Sofia University "St. Kliment Ohridski"

Master's Thesis

# Dynamic contact algebras and quantifier-free logics for space and time

*Plamen Dimitrov*

supervised by
Dimiter Vakarelov

March 22, 2018

# Contents

# Preface

Classical Euclidean geometry, one of the oldest and most established theories of space, is built on the primitive notion of *point*. In his book *"The Organization of Thought"*[5] Alfred Whitehead mentions that "it follows from the relative theory that a point should be definable in terms of the relations between material things". Whitehead claimed that the theory of space and time should be "point-free" in the sense that neither space points nor time points should be taken as the foundation element for the theory, the reasoning being that these notions are too abstract and have no analog in the real world. This, along with some works from de Laguna, Tarski and other authors, gave birth to the so-called Region Based Theory of Space (RBTS), sometimes called mereotopology - a "point-free" theory of space where the primitive notions are those of "region" and "contact" between regions. An exhaustive look into RBTS is given in [2] where the concepts of "contact relations" and "contact algebras" are explored.

A natural extension of the idea of a point-free theory of space is to try and develop a point-free theory of time where the notion of a time point(moment) is not primitive. Dynamic contact algebras, introduced by Vakarelov[7], are a generalization of contact algebras and are an attempt in that direction. They study regions changing in time and present formal explications of Whitehead's ideas of integrated point-free theory of space and time. The current work is a continuation of that effort and is structured as follows:

- Section 1 focuses on establishing the needed notation as well as presenting some already known facts about dynamic contact algebras (DCAs). We also give intuition about the concept of a dynamic contact algebra by presenting the standard model for DCAs as described in [7].
- Section 2 explores a new type of Kripke structures called *dynamic relational structures.* We introduce the notions of a *weak* and *strong* dynamic contact algebras and establish a new relational representation theorem for DCAs.
- Section 3 focuses on the more generic notion of a *basic dynamic contact algebra.* We delevop reprentation theory for basic DCAs and finite basic DCAs. We use the *p-morphism* technique adapted from modal logic to establish some relations between basic DCAs and the other types of DCAs.
- In Section 4 we introduce finitary quntifier-free logics for space and time based on the studied types of dynamic contact algebras. The logics are based on Modus Ponens and several non-standard rules of inference which replace the non-universal axioms of DCA. We prove the completeness of these logics in the respective class of DCAs. Combining the completeness results with the results from previous sections we conclude some interesting metalogical properties of the proposed systems.

# 1 A brief look into RBTS

This introductory section will be abundant on definitions of well-known entities that play crucial role in the region-based theory of space. In an attempt to render this work as self-contained as possible, we start be taking a look at the foundations of lattice theory. This transitions into a brief study of Boolean algebras and contact algebras. Ultimately, we explore the concept of a dynamic contact algebra and dedicate an entire subsection to build intuition about the nature of DCAs.

## 1.1 Facts about lattices and boolean algebras

A structure $(W, \leq)$, where $\leq$ is a binary relation on $W$, is called a *partially ordered set (poset)* iff for any $x, y \in W$:

$x \leq x$ (reflexivity)

$x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)

$x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)

The relation $\leq$ is called a *partial order* on $W$. Let $\emptyset \neq A \subseteq W$ be a non-empty subset of $W$. An element $a \in W$ is called an *upper bound* of $A$ if $\forall x \in A : x \leq a$. The element $a$ is called *least upper bound* of $A$ if $a$ is an upper bound of $A$ and for all other upper bounds $b$ of $A$ we have that $a \leq b$. Dually, we can define a *lower bound* of $A$ and *greatest lower bound* of $A$. An element $a \in W$ such that $\forall x \in W : x \leq a$ is called the *greatest* element of $W$. Similarly, an element $a \in W$ such that $\forall x \in W : a \leq x$ is called the *smallest* element of $W$.

