1 and Procedure 3.2 are quite common in manner, in particular, the way the input tree is being recursively traversed. □

Observation 3.2.2. *For every i, $1 \leq i \leq s$:*

- *W_i is defined and is completely determined at the step when terminal vertex w_i is being the current root vertex of the traversed by the procedure sub-tree of G*
- $W_i \neq \emptyset$

Proof. Procedure 3.2 does never backtrack. Furthermore, by 2 and 7, the procedure traverses every terminal vertex of G and only once. Then by definition of W_i, 0, 1 and, either by 2 or, by both 6 and remark 6.1, W_i is being completely determined at the step of traversing the particular terminal vertex w_i.

By the former, the definition of W_i, 1 and, either by 2 or by both 6 and remark 6.1, we imply W_i is non-empty set by definition as union of (non-empty) segments or rays (rays possibly appear when w_i is the initial input choice for root of the tree G).

□

Observation 3.2.3. *All elements in* $\{W_1, \ldots, W_s\}$ *are regular closed sets of* $\mathbb{R}^1$.

Proof. By definition Procedure 3.2 is applied on segment of $\mathbb{R}^1$ which is guaranteed as per Observation 3.2.1.

Consider W_i. First suppose w_i is not the initial input terminal vertex chosen as a root of the tree G. If the procedure finishes at 2 then, trivially, W_i is a *regular closed* set of $\mathbb{R}^1$ by the input segment being such. Otherwise, by 5 and 6, from the initial segment coloured already in w_i are being subtracted countably many segments the result of which is what is left coloured in w_i and by Observation 3.2.2 W_i is finally determined at this point. Then, by definition of W_i, it is clear that W_i is a *regular closed* set of $\mathbb{R}^1$.

Now, the case when w_i is the initial input terminal vertex, as per the *Application* of Procedure 3.2, then, by definition of W_i, W_i is as per the case when w_i is not the initial input terminal vertex union the closure of the regular closed connected subset of $\mathbb{R}^1$ C subtracted by the chosen as initial input segment. The latter is obviously a regular closed set of $\mathbb{R}^1$ hence, considering the former case, W_i is a finite union of regular closed sets of $\mathbb{R}^1$ thus W_i also being such.

□

Observation 3.2.4. *For the initial connected regular closed set C it holds:*

$$C = \bigcup_{i=1}^{s} W_i$$

Proof. By 0 initially C is coloured in w' being the initial root (terminal) vertex. Then by 1 and either by 2 or by both 6 and 6.1 at each recursive step of the procedure the whole C is kept coloured (not necessarily in the same colour). By Observation 3.2.2 each W_i is being determined at the step of traversing terminal vertex w_i and is non-empty. Furthermore, by C being a *regular closed* set and the definition of W_i we imply $W_i \subseteq C$ because all elements of W_i are elements of C. Hence, by the former, after traversing all terminal vertices, which implies completion of the procedure, the equality is satisfied by definition of W_i because eventually every element of C is in at least one W_i.

□

Observation 3.2.5.

$$Int(W_i) \cap Int(W_j) = \emptyset \qquad 1 \leq i < j \leq s$$

Proof. By Observation 3.2.2 $W_{i'}$ is completely determined either by 1, should the procedure terminate at 2, or by 6 and remark 6.1, should the procedure continue recursively by 7. Considering those, the initial input conditions and definition of $W_{i'}$, then $W_{i'}$ is a union of the *closure* of rays (as a remark rays may appear only if $w_{i'}$ is the initial input root of the tree G) and segments (possibly countably many) coloured in $w_{i'}$ which have no non-empty intersection with the *closure* of segments or rays coloured in different colour but the *boundary* points. Therefore, by $i \neq j$, no *interior* point of W_i is in W_j and vice versa, hence, the equality holds.

□

Observation 3.2.6. *w_i is a terminal vertex for G, $1 \leq i \leq s$, and v is a connector vertex. Then:*

$w_i \in Adj_G(v)$ implies $A_v \in W_i$.

Proof.

Case 1): v is direct descendant of w_i.

By 5 and 6 A_v is a boundary point for segment $[A_v, B_{i'}]$ (for appropriate i' as per 5) coloured in w_i. By Observation 3.2.2 and definition of W_i we have $A_v \in W_i$

Case 2): w_i is direct descendant of v.

Then by Observation 3.2.2, 6, 6.1 and definition of W_i follows that $A_v \in W_i$.

□

Observation 3.2.7. *Let w_i and w_j be distinct terminal vertices.*

If there is no connector vertex v such that $\{w_i, w_j\} \subseteq Adj_G(v)$ then W_i and W_j are not in binary contact (that is $W_i \cap W_j = \emptyset$).

Proof. 1) w_i or w_j is descendant of the other.

Without loss of generality assume w_j is descendant of w_i. Let v be the direct (connector) descendant of w_i on the (single by G being tree) path down to w_j. Let w_{j_1} be the (terminal) direct descendant of v. Apparently $w_{j_1} \neq w_j$, otherwise $\{w_i, w_j\} \subseteq Adj_G(v)$ which is a contradiction.

$$\begin{tikzcd} w_i \arrow[d] \\ v \arrow[d] \\ w_{j_1} \arrow[d] \\ \cdots \arrow[d] \\ w_j \end{tikzcd}$$

By 6 and 7 there is appropriate segment U on which as per 7 the procedure is applied recursively for root w_{j_1}. Remark that by Observation 3.2.2, definition of W_i and by considering 1 and 6 we imply $W_i \cap Int(U) = \emptyset$ because the only common points for W_i and U could be boundary points of both sets.

In this way down the path from w_{j_1} to w_j we obtain sequence $U_1, \ldots, U_r$ of segments such that:

- $U_1 \supseteq \ldots \supseteq U_r$
- $U = U_1$
- U_r is the initial segment for the application of the Procedure 3.2 for root w_j as per 7

By $w_{j_1} \neq w_j$ we have $r \geq 2$. Then by the last bullet, Observation 3.2.2 and considering 1, 2 and 6 we imply $W_j \subseteq U_{r-1}$. Furthermore, 3 guarantees $W_j \subseteq Int(U_{r-1})$. Recall $U_{r-1} \subseteq U_1 = U$ and $W_i \cap Int(U) = \emptyset$ hence $W_i \cap W_j = \emptyset$.

2) The first common ancestor of w_i and w_j is connector vertex.

Call it v. Then there is direct terminal descendant of v different from w_i and w_j and one of w_i and w_j is its descendant. Without loss of generality let w_i be descendant of that vertex w_{i_1} thus having $w_{i_1} \neq w_i$ and $w_{i_1} \neq w_j$. Let the first terminal descendant towards w_j be w_{j_1}, hence, non-necessarily different from w_j.

$$\begin{tikzcd}
 & & & v \arrow[dl, bend right] \arrow[dr, bend left] & & & \\
 & & w_{i_1} \arrow[dl, bend right] & & w_{j_1} \arrow[dr, bend left] & & \\
 & \cdots \arrow[dl, bend right] & & & & \cdots \arrow[dr, bend left] & \\
w_i & & & & & & w_j
\end{tikzcd}$$

As in case 1) here for both branches from w_{i_1} down to w_i and from w_{j_1} down to w_j we form sequences of segments $U^1_1, \ldots, U^1_{r_1}$ and $U^2_1, \ldots, U^2_{r_2}$ such that:

- $U^1_1 \supseteq \ldots \supseteq U^1_{r_1}$
- $U^2_1 \supseteq \ldots \supseteq U^2_{r_2}$
- $U^1_{r_1}$: the initial segment for w_i (as per 7)
- $U^2_{r_2}$: the initial segment for w_j (as per 7)
- U^1_1: the initial segment for w_{i_1} (as per 7)
- U^2_1: the initial segment for w_{j_1} (as per 7)

By $w_{i_1} \neq w_i$ as in 1) we obtain

- $r_1 \geq 2$
- $W_i \subseteq Int(U^1_{r_1-1})$

Therefore $W_i \subseteq Int(U^1_1)$.

2.1) $w_{j_1} \neq w_j$

Then $r_2 \geq 2$. In analogy as above $W_j \subseteq Int(U^2_{r_2-1})$ hence $W_j \subseteq Int(U^2_1)$. By 6 we imply $U^1_1 \cap U^2_1 = \emptyset$ by which follows that $W_i \cap W_j = \emptyset$.

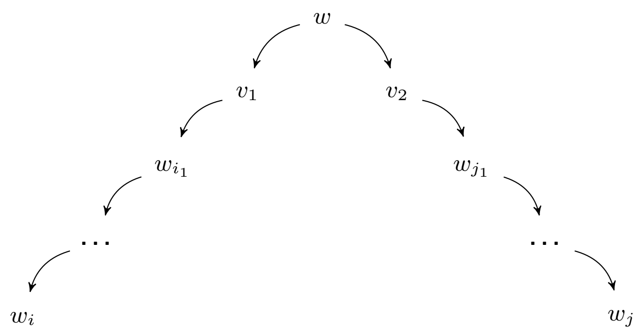
2.2) $w_{j_1} = w_j$

Then by Observation 3.2.2, 6 and definition of W_j it follows that W_j and U^1_1 may have common points only if being boundary points for both sets. Hence $W_j \cap Int(U^1_1) = \emptyset$. Recall $W_i \subseteq Int(U^1_1)$ then $W_i \cap W_j = \emptyset$.

3) The first common ancestor of w_i and w_j is terminal vertex.

Call it w. Therefore there are distinct v_1 and v_2 connector vertices and distinct w_{i_1} and w_{j_1} terminal vertices such that:

- v_1 and v_2 are direct descendants of w
- w_{i_1} is direct descendent of v_1
- w_{j_1} is direct descendent of v_2
- w_i is descendant of w_{i_1} or $w_i = w_{i_1}$
- w_j is descendant of w_{j_1} or $w_j = w_{j_1}$



$w_{i_1} \neq w_{j_1}$ because otherwise $w_i = w_j$ or one of w_i and w_j is descendant of the other, or the common ancestor of w_i and w_j is terminal vertex other than w eventually all in contradiction to 3) and the choice of w.

Then by 4 there are segments:

- $[B_{l_1-1}, B_{l_1}]$ for A_{v_1}
- $[B_{l_2-1}, B_{l_2}]$ for A_{v_2}

such that $(B_{l_1-1}, B_{l_1}) \cap (B_{l_2-1}, B_{l_2}) = \emptyset$. Furthermore, in analogy to the reasoning in the former cases, as per 5 and 6 there would be appropriate segments U_1 and U_2 for w_{i_1} and w_{j_1} respectively for which 7 is applied on. For them we have:

- $U_1 \subseteq (B_{l_1-1}, B_{l_1})$
- $U_2 \subseteq (B_{l_2-1}, B_{l_2})$

In the case when $w_i \neq w_{i_1}$ then, in analogy to the former cases, we infer $W_i \subseteq U_1$ hence $W_i \subseteq (B_{l_1-1}, B_{l_1})$. Otherwise, when $w_i = w_{i_1}$ by Observation 3.2.2, 5, 6, remark 6.1 and definition of $W_{i'}$ we imply $W_i \subseteq (B_{l_1-1}, B_{l_1})$. Thus, applying the same reasoning for w_j and w_{j_1}, eventually we obtain:

- $W_i \subseteq (B_{l_1-1}, B_{l_1})$
- $W_j \subseteq (B_{l_2-1}, B_{l_2})$

Therefore $W_i \cap W_j = \emptyset$.

□

Observation 3.2.8. *v is a connector vertex, $w_{i_1}, \ldots, w_{i_k}$ are terminal vertices (not necessarily distinct), $1 \leq i_j \leq s$.*
If $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$ then $W_{i_1}, \ldots, W_{i_k}$ are in k-ary contact .

Proof. By Observation 3.2.6 for any j, $1 \leq j \leq k$, $A_v \in W_{i_j}$ hence A_v is a witness to a k-ary contact of $W_{i_1}, \ldots, W_{i_k}$.

□

Observation 3.2.9. *If $W_{i_1}, \ldots, W_{i_k}$ are in k-ary contact then exists connector vertex v such that $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$.*

Proof. By induction on k. $k = 2$ is directly by Observation 3.2.7. The inductive step is exactly the same as already made in Observation 3.1.9.

□

Observation 3.2.10. *$W_{i_1}, \ldots, W_{i_k}$ are in k-ary contact*
iff
there exists connector vertex v such that $\{w_{i_1}, \ldots, w_{i_k}\} \subseteq Adj_G(v)$

Proof. The statement is the combined result of Observation 3.2.8 and Observation 3.2.9. □

Recall that by Observation 2.2.1 the acyclic n-graph G is a contact n-graph.

Claim 3.2.1. *Let $\mathcal{F}^c$ be the Kripke frame with carrier $\{W_1, \ldots, W_s\}$ and interpretation of the relation symbols of $L_{\mathcal{R}}$ the standard contact relation for the corresponding arity of the symbols. Let $\mathcal{F}$ be the contact n-frame induced by the contact n-graph G.*
Then $\mathcal{F}^c \cong \mathcal{F}$.

Proof. The proof is exactly the same as the one of the analogous Claim 3.1.1 but using Observation 3.2.10 instead of Observation 3.1.10.

□

Remark that the statement in the following observation is valid for any *regular closed* set. Moreover, we already have that all elements W_i are *regular closed* sets by Observation 3.2.3. Despite seemingly unnecessary extra effort, we cite and prove the observation for the sake of reference and for attaining better understanding why the statement is valid in the particular case.

Observation 3.2.11. *Let a be an arbitrary element of W_i.*
Then for every open $o \ni a$ is satisfied $o \cap Int(W_i) \neq \emptyset$.

Proof. Trivially for $a \in Int(W_i)$.

Consider a a boundary point of W_i. By ①, ⑤, ⑥ and ⑦ if the procedure finishes at ② then, by definition of W_i, the latter will be (the *closure* of) an infinite union of segments plus A_v. Otherwise, considering Observation 3.2.2 and the definition of W_i, the latter is formed by the choice in ④ and ⑤ and eventually by ⑥. Remark that, again, considering also the case when w_i is the initial input terminal vertex, W_i is (the closure) of a union of infinite number of segments as per ⑤ and ⑥, (finite number) of rays (because the initial input is connected set) and A_v. Therefore in any case the boundary points of W_i are either:

- boundary points for the segments or rays in the union
- A_v

In the first case the claim is true. In the second case, when $a = A_v$, then by ⑤ there are (infinitely many) segments from the union forming W_i which are in o. And this means their interior is in o so we conclude $o \cap Int(W_i) \neq \emptyset$.

□

3.3 Formal approach for 2-graphs and polytopes of $\mathbb{R}^1$

As per the introductory notes of Section 3.2, whenever three polytopes of $\mathbb{R}^1$ are in ternary contact then some two of them have intersection with non-empty *interior*. Furthermore, this was the reason in the general case for an acyclic n-graph to be not possible to obtain results analogous to those in Section 3.1 in the case of polytopes of $\mathbb{R}^1$.

Should we be able to define specific requirements for the acyclic n-graph by which to alleviate the mentioned obstructing condition then we may be able to attain the desired results for the class of n-graphs satisfying those requirements.

Informally, if we express the obstructing condition as a formula of $L_{\mathcal{R}}$ then it resembles the one in Claim 2.4.2 *(i)*. By Claim 2.4.2 we have that an n-frame in which such a formula is valid effectively is a 2-frame. Considering Claim 2.3.3 and making the parallel with Claim 3.1.1 and Claim 3.2.1 we conclude that if the acyclic n-graph is 2-graph then we may probably achieve the results as in Section 3.1 and Section 3.2 but for *polytopes* of $\mathbb{R}^1$.