**Definition 1.1** (Lattice)**.** The partially ordered set $(W, \leq, \cdot, +)$ is called a *lattice* if every two-element subset of $W$ has greatest lower bound and least upper bound. We'll use the notation $a \cdot b$ to denote the greatest lower bound of $\{a, b\}$ and $a + b$ to denote the least upper bound of $\{a, b\}$. A lattice which has a greatest element and a smallest element will be called a *bounded lattice.* We'll denote such lattices with $(W, \leq, 0, 1, \cdot, +)$, where 0 is the smallest and 1 is the greatest element. A lattice is called a *distributive lattice* if it satisfies the following additional conditions:

$(D)$ $a \cdot (b + c) = a \cdot b + a \cdot c$

$(\widehat{D})$ $a + (b \cdot c) = (a + b) \cdot (a + c)$

**Definition 1.2** (Boolean algebra)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *)$ be a structure where $(B, \leq, 0, 1, \cdot, +)$ is a bounded distributive lattice and $*$, called the *complementation* operation, satisfies the following axioms:

(*1) $a + a^* = 1$

(*2) $a \cdot a^* = 0$

Then $\underline{B}$ is called a *Boolean algebra.* If $0 \neq 1$ then $\underline{B}$ is called a *nondegenerate* Boolean algebra.

**Lemma 1.3** (Some properties of Boolean algebra)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *)$ be a Boolean algebra and $a, b \in B$. Then:

| | | |
|---|---|---|
| $a \cdot b \leq a$ | $a + a = a$ | $a^{**} = a$ |
| $a \cdot b \leq b$ | $a \cdot b = b \cdot a$ | $a \leq b \Leftrightarrow a \cdot b^* = 0$ |
| $a \leq a + b$ | $a + b = b + a$ | $a \leq b \Leftrightarrow b^* \leq a^*$ |
| $b \leq a + b$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ | |
| $a \cdot a = a$ | $(a + b) + c = a + (b + c)$ | |

**Definition 1.4** (Atom)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *)$ be a Boolean algebra. An element $p \in B$ is called an *atom* iff $p \neq 0$ and given any $q \in B$ such that $q \leq p$ we have $q = 0$ or $q = p$. Intuitively, atoms are minimal among the non-zero elements of a Boolean algebra.

**Definition 1.5** (Atomic Boolean algebra)**.** Let $\underline{B}$ be a Boolean algebra and let $A$ be the set of its atoms. We say that $\underline{B}$ is *atomic* iff for every non-zero element $p \in B$, there exists $a \in A$ such that $a \leq p$. Equivalently, every element $p \in B$ is the sum of the atoms $a$ such that $a \leq p$.

**Lemma 1.6.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *)$ be a Boolean algebra such that $B$ is a finite set. Then $\underline{B}$ is atomic.

**Definition 1.7** (Filter)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *)$ be a Boolean algebra. A subset $F$ of $B$ is called a *filter* if the following conditions hold:

(i) $1 \in F$

(ii) $x \leq y, x \in F \Rightarrow y \in F$

(iii) $x, y \in F \Rightarrow x \cdot y \in F$

If $0 \notin F$ then $F$ is called a *proper filter.*

**Remark.** Let $\underline{B}$ be a boolean algebra and let $a \in B$. Then the set $[a) = \{c : a \leq c\}$ is a filter.

**Definition 1.8** (Ultrafilter)**.** An *ultrafilter* is a proper filter $F$ having the following property:

$x + y \in F \Rightarrow x \in F$ or $y \in F$.

**Lemma 1.9.** Let $\underline{B}$ be a Boolean algebra and let $a, b \in B$ be such that $a \nleq b$. Then there exists an ultrafilter $U$ such that $a \in U$ and $b \notin U$.

## 1.2 Facts about contact algebras

**Definition 1.10** (Precontact algebra)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C)$ be a structure such that $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra and the relation $C \subseteq B \times B$ satisfies the following axioms:

$(C1)$ $aCb \Rightarrow a \neq 0$ and $b \neq 0$

$(C2)$ $aCb$, $a \leq a'$ and $b \leq b' \Rightarrow a'Cb'$

$(C3)$ $aC(b + c) \Rightarrow aCb$ or $aCc$

$(C3')$ $(a + b)Cc \Rightarrow aCc$ or $bCc$

Then the relation $C$ is called a *precontact relation* on $B$ or simply a *precontact* and the structure $\underline{B}$ is called a *precontact algebra*.