In this section we obtain the desired results for *polytopes* of $\mathbb{R}^1$ when the acyclic n-graph is a 2-graph, hence, it really is a sufficient requirement.

3.3.1 Procedure for polytopes of $\mathbb{R}^1$ for acyclic 2-graph

Assumptions:

- Given $G = (W, V, E)$ finite connected acyclic 2-graph, W, V and E non-empty. Hence $Adj_G(v) = 2$ for every $v \in V$.
- $W = \{w_1, \ldots, w_s\}$, $\overline{\overline{W}} = s$.
- As per now, we consider G as a *rooted tree* for particularly chosen *root vertex* as well as any sub-tree of G in which case the root will be clear by the context. All terms then like *predecessor*, *descendant* etc. will be relative to the currently considered (*rooted*) sub-tree.

Procedure 3.3. Polytopes in $\mathbb{R}^1$ for 2-graph
Input:

- $[S_0, S_1] \subseteq \mathbb{R}^1$: the current segment
- w' : the terminal vertex as the *root* of the sub-tree of the tree G being currently traversed

Procedure Steps:
1

- Consider $[S_0, S_1]$ as coloured in w'.

2

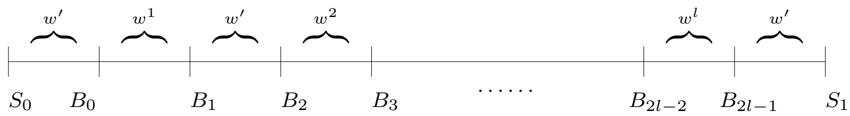
- If w' has no descendants then the current procedure recursive call finishes here.
- Otherwise:

Let the direct (hence connector) descendants of w' be $v_1, \ldots, v_l$. By every v_i of those being a 2-vertex then let their direct descendants (hence terminal) be $w^1, \ldots, w^l$ respectively.
3

- Choose l distinct non-intersecting proper segments in $[S_0, S_1]$ that is segments $[B_0, B_1], [B_2, B_3], \ldots, [B_{2l-2}, B_{2l-1}]$ such that:
  - $B_i < B_{i+1}$, for $0 \leq i < 2l - 1$
  - $S_0 < B_0$ and $B_{2l-1} < S_1$

4

- Colour segment $[B_{2i-2}, B_{2i-1}]$ in w^i $(1 \leq i \leq l)$



5

- Apply the procedure recursively on each segment $[B_{2i-2}, B_{2i-1}]$, $1 \leq i \leq l$:
  - $[S_0, S_1]$ is assigned $[B_{2i-2}, B_{2i-1}]$
  - w' is assigned w^i

Application:
By C let us denote $\mathbb{R}^1$. As a remark, it is sufficient C to be a non-empty *connected regular closed* subset of $\mathbb{R}^1$. Thus we will call C the *initial connected regular closed set* and in this way abstracting from which set we have exactly chosen it to be.

- Choose terminal vertex w' as the root of the tree G.

0

- Colour C in w'.

- Choose an arbitrary segment $[S_0, S_1] \subseteq C$.
- Apply the procedure on $[S_0, S_1]$ (the initial segment) and w' as the root of the tree G.

Completion:
Upon completion of the procedure define W_i for every i, $1 \leq i \leq s$:

- $W_i \coloneqq$ the closure of the union of the closures of all regions being coloured in w_i

Remark 3.3.1. Procedure 3.3 is valid for $\mathbb{R}^m$ for any $m \geq 1$. It is simply that the procedure should be applied on the $\mathbb{R}^1$ projection of $\mathbb{R}^m$ (or its considered connected regular closed subset).

3.3.2 Observations

Observation 3.3.1. *The following statements are immediately from the definition of Procedure 3.3:*

- *0 is correctly required as being consistent with both 0 (the initial input) and 5 (the recursive step).*
- *The procedure eventually completes.*

Proof note: The reasoning with respect to the completion of the procedure repeats the one in Observation 3.1.1. Informally, it is because both Procedure 3.1 and Procedure 3.3 are quite common in manner, in particular, the way the input tree is being recursively traversed.

As a short remark, the recursive step as per 5 is made on already coloured in w^i segment $[B_{2i-2}, B_{2i-1}]$ due to 4 hence consistent with requirement 1.

□

Observation 3.3.2. *For every i, $1 \leq i \leq s$:*

- *W_i is defined and is completely determined at the step when terminal vertex w_i is being the current root vertex of the traversed by the procedure sub-tree of G*
- $W_i \neq \emptyset$

Proof. Procedure 3.3 does never backtrack. Furthermore, by 2 and 5, the procedure traverses every terminal vertex of G and only once. Then by definition of W_i, 0, 1 and either by 2 or by 4 W_i is being completely determined at the step of traversing the particular terminal vertex w_i.

By the former, the definition of W_i, 1 and either by 2 or by both 3 and 4 we imply W_i is non-empty set by definition as a union of (non-empty) segments or rays (rays possibly appear when w_i is the initial input choice for root of the tree G).

□

Observation 3.3.3. *All elements in $\{W_1, \ldots, W_s\}$ are polytopes of $\mathbb{R}^1$.*

Proof. By Observation 3.3.1 ① guarantees the input segment as being such hence polytope of $\mathbb{R}^1$.

Considering Observation 3.3.2, if the procedure finishes at ② for input vertex w_i then only the current input segment is coloured w_i by ④ and ⑤. Hence, by definition of W_i, W_i is this segment itself thus polytope of $\mathbb{R}^1$. Remark that if w_i is the initial input vertex then G is degenerate graph consisting of the vertex w_i only. This is not 2-graph. Nevertheless the procedure gives trivial solution for such a graph/tree, namely, $C = W_1$, where w_1 is the only vertex of G. Think of such graph/tree as a particular/exceptional case then.

Now, when the procedure continues recursively as per ⑤, let the current vertex be w_i.

First, let w_i be not the initial input root vertex chosen for the tree G. By ③ and ④ and considering Observation 3.3.2 the coloured in w_i regions are effectively union of finitely many non-necessarily closed non-intersecting segments in $\mathbb{R}^1$. Therefore their closure is polytope of $\mathbb{R}^1$. By definition of W_i this closure is exactly W_i by which W_i is a polytope.

The case when w_i is the initial input root vertex chosen for the tree G then considering Observation 3.3.2 the regions coloured in w_i are the following. On one hand, those from the chosen as initial input segment from the regular closed connected set C after application of the procedure. Then, as per the former case, the closure of their union is a polytope. On the other hand, in w_i is coloured the remnant of C subtracted the chosen as initial input segment. The closure of the latter is a polytope as well. Therefore the closure of the union of all coloured in w_i sets is a polytope. By definition of W_i that union is exactly W_i.

□

Observation 3.3.4. *For the initial connected regular closed set C it holds:*

$$C = \bigcup_{i=1}^{s} W_i$$

Proof. By ⓪ initially C is coloured in w' being the initial root (terminal) vertex. Then by ① and, either by ② or by ④, at each recursive step of the procedure the whole C is kept coloured (not necessarily in the same colour). By Observation 3.3.2 each W_i is being determined at the step of traversing terminal vertex w_i and is non-empty. Furthermore, by C being a *regular closed* set and definition of W_i we imply $W_i \subseteq C$ as long as all elements of W_i are elements of C. Hence, by the former, after traversing all terminal vertices, which implies completion of the procedure, the equality is satisfied again by definition of W_i because eventually every element of C is in at least one W_i.

□

Observation 3.3.5.

$$Int(W_i) \cap Int(W_j) = \emptyset \qquad 1 \leq i < j \leq s$$

Proof. By Observation 3.3.2 any $W_{i'}$ is completely determined either by ①, should the procedure terminate at ②, or by ④, should the procedure continue recursively by ⑤. Considering those, the initial input conditions, ③ and definition of $W_{i'}$, then $W_{i'}$ is a union of the *closure* of rays (as a remark rays may appear only if $w_{i'}$ is the initial input root of the tree G) and segments, both finitely many, coloured in $w_{i'}$ which have no non-empty intersection with the *closure* of segments or rays coloured in different colour but their *boundary* points. Therefore, by $i \neq j$, no *interior* point of W_i is in W_j and vice versa, hence, the equality holds.

$\square$

Observation 3.3.6. *v is a connector vertex, $w_{i_1}, \dots, w_{i_k}$ are terminal vertices (not necessarily distinct), $1 \leq i_j \leq s$.*

If $\{w_{i_1}, \dots, w_{i_k}\} \subseteq Adj_G(v)$ then $W_{i_1}, \dots, W_{i_k}$ are in k-ary contact.

Proof. By G being a 2-graph then $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} \leq 2$. In the case when $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} = 1$ then by Observation 3.3.2 and definition of W_i trivially $W_{i_1}, \dots, W_{i_k}$ are in contact as all being the same non-empty set.

Let then $\{w_{i_1}, \dots, w_{i_k}\} = \{w_i, w_j\}$, for $i \neq j$. Without loss of generality assume w_j is descendant of w_i in the rooted tree G. Then by Observation 3.3.2, definition of W_i and by ③, ④ and ⑤ we conclude W_i and W_j are in contact as per the definition of the segments and colouring made in ③ and ④.

$\square$

Observation 3.3.7. *Let w_i and w_j be distinct terminal vertices.*

If there is no connector vertex v such that $\{w_i, w_j\} \subseteq Adj_G(v)$ then W_i and W_j are not in binary contact (that is $W_i \cap W_j = \emptyset$).

Proof. Neither w_i nor w_j is direct terminal descendant of the other otherwise there will be connector vertex v which will contradict the initial condition. Furthermore, w_i and w_j cannot have closest common predecessor connector vertex because the initial choice for a root of the tree G is terminal vertex thus such a connector vertex will have two distinct direct descendant (terminal) vertices plus one parent (terminal) vertex which is a contradiction to G being a 2-graph. Therefore we have the following two possible cases.

Case 1: w_i or w_j is descendant of the other.

Without loss of generality assume w_j is descendant of w_i. Let w_{j_1} be the first terminal descendant of w_i towards w_j (single path due to G tree). Hence $w_{j_1} \neq w_j$. By ③ and ⑤ there is proper segment $[B_{r_1-1}, B_{r_1}]$ on which ⑤ is applied for the vertex w_{j_1}. By Observation 3.3.2 and definition of W_i we infer W_i may have intersection points with $[B_{r_1-1}, B_{r_1}]$ only if being boundary for both sets. Hence $W_i \cap Int([B_{r_1-1}, B_{r_1}]) = \emptyset$.

In analogy to the choice of $[B_{r_1-1}, B_{r_1}]$ for w_{j_1} continuing downwards w_j we obtain sequence of segments $[B_{r_1-1}, B_{r_1}], \dots, [B_{r_t-1}, B_{r_t}]$, where $[B_{r_t-1}, B_{r_t}]$ is the segment on which ⑤ is applied for w_j. Hence, $t \geq 2$ due to $w_{j_1} \neq w_j$. Now again by Observation 3.3.2 and by definition of W_j we imply that $W_j \subseteq [B_{r_t-1}, B_{r_t}]$.

Remark that by $t \geq 2$ and by the choice of segments in ③ it is satisfied $[B_{r_{t'+1}-1}, B_{r_{t'+1}}] \subseteq Int([B_{r_{t'}-1}, B_{r_{t'}}])$. Hence, applying this inductively, we conclude $W_j \subseteq Int([B_{r_1-1}, B_{r_1}])$. Recall that $W_i \cap Int([B_{r_1-1}, B_{r_1}]) = \emptyset$ by which $W_i \cap W_j = \emptyset$.

Case 2: w_i, w_j have closest common predecessor terminal vertex.

Call it w. Let the direct connector descendants of w towards w_i and w_j be v_1 and v_2 respectively. Apparently $v_1 \neq v_2$, otherwise contradiction with the choice of w. Let then the direct descendants of v_1 and v_2 be w_{i_1} and w_{j_1} towards

w_i and w_j respectively. By definition w_i is descendant of w_{i_1} or $w_i = w_{i_1}$, the same applies for the relation between w_j and w_{j_1}.

As per the above reasoning, by ③ and ⑤ there are proper segments $[B_{r_1^1-1}, B_{r_1^1}]$ and $[B_{r_1^2-1}, B_{r_1^2}]$ on which ⑤ is applied for the vertices w_{i_1} and w_{j_1} respectively. Again, as per the above reasoning, we obtain sequences: $[B_{r_1^1-1}, B_{r_1^1}], \dots, [B_{r_{t_1}^1-1}, B_{r_{t_1}^1}]$ and $[B_{r_1^2-1}, B_{r_1^2}], \dots, [B_{r_{t_2}^2-1}, B_{{r^2}_{t_2}}]$, where $[B_{r_{t_1}^1-1}, B_{r_{t_1}^1}]$ is the segment on which ⑤ is applied for w_i and $[B_{r_{t_2}^2-1}, B_{r_{t_2}^2}]$ is the one for w_j. Hence, by Observation 3.3.2 and definition of W_i and W_j we have $W_i \subseteq [B_{r_{t_1}^1-1}, B_{r_{t_1}^1}]$ and $W_j \subseteq [B_{r_{t_2}^2-1}, B_{r_{t_2}^2}]$.

Again by ③ we remark that $[B_{r_{t_1'+1}^1-1}, B_{r_{t_1'+1}^1}] \subseteq [B_{r_{t_1'}^1-1}, B_{r_{t_1'}^1}]$ as well as $[B_{r_{t_2'+1}^2-1}, B_{r_{t_2'+1}^2}] \subseteq [B_{r_{t_2'}^2-1}, B_{r_{t_2'}^2}]$. Applying it inductively we obtain $W_i \subseteq [B_{r_1^1-1}, B_{r_1^1}]$ and $W_j \subseteq [B_{r_1^2-1}, B_{r_1^2}]$.

Considering ③ and by $v_1 \neq v_2$ we conclude $[B_{r_1^1-1}, B_{r_1^1}] \cap [B_{r_1^2-1}, B_{r_1^2}] = \emptyset$. This gives $W_i \cap W_j = \emptyset$.

□

Observation 3.3.8. *If $W_{i_1}, \dots, W_{i_k}$ are in k-ary contact then exists connector vertex v such that $\{w_{i_1}, \dots, w_{i_k}\} \subseteq Adj_G(v)$.*

Proof. If $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} = 1$ then the claim is satisfied by G connected n-graph.

Let then $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} > 1$. Assume $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} > 2$. Let w_{j_1}, w_{j_2} and w_{j_3} be the first three distinct among $w_{i_1}, \dots, w_{i_k}$. W_{j_1}, W_{j_2} and W_{j_3} are in contact as well as any two of them. Then by Observation 3.3.7 there are connector vertices v_1, v_2 and v_3 such that $\{w_{j_1}, w_{j_2}\} \subseteq Adj_G(v_3)$, $\{w_{j_1}, w_{j_3}\} \subseteq Adj_G(v_2)$ and $\{w_{j_2}, w_{j_3}\} \subseteq Adj_G(v_1)$. v_1, v_2, v_3 are distinct otherwise, by G being a 2-graph, $\overline{\overline{\{w_{j_1}, w_{j_2}, w_{j_3}\}}} \leq 2$ which is a contradiction with the choice of w_{j_1}, w_{j_2} and w_{j_3}. Nevertheless the trail w_{j_1}-v_3-w_{j_2}-v_1-w_{j_3}-v_2-w_{j_1} is a circuit which is a contradiction with G acyclic.

Therefore $\overline{\overline{\{w_{i_1}, \dots, w_{i_k}\}}} = 2$. Now we apply Observation 3.3.7 directly. □

Remark. Observation 3.3.8 can be proven exactly as Observation 3.1.9 but using Observation 3.3.7 in the base of the induction instead.

Observation 3.3.9. *$W_{i_1}, \dots, W_{i_k}$ are in k-ary contact*
iff
there exists connector vertex v such that $\{w_{i_1}, \dots, w_{i_k}\} \subseteq Adj_G(v)$

Proof. The statement is the combined result of Observation 3.3.6 and Observation 3.3.8. □

Claim 3.3.1. *Let $\mathcal{F}^c$ be the Kripke frame with carrier $\{W_1, \dots, W_s\}$ and interpretation of the relation symbols of $L_\mathcal{R}$ the standard contact relation for the corresponding arity of the symbols. Let $\mathcal{F}$ be the contact n-frame induced by the contact n-graph G.*

Then $\mathcal{F}^c \cong \mathcal{F}$.

Proof. The proof is exactly the same as the one of the analogous Claim 3.1.1 but only using Observation 3.3.9 instead of Observation 3.1.10.

□

4 p-morphic preimages of contact n-frames

In the former Section 3 for given contact n-graph were achieved useful results as per obtaining corresponding to the graph Kripke frame with standard contact semantics illustrative examples being Claim 3.1.1, Claim 3.2.1 and Claim 3.3.1. An essential property of the originating n-graph was to be *acyclic*. As long as the approaches from Section 3 could be applied on that class of contact n-graphs rational motivation is to elaborate on a facility that provides sensible association between arbitrary finite contact n-frames and those induced by acyclic contact n-graphs.

In this section we will demonstrate a formal procedure which for given contact n-frame induced by an arbitrary contact n-graph transforms the graph into an acyclic contact n-graph such that the induced by it contact n-frame is a *p-morphic preimage* of the originating contact n-frame.

4.1 Formal procedure on n-graphs

Consider an arbitrary *n-graph* $G = (W, V, E)$.

Procedure Step 4.1.

- Choose an arbitrary circuit from the graph G. Denote it by C.
- Choose an arbitrary *terminal* vertex from the circuit. Denote it by w.
- Choose one of the (two) adjacent *connector* (by Definition 2.2.1) vertices of w in the circuit C. Denote it by v.
- Remove the edge (v, w) from E.
- Add a new distinct *terminal* vertex w' to W.
- Add a new edge (v, w') to E.

Procedure 4.1.

- While there is a circuit in G apply Procedure Step 4.1

4.2 Observations

Consider arbitrary n-graph and denote it by the standard notation for a graph: $G = (V, E)$. Denote the terminal vertices of G by W hence $W \subseteq V$.

Consider single application of Procedure Step 4.1 over the n-graph G. Then the resulting graph $G' = (V', E')$ is as follows:

- $V' = V \cup \{w'\}$
- $E' = (E \setminus \{(v, w)\}) \cup \{(v, w')\}$

Observation 4.2.1. *G' has less circuits than G.*

Proof. Consider the intermediate graph $G'' = (V'', E'')$ as follows:

- $V'' = V$
- $E'' = E \setminus \{(v, w)\}$

The edge (v, w) then breaks at least the chosen circuit C hence the number of circuits of G'' is less than that of G.

For G' we have:

- $V' = V'' \cup \{w'\}$
- $E' = E'' \cup \{(v, w')\}$

Apparently, the degree of the new vertex w' is 1 hence w' cannot participate in a circuit. Therefore any circuit in G' should have already been in G''. This means the number of the circuits in G'' is preserved the same in G'.

Eventually, the number of the circuits in G' is less than their number in G. □

Observation 4.2.2. *G' is n-graph.*

Proof. Straightforward verification that the conditions in Definition 2.2.1 of n-graph satisfied by G are preserved also in G'. □

Due to Observation 4.2.2 from here on we will denote the resulting graphs from Procedure Step 4.1 by the standard notation we use for n-graphs. In particular, for the n-graph $G = (W, V, E)$, the result $G' = (W', V, E')$ of applying the procedure step on G is defined as:

- $W' = W \cup \{w'\}$
- V is the same in both G and G'
- $E' = (E \setminus \{(v, w)\}) \cup \{(v, w')\}$

Observation 4.2.3. *$G' = (W', V, E')$ is a contact n-graph should $G = (W, V, E)$ be a contact n-graph.*

Proof. Verification of the conditions as per Definition 2.2.2.

(0):

It is Observation 4.2.2.

(1):

Consider $G'' = (W, V, E'')$ such that:

- W and V are as in G
- $E'' = E \setminus \{(v, w)\}$

By this and G being *simple*, trivially, G'' also is. For G' we have:

- $W' = W \cup \{w'\}$
- V is the same as in G and G''
- $E' = E'' \cup \{(v, w')\}$

Remark that all edges of G' are in G'' but (v, w'). Furthermore, (v, w') is a single such edge by Procedure Step 4.1. Therefore G' is *simple* by G'' being such.

(2):

Let v', v'' arbitrary connector vertices for G' not necessarily different.

If v' and v'' are both other than v then they are not incident on the new edge (v, w') in G' with respect to G. Thus all their adjacent vertices are the same as those in G. Thus the condition is satisfied by G contact n-graph.

Let then one of the vertices be v. Without loss of generality let $v'' = v$. By $(v, w') \in E'$ then $w' \in Adj_{G'}(v)$.

For any connector vertex v' other than v we have $w' \notin Adj_{G'}(v')$. Therefore, in such a case, $Adj_{G'}(v) \nsubseteq Adj_{G'}(v')$.

Let $Adj_{G'}(v') \subseteq Adj_{G'}(v)$ and assume $v' \neq v$. As clarified, then $w' \notin Adj_{G'}(v')$. Hence $Adj_{G'}(v') \subseteq (Adj_{G'}(v) \setminus \{w'\})$. As long as $v' \neq v$ then $Adj_G(v') = Adj_{G'}(v')$. Remark that by definition $Adj_G(v) = (Adj_{G'}(v) \setminus \{w'\}) \cup \{w\}$. By these we imply $Adj_G(v') \subseteq Adj_G(v)$. Now, by G contact n-graph, it follows that $v' = v$ which is a contradiction to our assumption.

□

Let $G = (W, V, E)$ be a contact n-graph. As per Observation 4.2.3, let the resulting *contact n-graph* after applying once Procedure Step 4.1 on G be $G' = (W', V, E')$. Let Procedure Step 4.1 has used terminal vertex w_0 from the chosen circuit in G and the added new one be w'_0. Finally, let the used connector vertex be v. To rewrite it exactly as per above in such a case we have:

- $W' = W \cup \{w'_0\}$
- V is the same in both G and G'
- $E' = (E \setminus \{(v, w_0)\}) \cup \{(v, w'_0)\}$

As per Claim 2.3.3, consider the *induced* by G and G' *contact n-frames* $\mathcal{F}$ and $\mathcal{F}'$ respectively. Denote them by:

- $\mathcal{F} = <W, R_2, \ldots, R_n, \ldots>$
- $\mathcal{F}' = <W', R'_2, \ldots, R'_n, \ldots>$

Observation 4.2.4. *Let $f : W' \twoheadrightarrow W$ be defined as:*

$$f(w) = \begin{cases} w & w \neq w'_0 \\ w_0 & w = w'_0 \end{cases}$$

Then f is p-morphism from $\mathcal{F}'$ onto $\mathcal{F}$.

Proof.
Forward condition:

Let $<w_1, \ldots, w_k> \in R'_k$. Then by Definition 2.3.2 **(r)** either $w_1 = \ldots = w_k$ or exists $v' \in V$ such that $\{w_1, \ldots, w_k\} \subseteq Adj_{G'}(v')$.

Should $w_1 = \ldots = w_k$ then, trivially, $f(w_1) = \ldots = f(w_k)$ hence, by Definition 2.3.2 **(r)**, we have $<f(w_1), \ldots, f(w_k)> \in R_k$.

Let $\{w_1, \ldots, w_k\} \subseteq Adj_{G'}(v')$.

If $v' \neq v$ then, by definition, $Adj_{G'}(v') = Adj_G(v')$. Furthermore, by definition of E', none of $w_1, \ldots, w_k$ is w'_0. Hence $f(w_i) = w_i$ for all i, $1 \leq i \leq k$. It follows that $\{f(w_1), \ldots, f(w_k)\} \subseteq Adj_G(v')$ and by Definition 2.3.2 **(r)** $<f(w_1), \ldots, f(w_k)> \in R_k$.

Now let $v' = v$. Remark that, by definition of E', $w_0 \notin Adj_{G'}(v)$ hence $w_0 \neq w_i$ for all i, $1 \leq i \leq k$.

Consider w_i, $1 \leq i \leq k$. Remark that by definition of E and E' we conclude $Adj_{G'}(v) \setminus \{w_0'\} = Adj_G(v) \setminus \{w_0\}$. Then, if $w_i \neq w_0'$, on one hand, $f(w_i) = w_i$ and, on the other, $w_i \in Adj_G(v)$ thus $f(w_i) \in Adj_G(v)$. Otherwise, when $w_i = w_0'$ then $f(w_i) = w_0$ but $w_0 \in Adj_G(v)$ by the choice of v hence, again, $f(w_i) \in Adj_G(v)$. It follows that $\{f(w_1), \ldots, f(w_k)\} \subseteq Adj_G(v)$ which by Definition 2.3.2 **(r)** means $<f(w_1), \ldots, f(w_k)> \in R_k$.

Backward condition:

Let $<w_1, \ldots, w_k> \in R_k$. Then by Definition 2.3.2 **(r)** either $w_1 = \ldots = w_k$ or exists $v' \in V$ such that $\{w_1, \ldots, w_k\} \subseteq Adj_G(v')$.

Should $w_1 = \ldots = w_k$ then let $w \in W'$ be such that $f(w) = w_1 = \ldots = w_k$. Then, by Definition 2.3.2 **(r)** (in fact even by trivial reasons due to definition of *contact n-frames*) $<w, \ldots, w> \in R_k'$.

Let $\{w_1, \ldots, w_k\} \subseteq Adj_G(v')$.

If $v' \neq v$ then, again, by definition, $Adj_G(v') = Adj_{G'}(v')$. Furthermore, by $w_0' \notin W$, thus $w_0' \notin \{w_1, \ldots, w_k\}$ we have $f(w_i) = w_i$ by definition of f. Now by $\{w_1, \ldots, w_k\} \subseteq Adj_{G'}(v')$ and Definition 2.3.2 **(r)** we conclude $<w_1, \ldots, w_k> \in R_k'$.

Let $v' = v$. By definition of E' we have

$$Adj_{G'}(v) = (Adj_G(v) \setminus \{w_0\}) \cup \{w_0'\} \tag{1}$$

Let $g : W \rightarrowtail W'$ be defined as follows:

$$g(w) = \begin{cases} w & w \neq w_0 \\ w_0' & w = w_0 \end{cases}$$

Clearly g is an injection. Furthermore, remark that $w_0 \notin Range(g)$. By equation 1 and definition of g it follows that:

$$w \in Adj_G(v) \quad \textit{iff} \quad g(w) \in Adj_{G'}(v) \qquad \text{for } w \in W$$

Therefore $\{g(w_1), \ldots, g(w_k)\} \subseteq Adj_{G'}(v)$. Hence, by Definition 2.3.2 **(r)**:

$$<g(w_1), \ldots, g(w_k)> \in R_k'$$

We will demonstrate $<g(w_1), \ldots, g(w_k)>$ is a witness for which it is sufficient to show that for all i, $1 \leq i \leq k$, then $f(g(w_i)) = w_i$. This follows immediately from the more general observation, namely:

$$f(g(w)) = w \qquad \text{for } w \in W$$

If $w \neq w_0$ then $g(w) = w$. $w \in W$ thus $w \neq w_0'$. Therefore $f(g(w)) = f(w) = w$. Otherwise, when $w = w_0$ then $g(w_0) = w_0'$ hence $f(g(w_0)) = f(w_0') = w_0$.

□

Observation 4.2.5. *Procedure 4.1 applied on an arbitrary finite n-graph eventually finishes. Furthermore, the resulting graph is a finite acyclic n-graph.*

Proof. By Observation 4.2.1 upon each application of Procedure Step 4.1 the number of circuits in the resulting graph strongly decreases. The input graph is finite thus it has finite number of circuits. Therefore Procedure 4.1 performs only finite number of steps.

Procedure 4.1 terminates only when there are no circuits left in the graph hence trivially the resulting graph is acyclic.

By applying Observation 4.2.2 inductively, eventually, the resulting graph is n-graph.

By definition of Procedure Step 4.1 at each step are being added finitely many new vertices, in particular exactly one. Just as a note, the number of edges is preserved. Procedure 4.1 completes in finitely many steps hence the resulting graph is finite.

□

Remark 4.2.1. Consider arbitrary *finite contact n-graph.* By Observation 4.2.5 Procedure 4.1 finishes and the resulting graph is acyclic n-graph. Then by inductively applying Observation 4.2.3 it follows that the resulting graph is contact n-graph.

Furthermore, remark that by Observation 2.2.1 for arbitrary *finite n-graph* the result of Procedure 4.1 will again be *contact n-graph.*

The benefit of the former reasoning is in that by Observation 4.2.3 it demonstrates the n-graph is guaranteed *contact* within all the intermediate calls/substeps of Procedure Step 4.1.

Remark 4.2.2. Consider an arbitrary *finite contact n-graph G.* By Claim 2.3.3 let $\mathcal{F}$ be the *induced* by G *contact n-frame.* As per Observation 4.2.5 let G' be the resulting acyclic n-graph upon applying Procedure 4.1 on G. By Remark 4.2.1 G' is a *contact n-graph.* Let then, again as per Claim 2.3.3, $\mathcal{F}'$ be the *induced* by G' *contact n-frame.*

Remark 4.2.2 is the ground for stating the following:

Claim 4.2.1. *$\mathcal{F}$ is a p-morphic image of $\mathcal{F}'$.*

Proof. By Observation 4.2.5 Procedure 4.1 eventually finishes hence Procedure Step 4.1 is performed finitely many times. Let $G_0, G_1, \dots, G_r$ be the intermediate graphs being result of Procedure Step 4.1 within the finite run of Procedure 4.1 such that:

- $G_0 = G$
- $G_r = G'$
- G_{i+1} is the result of Procedure Step 4.1 on G_i, $0 \leq i \leq r-1$

By Observation 4.2.3 G_i is a *contact n-graph* for each i, $0 \leq i \leq r$. As per Claim 2.3.3 consider $\mathcal{F}_0, \dots, \mathcal{F}_r$ be the *induced contact n-frames* by $G_0, \dots, G_r$ respectively, that is:

- $G_0 \longrightarrow \mathcal{F}_0, \dots, G_r \longrightarrow \mathcal{F}_r$
- $\mathcal{F} = \mathcal{F}_0$
- $\mathcal{F}' = \mathcal{F}_r$

By Observation 4.2.4 $\mathcal{F}_i$ is a *p-morphic image* of $\mathcal{F}_{i+1}$ for $0 \leq i \leq r-1$. Denote by f_{i+1} an arbitrary *p-morphism* from $\mathcal{F}_{i+1}$ onto $\mathcal{F}_i$, for example the one as per Observation 4.2.4. Composition of p-morphisms is a p-morphism therefore, eventually, $\mathcal{F}_0$ is a *p-morphic image* of $\mathcal{F}_r$ by the *p-morphism*:

$$f = f_r \circ f_{r-1} \circ \ldots \circ f_1$$

□

Given is n-graph G. Let, again, G' be the result of applying once Procedure Step 4.1 on G.