**Lemma 1.11** (R-extension Lemma)**.** [4] Let $\underline{B}$ be a Boolean algebra and $R$ be a precontact relation on $B$. If $F$ and $G$ are filters of $B$ such that $F \times G \subseteq R$ then there are ultrafilters $U$ and $V$ such that $F \subseteq U$, $G \subseteq V$ and $U \times V \subseteq R$.

**Definition 1.12** (Contact algebra)**.** Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C)$ be a precontact algebra where the precontact $C$ satisfies the additional axioms:

$(C4)$ $aCb \Rightarrow bCa$

$(C5)$ $a \cdot b \neq 0 \Rightarrow aCb$

Then $C$ is called a *contact relation* or simply a *contact* and $\underline{B}$ is called a *contact algebra*. On the base of $(C4)$ only one of the axioms $(C3)$ and $(C3')$ is needed. Also, $(C5)$ is equivalent to $(C5')$ $a \neq 0 \Rightarrow aCa$. We'll denote by $\overline{C}$ the complement of $C$. In the context of contact algebras, the elements of the underlying Boolean algebra are called *regions* and are considered as abstractions for spacial bodies. Boolean operations between regions can be used to construct new regions. The 0 element of the Boolean algebra will be treated as a *non-existing region*. We'll say that a region $a$ *ontologically exists* or simply, *exists*, iff $a \neq 0$. If for two regions $a$ and $b$ we have that $a \leq b$ we'll say that $a$ is *part of* $b$.

We will be interested in contact and precontact algebras satisfying the following additional axiom: $(CE)$ If $a\overline{C}b$ then $(\exists c)(a\overline{C}c$ and $c^*\overline{C}b)$. We call this axiom the *Efremovich axiom*, because it is used in the definition of Efremovich proximity spaces.

The following construction from [4] gives an example of Boolean algebras with precontact relations. Let $(W, R)$ be a relational system where $W$ is a non-empty set and $R$ is a binary relation on $W$ (such pairs are called *adjacency spaces* in [4]). For subsets $a, b$ of $W$ define $aC_Rb$ iff there exist points $x \in a$ and $y \in b$ such that $xRy$. Then $C_R$ is a precontact relation. In [4] it is shown every precontact algebra is representable as precontact algebra over an adjacency space. The following fact is proved in [4]:

**Lemma 1.13.** (i) $C_R$ satisfies axiom (C4) iff $R$ is a symmetric relation in $W$

(ii) $C_R$ satisfies axiom (C5) iff $R$ is a reflexive relation in $W$

(iii) $C_R$ satisfies the Efremovich axiom $(CE)$ iff $R$ is a transitive relation in $W$

If $(W, R)$ is a relational system such that the relation $R$ is reflexive and symmetric then by Lemma 1.13 (i) and (ii) the precontact relation $C_R$ is a contact relation in the Boolean algebra of all subsets of $W$. Every contact algebra is representable as a contact algebra of this form (see [4]).

Let $(W, R, S)$ be a relational system with two relations. We consider the following two first-order conditions for $R$ and $S$ (henceforth called *compositional axioms*):

$(R \circ S \subseteq S)$ If $xRy$ and $ySz$, then $xSz$

$(S \circ R \subseteq S)$ If $xSy$ and $yRz$, then $xSz$

We consider also the following two conditions for precontact relations $C_R$ and $C_S$ similar to the Efremowich axiom $(CE)$:

$(C_R C_S)$ If $a\overline{C}_S b$, then there exists $c \subseteq W$ such that $a\overline{C}_R c$ and $c^*\overline{C}_S b$

$(C_S C_R)$ If $a\overline{C}_S b$, then there exists $c \subseteq W$ such that $a\overline{C}_S c$ and $c^*\overline{C}_R b$.

The proof of the following lemma is similar to the proof of Lemma 1.13 (iii):

**Lemma 1.14.** (i) The condition $(C_R C_S)$ is fulfilled between precontact relations $C_R$ and $C_S$ iff the condition $(R \circ S \subseteq S)$ is satisfied

(ii) The condition $(C_S C_R)$ is fulfilled between precontact relations $C_R$ and $C_S$ iff the condition $(S \circ R \subseteq S)$ is satisfied

## 1.3 Dynamic contact algebras

### 1.3.1 Abstract definition

**Definition 1.15** (Dynamic contact algebra)**.** A *dynamic contact algebra*(DCA) is any system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C^s, C^t, \mathcal{B}, TR, UTR, NOW)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra and the following properties hold:

(i) $C^s$ is a contact relation on $B$, which is called *space contact*

(ii) $C^t$ is a contact relation on $B$, called *time contact* which satisfies the following additional axioms:

$(C^s \Rightarrow C^t)$ $aC^s b \Rightarrow aC^t b$

$(C^t E)$ $a\overline{C^t}b \Rightarrow (\exists c)(a\overline{C^t}c$ and $c^*\overline{C^t}b)$ (*Efremovich axiom*)

(iii) $\mathcal{B}$ is a precontact relation on $B$ called the *precendence relation*

(iv) $TR$ and $UTR$ are subsets of $B$ called *time representatives* and *universal time representatives* respectively, satisfying the following axioms:

$(TR1)$ $c \in TR \Leftrightarrow c \neq 0$ and $(\forall a, b \in B)(aC^t c$ and $bC^t c \Rightarrow aC^t b)$

$(TR2)$ $c \in UTR \Leftrightarrow c \in TR$ and $c\overline{C^t}c^*$

$(TRC^t)$ $aC^t b \Rightarrow (\exists c \in UTR)(aC^t c$ and $bC^t c)$

$(TRC^s)$ $aC^sb \Rightarrow (\exists c \in UTR)((a \cdot c)C^sb)$

$(TR\mathcal{B}1)$ $c \in TR, c\mathcal{B}b$ and $aC^tc \Rightarrow a\mathcal{B}b$

$(TR\mathcal{B}2)$ $d \in TR, a\mathcal{B}d$ and $bC^td \Rightarrow a\mathcal{B}b$

$(TR\mathcal{B}3)$ $a\mathcal{B}b \Rightarrow (\exists c \in UTR)(c\mathcal{B}b$ and $aC^tc)$

$(TR\mathcal{B}4)$ $a\mathcal{B}b \Rightarrow (\exists d \in UTR)(a\mathcal{B}d$ and $bC^td)$

Below, $c(i)$ and $c(j)$ are arbitrary elements of $UTR$:

$(UTR\mathcal{B}11)$ $(\forall p \in B)(p\mathcal{B}c(i)$ or $p^*\mathcal{B}c(j))$ iff $(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ and $c(k)\mathcal{B}c(j))$

$(UTR\mathcal{B}12)$ $(\forall p \in B)(p\mathcal{B}c(i)$ or $c(j)\mathcal{B}p^*)$ iff $(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ and $c(j)\mathcal{B}c(k))$

$(UTR\mathcal{B}21)$ $(\forall p \in B)(c(i)\mathcal{B}p$ or $p^*\mathcal{B}c(j))$ iff $(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ and $c(k)\mathcal{B}c(j))$

$(UTR\mathcal{B}22)$ $(\forall p \in B)(c(i)\mathcal{B}p$ or $c(j)\mathcal{B}p^*)$ iff $(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ and $c(j)\mathcal{B}c(k))$

$(UTRNOW)$ $NOW \in UTR$

This definition is also called the *abstract definition* of a DCA. In the next section we'll look at the standard model of a DCA which will reveal the reasons this definition was coined the way it is.