Observation 4.2.6. *If G is connected graph, then also is G'.*

Proof. As formerly, denote $G = (W, V, E)$ and $G' = (W', V, E')$ such that:

- $W' = W \cup \{w'_0\}$
- V is the same in both G and G'
- $E' = (E \setminus \{(v, w_0)\}) \cup \{(v, w'_0)\}$

where v, w_0 and w'_0 are the choice of vertices as per Procedure Step 4.1.

Consider the intermediate graph $G'' = (W, V, E'')$ such that:

- W and V are the same in G and G''
- $E'' = E \setminus \{(v, w_0)\}$

Assume G'' be not connected. Then consider component $(W'' \cup V'')$ of G'', where $W'' \subseteq W$ and $V'' \subseteq V$. Consider the partitioning $(W'' \cup V'')$ and $(W \cup V) \setminus (W'' \cup V'')$ in G''. Remark that if there is an edge in G connecting vertices from both partitions other than (v, w_0) then this edge is also in G'' which is a contradiction. Nevertheless, by G connected, follows that there is an edge in G with vertices from both partitions. Therefore this edge certainly is (v, w_0) as the only possible. This means vertices v and w_0 are in different partitions with respect to the chosen partitioning. Recall that by the choice of v and w_0 in Procedure Step 4.1 there is a circuit in G which contains the edge (v, w_0). All edges from that circuit but (v, w_0) are in G''. Therefore there is a path in G'' connecting v and w_0. This is a contradiction because $(W'' \cup V'')$ is a component in G''.

Now, with respect to G'', for G' we have:

- $W' = W \cup \{w'_0\}$
- $E' = E'' \cup \{(v, w'_0)\}$

By this and G'' connected we imply G' is also connected.

□

Observation 4.2.7. *Upon applying Procedure 4.1 on a finite connected n-graph the resulting n-graph is also connected.*

Proof. By Observation 4.2.5 Procedure 4.1 finishes in finitely many steps and the result is an n-graph. Then, by applying Observation 4.2.6, inductively, we imply the resulting n-graph is also connected.

□

Let $G = (W, V, E)$ be an arbitrary contact n-graph and, again, $G' = (W', V, E')$, defined as above, be the result of applying once Procedure Step 4.1 on G.

Observation 4.2.8. *G' is a contact n-graph for the same n as for G.*

Proof. G' is a contact n-graph by Observation 4.2.3. By the equation $E' = (E \setminus \{(v, w_0)\}) \cup \{(v, w'_0)\}$ it is obvious that the degree of all connector vertices of G' is preserved exactly the same as of G.

□

Observation 4.2.9. *Upon applying Procedure 4.1 on a finite contact n-graph then the result is a finite contact n-graph for the same n as for the originating contact n-graph.*

Proof. By Observation 4.2.5 Procedure 4.1 finishes in finitely many steps and the resulting graph is finite. Furthermore, by Remark 4.2.1 that graph is a contact n-graph. Applying Observation 4.2.8 inductively we imply that the resulting contact n-graph is n-graph for the same n as the originating one. □

5 n-ary contact axioms

Following we define the axioms of the (logic of the) n-ary contact. In the later sections we will prove those axioms being sufficient for axiomatising particular classes of Boolean frames, namely, those defined on subalgebras of the Boolean algebras of the *polytopes* or the *regular closed* sets of $\mathbb{R}^m$ (see Section 1.6.2 and Section 1.6.4) and having as interpretation of the relation symbols the *standard contact semantics*. Here the validity of the axioms will be studied from *contact n-frames* perspective.

5.1 Axioms of the n-ary contact

(c1) $(\rho(P) = n,\, n \geq 0,\, \sigma : n \to n)$

$$P(x_1, \dots, x_n) \implies P(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

(c2) $(\rho(P) = n + 1,\, \rho(Q) = n,\, n \geq 0)$

$$P(x_1, x_1, x_2 \dots, x_n) \iff Q(x_1, x_2, \dots, x_n)$$

(c3) $(\rho(P) = 2)$

$$\neg(x \equiv 0) \implies P(x, x)$$

(c4) $(\rho(P) = 2)$

$$\neg(x \equiv 0) \wedge \neg(-x \equiv 0) \implies P(x, -x)$$

PRC1 $(\rho(P) = 3)$

$$P(x_1, x_2, x_3) \implies \neg(x_1 \cap x_2 \equiv 0) \vee \neg(x_2 \cap x_3 \equiv 0) \vee \neg(x_1 \cap x_3 \equiv 0)$$

5.2 Validity of the axioms in the contact n-frames

Claim 5.2.1. *Let $\mathcal{F}$ be a Kripke frame in which are valid **(c1)**, **(c2)** and **(c3)**.*
*Then $\mathcal{F}$ satisfies conditions **(a)**, **(b)** and **(c)** of Definition 2.1.1 of a contact n-frame.*

Proof. Let $\mathcal{F} = <S, I>$. We will demonstrate each of the conditions **(a)**, **(b)** and **(c)** of Definition 2.1.1.
(a):
Let $<s_1, \dots, s_k> \in I(P)$ and consider valuation:

$$\mathcal{V}(x) = \begin{cases} \{s_i\} & x = x_i \\ \text{arbitrary} & x \notin \{x_1, \dots, x_k\} \end{cases} \tag{2}$$

By $\mathcal{F} \Vdash$ **(c1)** then we imply $<s_{\sigma(1)}, \dots, s_{\sigma(k)}> \in I(P)$.
(b):
Consider relation symbols P and Q such that $\rho(P) = k+1$ and $\rho(Q) = k$. Let $<s_1, s_1, \dots, s_k> \in I(P)$. Take again valuation (2). By $\mathcal{F} \Vdash$ **(c2)** and $<\mathcal{F}, \mathcal{V}> \Vdash P(x_1, x_1, \dots, x_k)$ then $<\mathcal{F}, \mathcal{V}> \Vdash Q(x_1, \dots, x_k)$ thus $<s_1, \dots, s_k> \in I(Q)$.
The opposite direction is in analogy.
(c):

Consider arbitrary $s \in S$ and relation symbol P such that $\rho(P) = 2$. Let:

$$\mathcal{V}(y) = \begin{cases} \{s\} & x = y \\ \text{arbitrary} & x \neq y \end{cases}$$

By $\mathcal{F} \Vdash$ **(c3)** and $\mathcal{V}(x) \neq \emptyset$ it follows:

$$<\mathcal{F}, \mathcal{V}> \Vdash P(x, x)$$

Therefore $<s, s> \in I(P)$.

$\square$

Claim 5.2.2. *Let be given a Kripke frame satisfying conditions **(a)**, **(b)** and **(c)** of Definition 2.1.1 of a contact n-frame. If the Kripke frame is finite, then it is a contact n-frame (for appropriate n).*

Proof. Let $\mathcal{F} = <W, I>$, where $\overline{\overline{W}} = s < \omega$. By Definition 2.1.1 it remains to show $\mathcal{F}$ satisfies conditions **(d)**.

Let n be the greatest with the property that there are distinct $w_1, \ldots, w_n \in W$ such that $<w_1, \ldots, w_n> \in R_n$ for $\mathcal{F}$. Such n exists as by Remark 2.1.1 with sure $n \geq 1$ and by the finiteness of W $n \leq s$. Therefore, by definition, this n satisfies **(d.1)**.

Let $<w_1, \ldots, w_k> \in R_k$. If $k \leq n$ then apparently $\overline{\overline{\{w_1, \ldots, w_k\}}} \leq n$. Consider $k > n$ and assume $\overline{\overline{\{w_1, \ldots, w_k\}}} > n$. Take $n + 1$ distinct elements from $\{w_1, \ldots, w_k\}$. Without loss of generality consider them $w_1, \ldots, w_{n+1}$. Then by **(a)** for $\mathcal{F}$ we obtain $<\underbrace{w_1, \ldots, w_1}_{k-(n+1)}, w_1, \ldots, w_{n+1}> \in R_k$. Hence, by **(b)** applied $k-(n+1)$ times, we imply that $<w_1, \ldots, w_{n+1}> \in R_{n+1}$. This is a contradiction to the choice of n hence our assumption is wrong, by which **(d.2)** is satisfied.

$\square$

Proposition 5.2.3. *If **(c1)**, **(c2)** and **(c3)** are valid in a finite Kripke frame then the latter is a contact n-frame.*

Proof. By Claim 5.2.1 the Kripke frame satisfies conditions **(a)**, **(b)** and **(c)** of Definition 2.1.1 of a contact n-frame. Then, by Claim 5.2.2, it is a contact n-frame.

$\square$

6 Boolean frames and subframes. Finite Boolean algebras of regular closed sets of $\mathbb{R}^m$

In this section are discussed some properties of Boolean frames and intended Boolean algebras which play an essential role when considering the completeness features of the studied in the next section logic of n-ary contact.

6.1 Boolean subframes

Consider Boolean frames $\mathcal{B}$ and $\mathcal{B}_0$:

$$\begin{aligned}\mathcal{B} &= <A, 0_A, -_A, \cup_A, I> \\ \mathcal{B}_0 &= <A_0, 0_{A_0}, -_{A_0}, \cup_{A_0}, I_0>\end{aligned}$$

Definition 6.1.1. A Boolean frame $\mathcal{B}_0$ is called a *Boolean subframe* of $\mathcal{B}$, denoted by $\mathcal{B}_0 \subseteq \mathcal{B}$, if:

- $A_0 \subseteq A$ and A_0 is a *non-degenerate* Boolean algebra *subalgebra* of A
- For the n-ary relation symbol P and for every $a_1, \ldots, a_n$ of A_0 then it holds:

$$<a_1, \ldots, a_n> \in I_0(P) \quad \textit{iff} \quad <a_1, \ldots, a_n> \in I(P)$$

Claim 6.1.1. *Consider Boolean frames $\mathcal{B}_0$ and $\mathcal{B}$ such that $\mathcal{B}_0$ is a subframe of $\mathcal{B}$. Let $\mathcal{V}_0$ and $\mathcal{V}$ be valuations on $\mathcal{B}_0$ and $\mathcal{B}$ respectively. The following are satisfied:*

(i) For any Boolean term τ if $\mathcal{V}_0(x) = \mathcal{V}(x)$ for every x from $BV(\tau)$, then:

$$\widetilde{\mathcal{V}_0}(\tau) = \widetilde{\mathcal{V}}(\tau)$$

(ii) For any formula φ if $\mathcal{V}_0(x) = \mathcal{V}(x)$ for every x from $BV(\varphi)$, then:

$$<\mathcal{B}_0, \mathcal{V}_0> \Vdash \varphi \quad \text{iff} \quad <\mathcal{B}, \mathcal{V}> \Vdash \varphi$$

Proof. Denote:

$$\begin{aligned}\mathcal{B} &= <B, 0_B, -_B, \cup_B, I> \\ \mathcal{B}_0 &= <B_0, 0_{B_0}, -_{B_0}, \cup_{B_0}, I_0>\end{aligned}$$

(i):

The proof is by induction on the complexity of the Boolean term τ.

Trivially, $\widetilde{\mathcal{V}_0}(\tau) = \widetilde{\mathcal{V}}(\tau)$ when $\tau = x$ by definition. Furthermore, the case when $\tau = 0$, then $\widetilde{\mathcal{V}_0}(\tau) = 0_{B_0} = 0_B = \widetilde{\mathcal{V}}(\tau)$.

Consider: $\tau = -\tau_1$.

$$\widetilde{\mathcal{V}_0}(\tau) = \widetilde{\mathcal{V}_0}(-\tau_1) = -_{B_0}\widetilde{\mathcal{V}_0}(\tau_1)$$

By the inductive hypothesis $\widetilde{\mathcal{V}_0}(\tau_1) = \widetilde{\mathcal{V}}(\tau_1)$, hence, by B_0 subalgebra of B:

$$-_{B_0}\widetilde{\mathcal{V}_0}(\tau_1) = -_B\widetilde{\mathcal{V}}(\tau_1) = \widetilde{\mathcal{V}}(\tau)$$

Consider: $\tau = \tau_1 \cup \tau_2$.

$$\widetilde{\mathcal{V}_0}(\tau) = \widetilde{\mathcal{V}_0}(\tau_1 \cup \tau_2) = \widetilde{\mathcal{V}_0}(\tau_1) \cup_{B_0} \widetilde{\mathcal{V}_0}(\tau_2)$$

By inductive hypothesis and B_0 subalgebra of B it follows:

$$\widetilde{\mathcal{V}_0}(\tau_1) \cup_{B_0} \widetilde{\mathcal{V}_0}(\tau_2) = \widetilde{\mathcal{V}}(\tau_1) \cup_B \widetilde{\mathcal{V}}(\tau_2) = \widetilde{\mathcal{V}}(\tau_1 \cup \tau_2) = \widetilde{\mathcal{V}}(\tau)$$

(ii):
The proof is by induction on the complexity of the formula φ.
The case $\varphi = \bot$ is trivial.
Consider: $\varphi = (\tau_1 \equiv \tau_2)$

$$<\mathcal{B}_0, \mathcal{V}_0> \Vdash (\tau_1 \equiv \tau_2) \qquad \textit{iff} \qquad \widetilde{\mathcal{V}_0}(\tau_1) = \widetilde{\mathcal{V}_0}(\tau_2)$$

Then by *(i)*:

$$\textit{iff} \qquad \widetilde{\mathcal{V}}(\tau_1) = \widetilde{\mathcal{V}}(\tau_2) \qquad \textit{iff} \qquad <\mathcal{B}, \mathcal{V}> \Vdash (\tau_1 \equiv \tau_2)$$

Consider: $\varphi = P(\tau_1, \ldots, \tau_n)$

$$<\mathcal{B}_0, \mathcal{V}_0> \Vdash P(\tau_1, \ldots, \tau_n) \quad \textit{iff} \quad <\widetilde{\mathcal{V}_0}(\tau_1), \ldots, \widetilde{\mathcal{V}_0}(\tau_n)> \in I_0(P)$$

By *(i)*: $\widetilde{\mathcal{V}_0}(\tau_i) = \widetilde{\mathcal{V}}(\tau_i)$ for $1 \leq i \leq n$. Then, by $\mathcal{B}_0 \subseteq \mathcal{B}$:

$$\textit{iff} \quad <\widetilde{\mathcal{V}}(\tau_1), \ldots, \widetilde{\mathcal{V}}(\tau_n)> \in I(P) \quad \textit{iff} \quad <\mathcal{B}, \mathcal{V}> \Vdash P(\tau_1, \ldots, \tau_n)$$

The cases when $\varphi = \neg\varphi_1$ or $\varphi = \varphi_1 \vee \varphi_2$ follow directly by the inductive hypothesis.

□

Claim 6.1.2. *Every formula valid in a Boolean frame is also valid in all its subframes.*

Proof. Consider Boolean frames $\mathcal{B}_0$ and $\mathcal{B}$ such that $\mathcal{B}_0$ is a subframe of $\mathcal{B}$ and let φ be valid in $\mathcal{B}$. Let $\mathcal{V}_0$ be an arbitrary valuation on $\mathcal{B}_0$. By $\mathcal{B}_0 \subseteq \mathcal{B}$ then $\mathcal{V}_0$ is also a valuation on $\mathcal{B}$. Hence, by φ valid in $\mathcal{B}$, it follows that $<\mathcal{B}, \mathcal{V}_0> \Vdash \varphi$. By Claim 6.1.1 we obtain $<\mathcal{B}_0, \mathcal{V}_0> \Vdash \varphi$. $\mathcal{V}_0$ was an arbitrary valuation on $\mathcal{B}_0$ hence φ is valid in $\mathcal{B}_0$.

□

6.2 Finite Boolean algebras of polytopes or regular closed sets of $\mathbb{R}^m$

Definition 6.2.1. (BRC)
We say set W satisfies conditions **(BRC)** (for $\mathbb{R}^m$) if:

(i) Every element of W is a non-empty *regular closed* set of $\mathbb{R}^m$

(ii) $\cup W = \mathbb{R}^m$

(iii) For every $a, b \in W$, if $a \neq b$ then:

$$Int(a) \cap Int(b) = \emptyset$$

(iv) For every $a \in W$, for every $x \in a$ and for every open $o \ni x$ then:

$$o \cap Int(a) \neq \emptyset$$

(v) W is finite

Remark. Sets satisfying **(BRC)** exist. Trivial example is $W = \{\mathbb{R}^m\}$.

Remark. **(BRC)** condition *(ii)* could be required for particular *regular closed connected* subset of $\mathbb{R}^m$ instead of the whole $\mathbb{R}^m$.

Remark. Condition *(iv)* effectively is an implication of *(i)* as it is valid for any *regular closed* set of $\mathbb{R}^m$. We explicitly state this condition for convenience.

Claim 6.2.1. *Consider a set W satisfying **(BRC)**. Then for every A and B subsets of W the following are satisfied:*

- $\cup A \cup \cup B = \cup(A \cup B)$
- $Cl(Int(\cup A \cap \cup B)) = \cup(A \cap B)$
- $Cl(\mathbb{R}^m \setminus \cup A) = \cup(W \setminus A)$

Proof.

- $\cup A \cup \cup B = \cup(A \cup B)$

Trivially satisfied.

- $Cl(Int(\cup A \cap \cup B)) = \cup(A \cap B)$

Consider an arbitrary $x \in \cup(A \cap B)$. Then there is $e \in (A \cap B)$ such that $x \in e$. Hence x is in $\cup A$ and in $\cup B$ thus $x \in (\cup A \cap \cup B)$. Therefore:

$$\cup(A \cap B) \subseteq (\cup A \cap \cup B)$$

By the *monotonicity* as per Section 1.6.1 we imply:

$$Cl(Int(\cup(A \cap B))) \subseteq Cl(Int(\cup A \cap \cup B))$$

W is satisfying **(BRC)** conditions, hence, every element of W is a *regular closed* set. Therefore $\cup(A \cap B)$ is a union of finitely many *regular closed* sets, hence, a *regular closed* set, therefore:

$$\cup(A \cap B) = Cl(Int(\cup(A \cap B))),$$

by which:

$$\cup(A \cap B) \subseteq Cl(Int(\cup A \cap \cup B))$$

To prove the other direction, first, we will show the following helpful observation:

$$Int(\cup A \cap \cup B) \subseteq \cup(A \cap B) \tag{3}$$

Before that will demonstrate:

$$\text{If} \quad a \in A \text{ and } b \in B \text{ and } y \in Int(a) \quad \text{then} \quad \text{if } y \in b \text{ then } a = b \tag{4}$$

Assume $a \neq b$. By $y \in Int(a)$ there is open $o \ni y$ such that $o \subseteq Int(a)$. Then by $y \in b$ and by Definition 6.2.1 *(iv)* we imply $o \cap Int(b) \neq \emptyset$. This gives $Int(a) \cap Int(b) \neq \emptyset$ which is a contradiction with Definition 6.2.1 *(iii)*. Therefore $a = b$ which proves (4).

Now, for (3), let $z \in Int(\cup A \cap \cup B)$. Then there is $a \in A$ such that $z \in a$. By $z \in Int(\cup A \cap \cup B)$ there is an open $o \ni z$ such that $o \subseteq Int(\cup A \cap \cup B)$. By Definition 6.2.1 *(iv)* $o \cap Int(a) \neq \emptyset$. Let $y \in o \cap Int(a)$. Hence by $o \subseteq Int(\cup A \cap \cup B)$ there is $b \in B$ such that $y \in b$. Applying proprietary statement 4 on a, b and y we imply $a = b$, hence, $a \in B$. This proves (3).

Now, by (3) and *monotonicity* as per Section 1.6.1 we have:

$$Cl(Int(\cup A \cap \cup B)) \subseteq Cl(\cup(A \cap B))$$

$(A \cap B)$ is a finite set of *regular closed* sets, thus, a set of *closed* sets hence their union is a *closed* set. Then trivially:

$$Cl(\cup(A \cap B)) = \cup(A \cap B),$$

by which finally:

$$Cl(Int(\cup A \cap \cup B)) \subseteq \cup(A \cap B)$$

- $Cl(\mathbb{R}^m \setminus \cup A) = \cup(W \setminus A)$

By **(BRC)** conditions: $(\mathbb{R}^m \setminus \cup A) = (\cup W \setminus \cup A)$. Remark that for every $x \in (\cup W \setminus \cup A)$ we imply there is $b \in (W \setminus A)$ such that $x \in b$. Therefore $x \in \cup(W \setminus A)$, thus having:

$$(\mathbb{R}^m \setminus \cup A) \subseteq \cup(W \setminus A)$$

By **(BRC)** conditions $(W \setminus A)$ is a finite set of *regular closed* sets therefore $\cup(W \setminus A)$ is a *regular closed* set, in particular, it is a *closed* set then, by *monotonicity* as per Section 1.6.1 and the latter, subsequently:

$$Cl(\mathbb{R}^m \setminus \cup A) \subseteq Cl(\cup(W \setminus A)) = \cup(W \setminus A)$$

For the other direction, first, remark that by definition of *closure*:

$$Cl(\mathbb{R}^m \setminus \cup A) = \mathbb{R}^m \setminus Int(\mathbb{R}^m \setminus (\mathbb{R}^m \setminus \cup A)) = \mathbb{R}^m \setminus Int(\cup A)$$

Then we have to demonstrate:

$$\cup(W \setminus A) \subseteq \mathbb{R}^m \setminus Int(\cup A)$$

Let $x \in \cup(W \setminus A)$. Hence there is $a \in W \setminus A$ such that $x \in a$.

Assume $a \cap Int(\cup A) \neq \emptyset$. Take a witness $y \in a \cap Int(\cup A)$. By $y \in Int(\cup A)$ there is an open $o \ni y$ such that $o \subseteq Int(\cup A)$. By Definition 6.2.1 **(iv)** we have $o \cap Int(a) \neq \emptyset$. Take a witness z, hence, $z \in Int(a) \cap Int(\cup A)$. This means there is $b \in A$ such that $z \in b$. $z \in Int(a)$ thus there is an open $o' \ni z$ such that $o' \subseteq Int(a)$. By Definition 6.2.1 *(iv)* $o' \cap Int(b) \neq \emptyset$. This implies $Int(a) \cap Int(b) \neq \emptyset$. Applying Definition 6.2.1 *(iii)* the latter is possible only if $a = b$. This means $a \in W \setminus A$ and $a \in A$, which is a contradiction.

Therefore our assumption is wrong. Then, by $a \cap Int(\cup A) = \emptyset$, we imply $x \notin Int(\cup A)$. Thus $x \in \mathbb{R}^m \setminus Int(\cup A)$.

$\square$

Definition. For an arbitrary set S by $B_{RC}(S)$ denote:

$$B_{RC}(S) \risingdotseq \Big\{ \cup A \mid A \in \mathcal{P}(S) \Big\}$$

Claim 6.2.2. *The following statements hold:*

*(i) If a set W satisfies **(BRC)** conditions (Definition 6.2.1) then $B_{RC}(W)$ is a Boolean algebra subalgebra of the Boolean algebra of the regular closed sets of $\mathbb{R}^m$.*

(ii) Furthermore, if in addition every element of W is a polytope of $\mathbb{R}^m$ then $B_{RC}(W)$ is a Boolean algebra subalgebra of the Boolean algebra of the polytopes of $\mathbb{R}^m$.

Proof. Consider $\cup A \in B_{RC}(W)$, where $A \in \mathcal{P}(W)$. By definition every $a \in A$ is either a *regular closed* set of $\mathbb{R}^m$ as per *(i)* or a *polytope* of $\mathbb{R}^m$ as per *(ii)*. Hence such one also is $\cup A$.

Consider the structure:

$$B_{RC} = <B_{RC}(W), -_{RC}, \cup_{RC}, \cap_{RC}>,$$

where $-_{RC}$, $\cup_{RC}$ and $\cap_{RC}$ are as per Section 1.6.2. Then, by Claim 6.2.1, for arbitrary $\cup A$ and $\cup B$ from $B_{RC}(W)$, where A and B are elements of $\mathcal{P}(W)$ we have:

- $\cup A \cup_{RC} \cup B = \cup(A \cup B)$
- $\cup A \cap_{RC} \cup B = \cup(A \cap B)$
- $-_{RC} \cup A = \cup(W \setminus A)$

By this we imply that B_{RC} is closed under the operations of the Boolean algebra of either the *regular closed* sets for (i) or the *polytopes* for (ii) of $\mathbb{R}^m$. Therefore B_{RC} is a Boolean algebra subalgebra of either the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ for (i) or the *polytopes* of $\mathbb{R}^m$ for (ii).

□

Remark 6.2.1. Consider the Boolean algebra as per Claim 6.2.2 (either *(i)* or *(ii)*) $B_{RC}(W)$. Remark that as per Section 1.6.4:

- The *zero* of the Boolean algebra is 0_{RC}, namely, the empty set.
- The *unit* of the Boolean algebra is 1_{RC}, namely, $\mathbb{R}^m$.

Definition. For an arbitrary set S consider the structure:

$$\mathcal{B}_{RC}(S) = <B_{RC}(S), 0_{B_{RC}}, -_{B_{RC}}, \cup_{B_{RC}}, I_{RC}>$$

defined as follows:

- $0_{B_{RC}} = \emptyset$
- $-_{B_{RC}}(\cup A) = \cup(S \setminus A)$
- $\cup A \cup_{B_{RC}} \cup B = \cup(A \cup B)$

- For the n-ary relation symbol P:

$$<a_1,\ldots,a_n> \in I_{RC}(P) \qquad \textit{iff} \qquad a_1 \cap \ldots \cap a_n \neq \emptyset$$

Claim 6.2.3. *If a set W satisfies **(BRC)** conditions (Definition 6.2.1) then $\mathcal{B}_{RC}(W)$ is a Boolean frame.*

Proof. By Claim 6.2.2 and Remark 6.2.1 then $B_{RC}(W)$ is a Boolean algebra subalgebra of the Boolean algebra of either the *polytopes* (should all elements of W be *polytopes*) or the *regular closed* sets of $\mathbb{R}^m$. Furthermore, remark that $B_{RC}(W)$ is a non-degenerate algebra *iff* $W \neq \emptyset$. The latter is obtained by **(BRC)** Definition 6.2.1 *(ii)*. It remains to show the interpretation I_{RC} satisfies the conditions for a Boolean frame.

Let $<a_1,\ldots,a_n> \in I_{RC}(P)$. Then, by definition, $a_1 \cap \ldots \cap a_n \neq \emptyset$. Hence every $a_i \neq \emptyset = 0_{B_{RC}}$.

Furthermore, we have the following:

$$\begin{array}{ll}
 & <a_1,\ldots,a'_i \cup a''_i,\ldots,a_n> \in I_{RC}(P) \\
\textit{iff} & a_1 \cap \ldots \cap (a'_i \cup a''_i) \cap \ldots \cap a_n \neq \emptyset \\
\textit{iff} & a_1 \cap \ldots \cap a'_i \cap \ldots \cap a_n \neq \emptyset \quad \text{or} \quad a_1 \cap \ldots \cap a''_i \cap \ldots \cap a_n \neq \emptyset \\
\textit{iff} & <a_1,\ldots,a'_i,\ldots,a_n> \in I_{RC}(P) \quad \text{or} \quad <a_1,\ldots,a''_i,\ldots,a_n> \in I_{RC}(P)
\end{array}$$

□

6.3 Associations between finite Boolean and Kripke frames

Claim 6.3.1. *If $\mathcal{B}_0$ is a finite Boolean frame then it exists a finite Kripke frame $\mathcal{F}$ such that:*

$$B(\mathcal{F}) \cong \mathcal{B}_0$$

Proof. Denote:

$$\mathcal{B}_0 = <B_0, 0_{B_0}, -_{B_0}, \cup_{B_0}, I_0>$$

The Boolean algebra B_0 is finite then it is atomic. Then denote by W the set of all the atoms of B_0. Furthermore, as per normal, consider the ordering relation:

$$a \leq_{B_0} b \;\leftrightharpoons\; a \cup_{B_0} b = b \quad (\text{or equivalently: } a \cap_{B_0} b = a)$$

Consider the structure:

$$\mathcal{F} = <W, I>$$

where for any n-ary relation symbol P and $a_1,\ldots,a_n$ from W is satisfied:

$$<a_1,\ldots,a_n> \in I(P) \qquad \textit{iff} \qquad <a_1,\ldots,a_n> \in I_0(P)$$

Remark that $\mathcal{F}$ is finite Kripke frame. Denote the Boolean frame over $\mathcal{F}$ (as per Section 1.3.3 equivalently [1] Section 4, "*Correspondence*") defined as:

$$B(\mathcal{F}) = <\mathcal{P}(W), \emptyset, \setminus_W, \cup, I_B>$$

where (recall by definition):

- $<\mathcal{P}(W), \emptyset, \setminus_W, \cup>$ is the Boolean algebra of all subsets of W ($\setminus_W$ is the set theoretical difference with respect to the set W).

- For any $A^1, \ldots, A^n \in \mathcal{P}(W)$:

  $<A^1, \ldots, A^n> \in I_B(P)$
  iff
  exists $a_1 \in A^1, \ldots,$ exists $a_n \in A^n$ such that $<a_1, \ldots, a_n> \in I(P)$

Consider $f : \mathcal{P}(W) \to B_0$ defined as:

$$f(A) \coloneqq \cup_{B_0} A$$

Remark that by W finite then f is well defined.

Denote:

$$A_b = \left\{ a \in W \;\middle|\; a \leq_{B_0} b \right\}$$

Observation:

$$b = \cup_{B_0} A_b \tag{5}$$

Proof of the observation:

By definition of A_b we have $\cup_{B_0} A_b \leq_{B_0} b$. Let $b_0 = \cup_{B_0} A_b$ and assume $b \neq b_0$. Then, by $b_0 \leq_{B_0} b$, we imply $b \nleq_{B_0} b_0$. Hence $b \cap_{B_0} (-_{B_0} b_0) \neq 0_{B_0}$. B_0 is atomic Boolean algebra then there is $a \in W$ such that $a \leq_{B_0} b \cup_{B_0} (-_{B_0} b_0)$. Then, on one hand, this means $a \leq_{B_0} b$. It follows then $a \in A_b$, by which we imply $a \leq_{B_0} b_0$. On the other hand, $a \leq_{B_0} -_{B_0} b_0$ and this is a contradiction because by a atom then $a \neq 0_{B_0}$. This proves (5).

- f is *surjection*.

For any $b \in B_0$ applying (5): $b = \cup_{B_0} A_b = f(A_b)$.

- f is *injection*.

Let $A', A'' \in \mathcal{P}(W)$ and $A' \neq A''$. Without loss of generality let $a \in A'$ and $a \notin A''$. Assume $f(A') = f(A'')$. $a \in A'$ then $a \leq_{B_0} (\cup_{B_0} A')$. $f(A') = f(A'')$, consequently $a \leq_{B_0} (\cup_{B_0} A'')$. $a \notin A''$ and a is atom, hence, for every $b \in A''$ we have $a \cap_{B_0} b = 0_{B_0}$ because $a \neq b$ and b is also an atom. Considering $a \leq_{B_0} (\cup_{B_0} A'')$:

$$a = a \cap_{B_0} (\cup_{B_0} A'') = \underset{b \in A''}{\cup_{B_0}} \underbrace{(a \cap_{B_0} b)}_{=0_{B_0}} = 0_{B_0}$$

This is a contradiction with a atom. Therefore $f(A') \neq f(A'')$.

Finally we obtained that f is *bijection*.

- f is *Boolean isomorphism*.

  - Will show: $f(W \setminus A) = -_{B_0} f(A)$

Let $b = f(W \setminus A) = \cup_{B_0}(W \setminus A)$. By (5) and f bijection follows: $W \setminus A = A_b$.

Consider $A_{(-_{B_0} b)}$.

If $a \in A_b$ then $a \leq_{B_0} b$, thus $a \nleq_{B_0} (-_{B_0} b)$, otherwise $a = 0_{B_0}$ which contradicts with a atom. By the latter $a \notin A_{(-_{B_0} b)}$. On the other hand, if $a \notin A_b$ then $a \nleq_{B_0} b$. By a atom we imply $a \leq_{B_0} (-_{B_0} b)$ by which $a \in A_{(-_{B_0} b)}$.

In detail, to show the implication $a \leq_{B_0} (-_{B_0} b)$, assume $a \cap_{B_0} b \neq 0_{B_0}$. Then, by $(a \cap_{B_0} b) \leq_{B_0} a$ and a atom it follows $a \cap_{B_0} b = a$. This is a contradiction with $a \nleq_{B_0} b$. Hence $a \cap_{B_0} b = 0_{B_0}$ by which we obtain what we wanted, namely: $a \leq_{B_0} (-_{B_0} b)$.
Finally, for every $a \in W$:

$$a \notin A_b \qquad \textit{iff} \qquad a \in A_{(-_{B_0} b)}$$

This means:

$$A_{-_{B_0} b} = W \setminus A_b = W \setminus (W \setminus A) = A$$

Now, by (5) trivially we imply:

$$f(W \setminus A) = b = -_{B_0}(-_{B_0} b) = -_{B_0}(\cup_{B_0} A_{(-_{B_0} b)}) = -_{B_0} f(A_{(-_{B_0} b)}) = -_{B_0} f(A)$$

- Will show: $f(A' \cup A'') = f(A') \cup_{B_0} f(A'')$

By (5) we have:

$$f(A') = \cup_{B_0} A' \leq_{B_0} \cup_{B_0}(A' \cup A'') = f(A' \cup A'')$$

The same for $f(A'')$, therefore:

$$(f(A') \cup_{B_0} f(A'')) \leq_{B_0} f(A' \cup A'')$$

For the other direction, let $a \leq_{B_0} f(A' \cup A'')$, where a is atom. Then $a \leq_{B_0} \cup_{B_0}(A' \cup A'')$.

Assume that $a \notin (A' \cup A'')$. This means $a \neq b$ for every $b \in (A' \cup A'')$. By $(a \cap_{B_0} b) \leq_{B_0} a$ and a atom then $a \cap_{B_0} b = 0_{B_0}$ or $a \cap_{B_0} b = a$. The latter means $a \leq_{B_0} b$ which by b atom and $a \neq 0_{B_0}$ follows that $a = b$ which is a contradiction. Therefore $a \cap_{B_0} b = 0_{B_0}$ for every $b \in (A' \cup A'')$. Then:

$$a = a \cap_{B_0} (\cup_{B_0}(A' \cup A'')) = \bigcup_{B_0 \atop b \in (A' \cup A'')} \underbrace{(a \cap_{B_0} b)}_{=0_{B_0}} = 0_{B_0}$$

This is a contradiction. Therefore $a \in (A' \cup A'')$. Then $a \in A'$ or $a \in A''$, by which $a \leq_{B_0} f(A')$ or $a \leq_{B_0} f(A'')$, hence $a \leq_{B_0} (f(A') \cup_{B_0} f(A''))$. Now by this and (5):

$$f(A' \cup A'') = \cup_{B_0} A_{f(A' \cup A'')} \leq_{B_0} (f(A') \cup_{B_0} f(A''))$$

- f is isomorphism between Boolean frames, namely:

$$<A^1, \ldots, A^n> \in I_B(P) \qquad \textit{iff} \qquad <f(A^1), \ldots, f(A^n)> \in I_0(P)$$

Let $<A^1, \ldots, A^n> \in I_B(P)$. Then exists $a_1 \in A^1$, ..., exists $a_n \in A^n$ such that $<a_1, \ldots, a_n> \in I(P)$ hence, by definition, $<a_1, \ldots, a_n> \in I_0(P)$. By $\mathcal{B}_0$ *Boolean frame* we imply $<\cup_{B_0} A^1, \ldots, \cup_{B_0} A^n> \in I_0(P)$.

Now, let $<\cup_{B_0} A^1, \ldots, \cup_{B_0} A^n> \in I_0(P)$. By $\mathcal{B}_0$ *Boolean frame* and *finite* then for some $a_1 \in A^1$, ..., for some $a^n \in A^n$ we have $<a_1, \ldots, a_n> \in I_0(P)$. For any i, $1 \leq i \leq n$, $A^i \subseteq W$ then, by definition, $<a_1, \ldots, a_n> \in I(P)$ by which $<A^1, \ldots, A^n> \in I_B(P)$.

$\square$

Claim 6.3.2. *Let the set W satisfy **(BRC)** conditions (Definition 6.2.1) and let $\mathcal{F} = <W, I>$ be a Kripke frame such that the interpretation I interprets the n-ary relation symbol P with the standard contact relation $\mathcal{C}_n$, namely:*

$$<a_1, \ldots, a_n> \in I(P) \qquad \text{iff} \qquad a_1 \cap \ldots \cap a_n \neq \emptyset$$

Then:

$$\mathcal{B}_{RC}(W) \cong B(\mathcal{F})$$

Proof. First, remark that by Claim 6.2.3 $\mathcal{B}_{RC}$(W) is a Boolean frame.

Denote $B(\mathcal{F})$ as in the proof of Claim 6.3.1:

$$B(\mathcal{F}) = <\mathcal{P}(W), \emptyset, \setminus_W, \cup, I_B>$$

Consider $f : \mathcal{P}(W) \to B_{RC}(W)$ defined as:

$$f(A) \ \leftrightharpoons\ \cup A$$

Remark that by definition of $B_{RC}(W)$ f is well defined.

- f is *surjection.*

By definition of $B_{RC}(W)$ f is *onto.*

- f is *injection.*

Let $A, B \in \mathcal{P}(W)$ and $A \neq B$. Without loss of generality let $a \in A$ and $a \notin B$. Assume $f(A) = f(B)$, meaning $\cup A = \cup B$. $a \in A$ then $a \subseteq \cup A$ hence $Int(a) \subseteq Int(\cup A)$. By Definition 6.2.1 *(i)* and *(iv)* $Int(a)$ is *non-empty*, thus, consider arbitrary $x \in Int(a)$. Then there is open $o \ni x$ such that $o \subseteq Int(a)$. $x \in Int(a)$ then $x \in \cup A$. $\cup A = \cup B$ then there is $b \in B$ such that $x \in b$. By Definition 6.2.1 *(iv)* $o \cap Int(b) \neq \emptyset$. Therefore $Int(a) \cap Int(b) \neq \emptyset$. By Definition 6.2.1 *(iii)* it follows that $a = b$ thus $a \in B$, which is a contradiction. Our assumption is wrong.

- f is *Boolean isomorphism.*

By Claim 6.2.3 the following are satisfied:

$$f(W \setminus A) = \cup(W \setminus A) = -_{B_{RC}}(\cup A) = -_{B_{RC}} f(A)$$
$$f(A \cup B) = \cup(A \cup B) = (\cup A) \cup_{B_{RC}} (\cup B) = f(A) \cup_{B_{RC}} f(B)$$

- f is isomorphism between Boolean frames.

The following equivalences hold:

$<A_1, \ldots, A_n> \in I_B(P)$

iff

exists $a_1 \in A_1, \ldots,$ exists $a_n \in A_n$ such that $<a_1, \ldots, a_n> \in I(P)$

iff

exists $a_1 \in A_1, \ldots,$ exists $a_n \in A_n$ such that $a_1 \cap \ldots \cap a_n \neq \emptyset$

iff

$\cup A_1 \cap \ldots \cap \cup A_n \neq \emptyset$

iff

$<\cup A_1, \ldots, \cup A_n> \in I_{RC}(P)$

□

7 Boolean logic of n-ary contact. Completeness

In this section the formal system of the logic of n-ary contact is defined. The intended result is to show its completeness with respect to certain classes of Boolean frames, namely, those with interpretation the *standard contact semantics* and carrier subalgebra of the Boolean algebra of either the *polytopes* or the *regular closed* sets of $\mathbb{R}^m$.

7.1 Boolean semantics and axiomatisation

As mentioned, the n-ary contact logic semantically will be considered in certain classes of Boolean frames.

7.1.1 Boolean frames of n-ary contact

Consider $\mathbb{R}^m$ for particular $m \geq 1$.

Definition 7.1.1.

- $\mathbf{PRC(\mathbb{R}^m)}$ $\leftrightharpoons$ the class of Boolean frames with carrier (non-degenerate) subalgebra of the Boolean algebra of the *polytopes* of $\mathbb{R}^m$ and interpretation of the relation symbols the standard *contact relation*
- $\mathbf{RC(\mathbb{R}^m)}$ $\leftrightharpoons$ the class of Boolean frames with carrier (non-degenerate) subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ and interpretation of the relation symbols the standard *contact relation*

Apparently, every Boolean frame of $\mathbf{PRC(\mathbb{R}^m)}$ also is of $\mathbf{RC(\mathbb{R}^m)}$.

As an example, consider $\mathcal{B} \in \mathbf{PRC(\mathbb{R}^m)}$. Denote:

$$\mathcal{B} = <B, 0_B, -_B, \cup_B, I>$$

Then:

- B is a (non-degenerate) subalgebra of the Boolean algebra of the polytopes of $\mathbb{R}^m$
- For the k-ary relation symbol P then $I(P) = \mathcal{C}_k$. In particular, this means:

$$<a_1, \ldots, a_k> \in I(P) \qquad \textit{iff} \qquad a_1 \cap \ldots \cap a_k \neq \emptyset$$

In analogy to the former, for the case when $\mathcal{B} \in \mathbf{RC(\mathbb{R}^m)}$ the difference is only that B is (non-degenerate) subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$.

The following claim proves that the definition of the classes $\mathbf{PRC(\mathbb{R}^m)}$ and $\mathbf{RC(\mathbb{R}^m)}$ is correct.

Claim 7.1.1. *Consider:*

$$\mathcal{B} = <B, 0_B, -_B, \cup_B, I>,$$

where (for particular $m \geq 1$):

- *B is a (non-degenerate) subalgebra of either the Boolean algebra of the polytopes of $\mathbb{R}^m$ or the Boolean algebra of the regular closed sets of $\mathbb{R}^m$.*
- *For the k-ary relation symbol P:*

$$<a_1, \ldots, a_k> \in I(P) \qquad \text{iff} \qquad a_1 \cap \ldots \cap a_k \neq \emptyset$$

Then $\mathcal{B}$ is a Boolean frame.

Proof. Let $<a_1, \ldots, a_k> \in I(P)$. By definition: $a_1 \cap \ldots \cap a_k \neq \emptyset$. Therefore $a_i \neq \emptyset = 0_B$ for any i, $1 \leq i \leq k$.

Furthermore, by definition, the following equivalences hold:

$$<a_1, \ldots, (a_i' \cup_B a_i''), \ldots, a_k> \in I(P)$$
iff
$$a_1 \cap \ldots \cap (a_i' \cup_B a_i'') \cap \ldots \cap a_k \neq \emptyset$$
iff
$$a_1 \cap \ldots \cap (a_i' \cup a_i'') \cap \ldots \cap a_k \neq \emptyset$$
iff
$$a_1 \cap \ldots \cap a_i' \cap \ldots \cap a_k \neq \emptyset \quad \text{or} \quad a_1 \cap \ldots \cap a_i'' \cap \cdots \cap a_k \neq \emptyset$$
iff
$$<a_1, \ldots, a_i', \ldots, a_k> \in I(P) \quad \text{or} \quad <a_1, \ldots, a_i'', \ldots, a_k> \in I(P)$$

□

As per Section 1.6.4 and Section 1.6.2 $PRC(m)$ and $RC(m)$ are Boolean algebras, namely, the Boolean algebra of the *polytopes* and the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ respectively.

Definition.

- Denote by $\mathcal{PRC}(\mathbb{R}^m)$ the Boolean frame with a carrier the Boolean algebra of the *polytopes* of $\mathbb{R}^m$.
- Denote by $\mathcal{RC}(\mathbb{R}^m)$ the Boolean frame with a carrier the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$.

In particular:

$$\mathcal{PRC}(\mathbb{R}^m) = <PRC(\mathbb{R}^m), 0_{RC}, -_{RC}, \cup_{RC}, I>$$
$$\mathcal{RC}(\mathbb{R}^m) = <RC(\mathbb{R}^m), 0_{RC}, -_{RC}, \cup_{RC}, I>,$$

where I interprets the k-ary relation symbol P as the standard *contact relation*, namely, $I(P) = \mathcal{C}_k$.

Remark that, by Definition 7.1.1 it trivially follows that:

- $\mathcal{PRC}(\mathbb{R}^m)$ is from the class $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$.
- $\mathcal{RC}(\mathbb{R}^m)$ is from the class $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$.

Furthermore, by Definition 6.1.1 it directly follows that:

- Every Boolean frame $\mathcal{B}$ of the class $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$ is a subframe of $\mathcal{PRC}(\mathbb{R}^m)$.
- Every Boolean frame $\mathcal{B}$ of the class $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$ is a subframe of $\mathcal{RC}(\mathbb{R}^m)$.

Then the following claim holds:

Claim 7.1.2. *For an arbitrary formula φ of $L_{\mathcal{R}}$:*

(i) φ is valid in the class $\boldsymbol{PRC(}\mathbb{R}^m\boldsymbol{)}$ iff *φ is valid in the Boolean frame $\mathcal{PRC}(\mathbb{R}^m)$*

(ii) φ is valid in the class $\boldsymbol{RC(}\mathbb{R}^m\boldsymbol{)}$ iff *φ is valid in the Boolean frame $\mathcal{RC}(\mathbb{R}^m)$*

Proof. For (i), consider φ be valid in $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$. Then φ is valid in $\mathcal{PRC}(\mathbb{R}^m)$ as a member of the class $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$. Let now φ be valid in $\mathcal{PRC}(\mathbb{R}^m)$. Consider an arbitrary Boolean frame $\mathcal{B}$ of $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$. Recall that any Boolean frame of $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$ is a *subframe* of $\mathcal{PRC}(\mathbb{R}^m)$. Then, by Claim 6.1.2, it follows that φ is valid in $\mathcal{B}$. Therefore φ is valid in $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$. The same reasoning applies for (ii) as well.

□

Definition.

- For an arbitrary Boolean frame $\mathcal{B}$ denote by $\mathcal{L}(\mathcal{B})$ the logic of the Boolean frame $\mathcal{B}$, namely, all formulas valid in $\mathcal{B}$.
- For an arbitrary class of Boolean frames C denote by $\mathcal{L}(C)$ the logic of the class C, namely, all formulas valid in the class C.

Then Claim 7.1.2 says that:

(i) $\mathcal{L}(\mathbf{PRC(}\mathbb{R}^m\mathbf{)}) = \mathcal{L}(\mathcal{PRC}(\mathbb{R}^m))$

(ii) $\mathcal{L}(\mathbf{RC(}\mathbb{R}^m\mathbf{)}) = \mathcal{L}(\mathcal{RC}(\mathbb{R}^m))$

7.1.2 Formal system of logic of n-ary contact

Consider the axiom schemes as in Section 5.1.

Definition 7.1.2.

- *Cont* $\leftrightharpoons$ the axioms **(c1)**, **(c2)**, **(c3)** and **(c4)**
- $Cont + PRC1$ $\leftrightharpoons$ *Cont* plus the axioms **PRC1**.

We adopt the formal logical system as stated in Section 1.3.4 literally being [1], Section 7.1, *"Axiomatization"*. Henceforth, will consider the formal systems:

$$\mathcal{L}_{Cont} \quad \text{and} \quad \mathcal{L}_{Cont+PRC1}$$

7.2 Correctness

Proposition 7.2.1. *(Correctness in* ***RC(***$\mathbb{R}^m$***)****,* $m \geq 1$*)*
For every formula φ *of the language* $L_{\mathcal{R}}$*:*

$$\vdash_{\mathcal{L}_{Cont}} \varphi \qquad \textit{implies} \qquad \Vdash_{\boldsymbol{RC(}\mathbb{R}^m\boldsymbol{)}} \varphi \qquad , \quad m \geq 1$$

Proof. Consider deduction $\varphi_1, \ldots, \varphi_k$ in $\mathcal{L}_{Cont}$ for φ, where $\varphi_k = \varphi$. By induction on the length of the deduction sequence will show that every element of it is valid in $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$ which proves the Proposition.

Consider the induction step case, namely, when φ_i is obtained via *M.P.* by φ_j and $(\varphi_j \implies \varphi_i)$ as elements in the deduction sequence before φ_i. By induction hypothesis for every Boolean frame $\mathcal{B}$ from $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$ and valuation $\mathcal{V}$ on $\mathcal{B}$ we have: $<\mathcal{B},\mathcal{V}> \Vdash \varphi_j$ and $<\mathcal{B},\mathcal{V}> \Vdash (\varphi_j \implies \varphi_i)$. Hence, trivially, $<\mathcal{B},\mathcal{V}> \Vdash \varphi_i$. Therefore φ_i is valid in $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$.

As an induction base, we need to verify each of the axiom groups (1) to (7) for $\mathcal{L}_{Cont}$ as per Section 1.3.4 (equivalently [1], Section 7.1, *"Axiomatization"*).

(1) to (6) are satisfied for every Boolean frame. It remains to show (7) which is to demonstrate all **(c1)** to **(c4)** are valid in $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$.

Consider $\mathcal{B}$ of $\mathbf{RC(}\mathbb{R}^m\mathbf{)}$ and an arbitrary valuation $\mathcal{V}$ on $\mathcal{B}$.

(c1):

Let $<\mathcal{B},\mathcal{V}> \Vdash P(x_1, \ldots, x_n)$. Given $\sigma : n \to n$. Then, by definition we have:

$$\widetilde{\mathcal{V}}(x_1) \cap \ldots \cap \widetilde{\mathcal{V}}(x_n) \neq \emptyset$$

thus:

$$\widetilde{\mathcal{V}}(x_{\sigma(1)}) \cap \ldots \cap \widetilde{\mathcal{V}}(x_{\sigma(n)}) \neq \emptyset$$

by which:

$$<\mathcal{B},\mathcal{V}> \Vdash P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

(c2):

$$\begin{aligned} &<\mathcal{B},\mathcal{V}> \Vdash P(x_1, x_1, \ldots, x_n) \\ &\textit{iff} \\ &\widetilde{\mathcal{V}}(x_1) \cap \ldots \cap \widetilde{\mathcal{V}}(x_n) \neq \emptyset \\ &\textit{iff} \\ &<\mathcal{B},\mathcal{V}> \Vdash Q(x_1, \ldots, x_n) \end{aligned}$$

(c3):

Let $<\mathcal{B},\mathcal{V}> \Vdash \neg(x \equiv 0)$. Hence $\widetilde{\mathcal{V}}(x) \neq 0_B = \emptyset$. Then, apparently $\widetilde{\mathcal{V}}(x) \cap \widetilde{\mathcal{V}}(x) \neq \emptyset$, by which: $<\mathcal{B},\mathcal{V}> \Vdash P(x, x)$.

(c4):

Let:

$$<\mathcal{B},\mathcal{V}> \Vdash (\neg(x \equiv 0) \wedge \neg(-x \equiv 0))$$

By this we imply:

$$\widetilde{\mathcal{V}}(x) \neq 0_B = \emptyset \quad \text{and} \quad \widetilde{\mathcal{V}}(-x) \neq 0_B = \emptyset$$

By B, the carrier of the Boolean frame $\mathcal{B}$, being a subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^m$ then:

$$\mathbb{R}^m = \widetilde{\mathcal{V}}(1) = \widetilde{\mathcal{V}}(x \cup -x) = \widetilde{\mathcal{V}}(x) \cup \widetilde{\mathcal{V}}(-x)$$

$\widetilde{\mathcal{V}}(x)$ and $\widetilde{\mathcal{V}}(-x)$ are elements of the Boolean algebra B hence they are *closed* sets by definition. Furthermore, $\mathbb{R}^m$ is *connected* set (see Section 1.5.2). Therefore, by $\widetilde{\mathcal{V}}(x)$ and $\widetilde{\mathcal{V}}(-x)$ being non-empty *closed* sets and $\mathbb{R}^m = \widetilde{\mathcal{V}}(x) \cup \widetilde{\mathcal{V}}(-x)$ we imply:

$$\widetilde{\mathcal{V}}(x) \cap \widetilde{\mathcal{V}}(-x) \neq \emptyset$$

This gives:

$$<\mathcal{B}, \mathcal{V}> \Vdash P(x, -x)$$

$\square$

Proposition 7.2.2. *(Correctness in* ***PRC(***$\mathbb{R}^m$***)****,* $m \geq 1$*)*
For every formula φ *of the language* $L_{\mathcal{R}}$*:*

$$\vdash_{\mathcal{L}_{Cont}} \varphi \qquad \textit{implies} \qquad \Vdash_{\boldsymbol{PRC(}\mathbb{R}^m\boldsymbol{)}} \varphi \qquad , \quad m \geq 1$$

Proof. Directly by Proposition 7.2.1 and the fact that the elements of the class **PRC(**$\mathbb{R}^m$**)** are elements of **RC(**$\mathbb{R}^m$**)**.

$\square$

Proposition 7.2.3. *(Correctness in* ***PRC(***$\mathbb{R}^1$***)****)*
For every formula φ *of the language* $L_{\mathcal{R}}$*:*

$$\vdash_{\mathcal{L}_{Cont+PRC1}} \varphi \qquad \textit{implies} \qquad \Vdash_{\boldsymbol{PRC(}\mathbb{R}^1\boldsymbol{)}} \varphi$$

Proof. By Proposition 7.2.2 and the proof of Proposition 7.2.1 it only remains to show that **PRC1** is valid in **PRC(**$\mathbb{R}^1$**)**.

Consider arbitrary Boolean frame $\mathcal{B}$ from **PRC(**$\mathbb{R}^1$**)**. Let $\mathcal{V}$ be arbitrary valuation on $\mathcal{B}$.

Let $<\mathcal{B}, \mathcal{V}> \Vdash P(x_1, x_2, x_3)$. Hence:

$$\widetilde{\mathcal{V}}(x_1) \cap \widetilde{\mathcal{V}}(x_2) \cap \widetilde{\mathcal{V}}(x_3) \neq \emptyset$$

Let $A_1 = \widetilde{\mathcal{V}}(x_1)$, $A_2 = \widetilde{\mathcal{V}}(x_2)$, $A_3 = \widetilde{\mathcal{V}}(x_3)$. Assume that for every A_i and A_j from $\{A_1, A_2, A_3\}$, $i \neq j$, we have:

$$Int(A_i) \cap Int(A_j) = \emptyset$$

Let $b \in A_1 \cap A_2 \cap A_3$. Without loss of generality suppose $b \in Int(A_1)$. Then there is open segment $o \ni b$ such that $o \subseteq Int(A_1)$. $b \in A_2$ and A_2 is a polytope of $\mathbb{R}^m$, hence, $o \cap Int(A_2) \neq \emptyset$ by which $Int(A_1) \cap Int(A_2) \neq \emptyset$. This is a contradiction with the main assumption.

Therefore b is boundary point for all A_1, A_2 and A_3. All those elements are polytopes then b is a boundary point for any of the finitely many closed segments and rays the element being union of. Without loss of generality assume the segment or the ray that b belongs to in A_1 is on the "left" of b meaning in the interval $(-\infty, b]$. By the same reasoning if the closed segment or ray that b belongs to in A_2 is also in the interval $(-\infty, b]$ then apparently those segments or rays from A_1 and A_2 respectively will have non-empty intersection of their interiors. This means $Int(A_1) \cap Int(A_2) \neq \emptyset$ which is a contradiction with the main assumption. Therefore, the segment or ray for A_2 is in the interval $[b, +\infty)$. Nevertheless for A_3, having the same reasoning, then its segment or ray will either be in $(-\infty, b]$ thus $Int(A_1) \cap Int(A_3) \neq \emptyset$ or in $[b, +\infty)$ thus $Int(A_2) \cap Int(A_3) \neq \emptyset$. Therefore our main assumption is wrong.

Now, without loss of generality, let $Int(A_1) \cap Int(A_2) \neq \emptyset$. A_1 and A_2 can be presented as finite unions of *non-intersecting* closed segments or rays. Therefore the union of their interiors is the interior of A_1 and A_2 respectively. By $Int(A_1) \cap Int(A_2) \neq \emptyset$ this means there will be point from the interior of closed segment or ray of the union for A_1 that is in the interior of closed segment or ray of the union for A_2. Denote those segments or rays s_1 for A_1 and s_2 for A_2. We have that $s_1 \cap s_2 \subseteq A_1 \cap A_2$ hence $Int(s_1 \cap s_2) \subseteq Int(A_1 \cap A_2)$. By s_1 and s_2 closed segments or rays and there is b' such that $b' \in Int(s_1)$ and $b' \in Int(s_2)$ then apparently $Int(s_1 \cap s_2)$ is non-empty. By all these we imply:

$$Int(A_1 \cap A_2) \neq \emptyset$$

Therefore:

$$\widetilde{\mathcal{V}}(x_1) \cap_B \widetilde{\mathcal{V}}(x_2) = Cl(Int(\widetilde{\mathcal{V}}(x_1) \cap \widetilde{\mathcal{V}}(x_2))) \neq \emptyset = 0_B$$

Hence:

$$<\mathcal{B}, \mathcal{V}> \Vdash \neg(x_1 \cap x_2 \equiv 0)$$

By this we obtain:

$$<\mathcal{B}, \mathcal{V}> \Vdash \mathbf{PRC1}$$

$\square$

7.3 Completeness

Proposition 7.3.1. *For every formula φ of the language $L_{\mathcal{R}}$:*

$$\Vdash_{\boldsymbol{PRC}(\mathbb{R}^m)} \varphi \quad \textit{implies} \quad \vdash_{\mathcal{L}_{Cont}} \varphi \qquad , \quad m \geq 2$$

Proof. Assume:

$$\nvdash_{\mathcal{L}_{Cont}} \varphi \tag{6}$$

Then, by Proposition 1.3.3 (recall it being an inference of "Proposition 26" in [1] *"Boolean logics with relations"*) we have:

$$\nVdash_{C^B_{Cont}} \varphi \tag{7}$$

Therefore it exists Boolean frame $\mathcal{B}$:

$$\mathcal{B} \text{ is from } C^B_{Cont} \tag{8}$$

and valuation $\mathcal{V}$ on $\mathcal{B}$ such that:

$$<\mathcal{B}, \mathcal{V}> \Vdash \neg\varphi$$

Let $\mathcal{B}$ has carrier B. Let:

$$A \leftrightharpoons \left\{ \mathcal{V}(x) \mid x \in BV(\varphi) \right\}$$

Let B_0 be the subalgebra of B *generated* by A. Remark then B_0 is a *non-degenerate* Boolean algebra. In particular, the *zero* and the *unit* of B are those in B_0. Furthermore, B is non-degenerate by definition thus the *zero* and the *unit* are not equal. This is a sufficient condition for B_0 to be *non-degenerate*. As per Definition 6.1.1, consider then the Boolean frame $\mathcal{B}_0$ a subframe of $\mathcal{B}$ and with carrier B_0:

$$\mathcal{B}_0 \subseteq \mathcal{B}$$

Let $\mathcal{V}_0$ be a valuation on $\mathcal{B}_0$ such that $\mathcal{V}_0(x) = \mathcal{V}(x)$ for all $x \in BV(\varphi)$ and $\mathcal{V}_0(x)$ be arbitrary for any $x \notin BV(\varphi)$. Then by Claim 6.1.1:

$$<\mathcal{B}_0, \mathcal{V}_0> \Vdash \neg\varphi$$

By (8) we have:

$$\mathcal{B} \Vdash Cont \tag{9}$$

Now, by $\mathcal{B}_0 \subseteq \mathcal{B}$ and by Claim 6.1.2:

$$\mathcal{B}_0 \Vdash Cont \tag{10}$$

Remark that by A finite then the Boolean algebra B_0 is finite. Then $\mathcal{B}_0$ is finite and by Claim 6.3.1 it exists finite Kripke frame $\mathcal{F}_0$ such that:

$$B(\mathcal{F}_0) \cong \mathcal{B}_0$$

By (10) and "Proposition 5" in [1] *"Boolean logics with relations"* cited as Proposition 1.3.1 we imply:

$$\mathcal{F}_0 \Vdash Cont \tag{11}$$

Consider the valuation $\mathcal{V}_0'$ on $B(\mathcal{F}_0)$ corresponding to $\mathcal{V}_0$ on $\mathcal{B}_0$ by the isomorphism between $B(\mathcal{F}_0)$ and $\mathcal{B}_0$. Then:

$$<B(\mathcal{F}_0), \mathcal{V}_0'> \Vdash \neg\varphi$$

As per Proposition 1.3.1 $\mathcal{V}_0'$ effectively is valuation on $\mathcal{F}_0$ and by this same proposition we imply:

$$<\mathcal{F}_0, \mathcal{V}_0'> \Vdash \neg\varphi$$

Recall (11). Then by $\mathcal{F}_0 \Vdash$ **(c1)**, $\mathcal{F}_0 \Vdash$ **(c2)**, $\mathcal{F}_0 \Vdash$ **(c3)** and $\mathcal{F}_0$ finite applying Proposition 5.2.3 we obtain $\mathcal{F}_0$ is contact n-frame.

Remark that the case when B_0 is the minimal non-degenerate Boolean algebra then $\mathcal{F}_0$ has carrier singleton. In particular this means a contact 1-frame. Despite the steps to follow are valid for this Kripke frame to avoid formal conflicts with definition of n-graph (in particular $n \geq 2$) will consider this case a bit later separately. Consider then the carrier of $\mathcal{F}_0$ with cardinality greater than 1. Now by $\mathcal{F}_0 \Vdash$ **(c4)** and Claim 2.4.1 it follows that the induced by $\mathcal{F}_0$ contact n-graph G_0:

$$G_0 \text{ is connected} \tag{12}$$

Apply Procedure 4.1 on the connected contact n-graph G_0. Let the resulting graph be G'. By Observation 4.2.5:

$$G' \text{ is acyclic}$$

By Observation 4.2.7:

$$G' \text{ is connected} \tag{13}$$

Consider the induced frame by G_0. Then by Claim 2.3.4 it effectively is $\mathcal{F}_0$. Now, as per Claim 2.3.3 consider the induced by G' contact n-frame $\mathcal{F}'$. Then the conditions for Claim 4.2.1 are satisfied. Therefore:

$$\mathcal{F}' \text{ is } \textit{p-morphic} \text{ preimage of } \mathcal{F}_0$$

Let f be p-morphism from $\mathcal{F}'$ onto $\mathcal{F}_0$. Consider valuation $\mathcal{V}'$ on $\mathcal{F}'$ such that:

$$s \in \mathcal{V}'(x) \qquad \textit{iff} \qquad f(s) \in \mathcal{V}'_0(x)$$

(trivially such valuation exists). Then by $<\mathcal{F}_0, \mathcal{V}'_0> \Vdash \neg\varphi$ and p-morphisms properties we imply:

$$<\mathcal{F}', \mathcal{V}'> \Vdash \neg\varphi$$

Recall that:

$$G' \text{ is acyclic and connected} \tag{14}$$

Then:

$$\text{Apply Procedure 3.1 on } G' \text{ (as per Remark 3.1.1)} \tag{15}$$

Hence we obtain partitioning $S = \{W_1, \ldots, W_s\}$ of $\mathbb{R}^m$, where:

$$W_i \text{ are } \textit{polytopes} \text{ of } \mathbb{R}^m \tag{16}$$

Consider:

$$\mathcal{F}_{RC} = <S, I>$$

being the Kripke frame with carrier S and I mapping the k-ary relational symbol, $k \geq 1$, to the k-ary contact relation $\mathcal{C}_k$. In particular, this means for the k-ary relation symbol P:

$$<W_{i_1}, \ldots, W_{i_k}> \in I(P) \qquad \textit{iff} \qquad W_{i_1} \cap \ldots \cap W_{i_k} \neq \emptyset$$

Then by Claim 3.1.1:

$$\mathcal{F}_{RC} \cong \mathcal{F}' \tag{17}$$

Furthermore, by $<\mathcal{F}', \mathcal{V}'> \Vdash \neg\varphi$, then for the valuation $\mathcal{V}_{RC}$ on $\mathcal{F}_{RC}$ corresponding to $\mathcal{V}'$ due to the isomorphism between $\mathcal{F}_{RC}$ and $\mathcal{F}'$ we imply:

$$<\mathcal{F}_{RC}, \mathcal{V}_{RC}> \Vdash \neg\varphi$$

We obtain:

$$S \text{ satisfies Definition 6.2.1 and every element of it is a polytope} \tag{18}$$

due to the following observations:

- By Observation 3.1.3 every element of S is a *polytope*
- By Observation 3.1.4: $\cup S = \mathbb{R}^m$
- By Observation 3.1.5: $Int(W_{i_1}) \cap Int(W_{i_2}) = \emptyset$, $i_1 \neq i_2$
- By $W_i \in S$ is a polytope then whichever point x of W_i is taken then any open $o \ni x$ will have the property: $o \cap Int(W_i) \neq \emptyset$
- (by definition) S is finite

Now, the case when $\mathcal{F}_0$ is with carrier singleton then consider $S = \{\mathbb{R}^m\}$. Let the carrier of $\mathcal{F}_0$ be $\{w\}$. Then

$$g(w) \;\leftrightharpoons\; \mathbb{R}^m$$

is isomorphism between $\mathcal{F}_0$ and $\mathcal{F}_{RC}$. Trivially g is bijection. Recall that $\mathcal{F}_0$ satisfies the conditions for n-frame. Consider arbitrary k-ary relation symbol P. For $k \geq 2$ follows that by applying Definition 2.1.1 **(c)** and $k-2$ times **(b)** that $<w, \ldots, w>$ is in the relation of the interpretation of P. The same follows for the case $k < 2$ again from **(c)** and applying **(b)**. By definition of I for $\mathcal{F}_{RC}$ then, again, we imply $<\mathbb{R}^m, \ldots, \mathbb{R}^m> \in I(P)$ for any relation symbol P. This means g trivially is isomorphism between the Kripke frames $\mathcal{F}_0$ and $\mathcal{F}_{RC}$. Now for the valuation $\mathcal{V}_{RC}$ corresponding to $\mathcal{V}'_0$ by the isomorphism g then again: $<\mathcal{F}_{RC}, \mathcal{V}_{RC}> \Vdash \neg\varphi$. Finally, remark that the conditions of Definition 6.2.1 are trivially satisfied by S.

Therefore, in all cases, $\mathcal{F}_{RC}$ satisfies the conditions in Claim 6.3.2 by which:

$$\mathcal{B}_{RC}(S) \cong B(\mathcal{F}_{RC})$$

By that isomorphism, again, there is an appropriate valuation $\mathcal{V}'_{RC}$ on $\mathcal{B}_{RC}(S)$ corresponding to $\mathcal{V}_{RC}$ on $B(\mathcal{F}_{RC})$ the latter being the one on $\mathcal{F}_{RC}$ as per Proposition 1.3.1. Then, by the same proposition and the isomorphism, we imply:

$$<\mathcal{B}_{RC}(S), \mathcal{V}'_{RC}> \;\Vdash\; \neg\varphi$$

By Claim 6.2.2:

$$B_{RC}(S) \text{ is a subalgebra of the Boolean algebra of the } \textit{polytopes} \text{ of } \mathbb{R}^m \text{ by } \textit{(ii)} \tag{19}$$

Therefore:

$$\mathcal{B}_{RC}(S) \in \mathbf{PRC}(\mathbb{R}^m) \tag{20}$$

This means then φ is valid in $\mathcal{B}_{RC}$(S), hence, in particular:

$$<\mathcal{B}_{RC}(S), \mathcal{V}'_{RC}> \Vdash \varphi$$

This is a contradiction. Therefore our assumption is wrong, hence:

$$\vdash_{\mathcal{L}_{Cont}} \varphi \tag{21}$$

□

Proposition 7.3.2. *For every formula φ of the language $L_{\mathcal{R}}$:*

$$\Vdash_{\boldsymbol{RC}(\mathbb{R}^m)} \varphi \quad \textit{implies} \quad \vdash_{\mathcal{L}_{Cont}} \varphi \qquad , \quad m \geq 1$$

Proof. Every element of $\mathbf{PRC}(\mathbb{R}^m)$ is element of $\mathbf{RC}(\mathbb{R}^m)$. Then for $m \geq 2$ it is direct implication of Proposition 7.3.1. For $m = 1$ the proof is analogous to the one of Proposition 7.3.1. Will illustrate the points of deviation.

(15):

Apply Procedure 3.2 on G' (as per Remark 3.2.1).

(16):

W_i are *regular closed* sets of $\mathbb{R}^1$

(17):

By Claim 3.2.1
(18):
S satisfies Definition 6.2.1 and every element of it is a *regular closed* set of $\mathbb{R}^1$ due to the following observations:

- By Observation 3.2.3 every element of S is a *regular closed* set of $\mathbb{R}^1$.
- By Observation 3.2.4: $\cup S = \mathbb{R}^1$
- By Observation 3.2.5: $Int(W_{i_1}) \cap Int(W_{i_2}) = \emptyset$, $i_1 \neq i_2$.
- By Observation 3.2.11 for every $W_i \in S$ then whichever point x of W_i is taken then any open $o \ni x$ will have the property: $o \cap Int(W_i) \neq \emptyset$

(19):
$B_{RC}(S)$ is a subalgebra of the Boolean algebra of the *regular closed* sets of $\mathbb{R}^1$ by *(i)*
(20):
$\mathcal{B}_{RC}(S) \in \mathbf{RC}(\mathbb{R}^1)$

$\square$

Proposition 7.3.3. *For every formula φ of the language $L_{\mathcal{R}}$:*

$$\Vdash_{\boldsymbol{PRC}(\mathbb{R}^1)} \varphi \quad \textit{implies} \quad \vdash_{\mathcal{L}_{Cont+PRC1}} \varphi$$

Proof. The proof is analogous to the one of Proposition 7.3.1. Will illustrate the points of deviation.

(6):
Assume: $\nvdash_{\mathcal{L}_{Cont+PRC1}} \varphi$
(7):
$\nVdash_{C^B_{Cont+PRC1}} \varphi$
(8):
$\mathcal{B}$ is from $C^B_{Cont+PRC1}$
(9):
$\mathcal{B} \Vdash Cont + PRC1$
(10):
$\mathcal{B}_0 \Vdash Cont + PRC1$
(11):
$\mathcal{F}_0 \Vdash Cont + PRC1$
(12):
Furthermore, by $\mathcal{F}_0 \Vdash$ **PRC1** then applying Claim 2.4.2 we imply that $\mathcal{F}_0$ is a contact n-frame for $n \leq 2$. Hence G_0 is a contact n-graph for $n \leq 2$.
(13):
Furthermore, by Observation 4.2.9, G' is a contact n-graph also for $n \leq 2$.
(14):
Recall as well that G' is a contact n-graph also for $n \leq 2$.
(15):
Apply Procedure 3.3 on G'.
(16):
W_i are *polytopes* of $\mathbb{R}^1$
(17):

By Claim 3.3.1
(18):
S satisfies Definition 6.2.1 and every element of it is a *polytope* due to the following observations:

- By Observation 3.3.3 every element of S is a *polytope* of $\mathbb{R}^1$.
- By Observation 3.3.4: $\cup S = \mathbb{R}^1$
- By Observation 3.3.5: $Int(W_{i_1}) \cap Int(W_{i_2}) = \emptyset$, $i_1 \neq i_2$.

(19):
$B_{RC}(S)$ is a subalgebra of the Boolean algebra of the *polytopes* of $\mathbb{R}^1$ by *(ii)*
(20):
$\mathcal{B}_{RC}(S) \in \mathbf{PRC(}\mathbb{R}^1\mathbf{)}$
(21):
$\vdash_{\mathcal{L}_{Cont+PRC1}} \varphi$

$\square$

7.4 Corollary Notes

By the pair of propositions Proposition 7.2.2 and Proposition 7.3.1 as well as the pair of propositions Proposition 7.2.1 and Proposition 7.3.2 we imply:

Corollary 7.4.1. *Completeness of formal system $\mathcal{L}_{Cont}$:*

- *The logic of the formal system $\mathcal{L}_{Cont}$ is the logic of the class of Boolean frames $\boldsymbol{PRC(}\mathbb{R}^m\boldsymbol{)}$, for any particular $m \geq 2$*
- *The logic of the formal system $\mathcal{L}_{Cont}$ is the logic of the class of Boolean frames $\boldsymbol{RC(}\mathbb{R}^m\boldsymbol{)}$, for any particular $m \geq 1$*

Now, by Corollary 7.4.1 and considering Claim 7.1.2, the following corollary holds:

Corollary 7.4.2. *Characterisation of the formal system $\mathcal{L}_{Cont}$:*
The following logics are equivalent:

(i) *The logic of the formal system $\mathcal{L}_{Cont}$.*

(ii) $\mathcal{L}(\boldsymbol{PRC(}\mathbb{R}^m\boldsymbol{)})$, *where* $m \geq 2$.

(iii) $\mathcal{L}(\boldsymbol{RC(}\mathbb{R}^m\boldsymbol{)})$, *where* $m \geq 1$.

(iv) $\mathcal{L}(\mathcal{PRC}(\mathbb{R}^m))$, *where* $m \geq 2$.

(v) $\mathcal{L}(\mathcal{RC}(\mathbb{R}^m))$, *where* $m \geq 1$.

(vi) $\mathcal{L}(\{\mathcal{PRC}(\mathbb{R}^m) \mid m \in X\})$, *where X is a non-empty subset of the set of the natural numbers greater or equal* 2.

(vii) $\mathcal{L}(\{\mathcal{RC}(\mathbb{R}^m) \mid m \in X\})$, *where X is a non-empty subset of the set of the natural numbers greater or equal* 1.

(viii) $\mathcal{L}(\cup\{\boldsymbol{PRC(}\mathbb{R}^m\boldsymbol{)} \mid m \in X\})$, *where X is a non-empty subset of the set of the natural numbers greater or equal* 2.

(ix) $\mathcal{L}(\cup\{\boldsymbol{RC(}\mathbb{R}^m\boldsymbol{)} \mid m \in X\})$, *where X is a non-empty subset of the set of the natural numbers greater or equal* 1.

Now, consider an arbitrary *connected* topological space $\mathbb{T}$. Denote by $\mathcal{RC}(\mathbb{T})$ the Boolean frame with a carrier the Boolean algebra $RC(\mathbb{T})$ and interpretation of the relation symbols $\mathcal{C}_n^{\mathbb{T}}$ for the appropriate arity of the relations.

Remark that $\mathcal{RC}(\mathbb{T})$ is correct with respect to $\mathcal{L}_{Cont}$ as long as the reasoning is exactly as the one demonstrated in Proposition 7.2.1.

Denote by $\mathbf{RC}_{conn}$ the class of the Boolean frames $\mathcal{RC}(\mathbb{T})$ for any *connected* topological space $\mathbb{T}$. Trivially, $\mathcal{RC}(\mathbb{R}^m)$ is an element of the class $\mathbf{RC}_{conn}$. Therefore, by Corollary 7.4.2, we imply that $\mathbf{RC}_{conn}$ is complete with respect to $\mathcal{L}_{Cont}$. Then, in addition to Corollary 7.4.2, we also have:

Corollary 7.4.3. *The following logics are equivalent:*

(i) *The logic of the formal system $\mathcal{L}_{Cont}$.*

(ii) $\mathcal{L}(\boldsymbol{RC}_{conn})$

By Corollary 7.4.2, non-formally, the classes $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$ for $m \geq 2$ and $\mathbf{RC(}\mathbb{R}^n\mathbf{)}$ for $n \geq 1$ are "indistinguishable" with respect to the set of formulas valid in them.

The difference between the logics of $\mathbf{PRC(}\mathbb{R}^1\mathbf{)}$ and $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$ for $m \geq 2$ is demonstrated by the pair Proposition 7.2.3 and Proposition 7.3.3:

Corollary 7.4.4. *Completeness of the formal system $\mathcal{L}_{Cont+PRC1}$*

- *The logic of the formal system $\mathcal{L}_{Cont+PRC1}$ is the logic of the class of Boolean frames $\boldsymbol{PRC(}\mathbb{R}^1\boldsymbol{)}$*

Now, by Corollary 7.4.4 and considering Claim 7.1.2, we imply:

Corollary 7.4.5. *Characterisation of formal system $\mathcal{L}_{Cont+PRC1}$:*
The following logics are equivalent:

(i) *The logic of the formal system $\mathcal{L}_{Cont+PRC1}$*

(ii) $\mathcal{L}(\boldsymbol{PRC(}\mathbb{R}^1\boldsymbol{)})$

(iii) $\mathcal{L}(\mathcal{PRC}(\mathbb{R}^1))$

By Corollary 7.4.4, non-formally, the class of Boolean frames $\mathbf{PRC(}\mathbb{R}^1\mathbf{)}$ is "distinguishable" from the classes $\mathbf{PRC(}\mathbb{R}^m\mathbf{)}$ for $m \geq 2$ and $\mathbf{RC(}\mathbb{R}^n\mathbf{)}$ for $n \geq 1$ due to the property of $\mathbf{PRC(}\mathbb{R}^1\mathbf{)}$ being that (the axiom) **PRC1** is valid in it.

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