**Remark.** The implications from right to left of the axioms $UTR\mathcal{B}11$, $UTR\mathcal{B}12$, $UTR\mathcal{B}121$ and $UTR\mathcal{B}22$ are provable by some (universal) axioms of DCA and, hence, are superfluous. As an example of the proof let's consider the following formula, which implies the implication from the right to the left part of axiom $(UTR\mathcal{B}21)$:

If $c \in UTR$, $a\mathcal{B}c$, and $c\mathcal{B}b$, then $a\mathcal{B}p$ or $p^*\mathcal{B}b$.

Suppose that this implication is not true. Then we have: (1) $c \in UTR$, (2) $a\mathcal{B}c$, (3) $c\mathcal{B}b$, (4) $a\overline{\mathcal{B}}p$ and (5) $p^*\overline{\mathcal{B}}b$. From (2) and (4) we get (by axioms (TR2) and $(TR\mathcal{B}2)$) (6) $c\overline{C^t}p$. Similarly, using $(TR\mathcal{B}1)$ and (3) and (5) we get (7) $c\overline{C^t}p^*$. By the contact axioms of $C^t$ we obtain from (6) and (7) $c\overline{C^t}(p+p^*)$ and $c\overline{C^t}1$, which implies $c = 0$. But (1) implies $c \neq 0$ - a contradiction.

Since DCAs are algebraic structures we adopt for them the standard definitions for *subalgebra, homomorphism, isomorphism* and *isomorphic embedding.* It can be noted from axioms $(TR1)$ and $(TR2)$ that the sets $TR$ and $UTR$ are definable with first-order formulas of the relation $C^t$. We, however, include those notions in the signature of a DCA since we want to show that they are preserved in the representation theory of DCAs.

**Lemma 1.16** (UTR properties)**.** Let $\underline{B}$ be a DCA. Then:

(i) if $c \in UTR$, then $aC^tc \Leftrightarrow a \cdot c \neq 0 \Leftrightarrow aC^sc$ for any $a \in B$

(ii) $aC^sb$ iff $(\exists c \in UTR)((a \cdot c)C^s(b \cdot c))$

(iii) $aC^tb$ iff $(\exists c \in UTR)((a \cdot c)C^t(b \cdot c))$

(iv) $a\mathcal{B}b$ iff $(\exists c, d \in UTR)((a \cdot c)\mathcal{B}(b \cdot d))$

### 1.3.2 Snapshot model

In classical physics, the properties of changing objects are defined as functions of time. This motivates that time is given by a set of time points which has a specific arithmetic structure. Often, this structure is an abstract relational system of the form $(T, \prec)$, where $T$ is a non-empty set of *time points* (also called *moments of time*) and $\prec$ is a binary relation on $T$ such that $m \prec n$ means that $m$ *is before* $n$. This intuition motivates to call $\prec$ *before-after* relation or *time order*.

Suppose that we want to describe a dynamic environment consisting of regions changing in time. First, assume that we are given a time structure $\underline{T} = (T, \prec)$ and we want to know what is the spatial configuration of regions at each moment of time $m \in T$. We assume that for each $m \in T$ the spatial configuration of the regions forms a contact algebra $(\underline{B}_m, C_m)$=$(B_m, \leq_m, 0_m, 1_m, \cdot_m, +_m, *_m, C_m)$, called a *coordinate contact algebra*. We can view the contact algebra $(\underline{B}_m, C_m)$ as a snapshot of the spacial configuration at moment $m$ (hence the name of the construction). We identify a given changing region $a$ with the series $< a_m >_{m\in T}$ of snapshots and call such a series a *dynamic region*. In a sense, this series can be considered also as a trajectory or time history of $a$. If $a =< a_m >_{m\in T}$ is a given dynamic region then $a_m$ can be considered as "$a$ at the time point $m$". The static region $a_m$ will also be called the $m$-th coordinate of $a$. For instance, the expression $a_m \neq 0_m$ means that $a$ exists at the time point $m$ and the expression $a_m C_m b_m$ means that $a$ and $b$ are in a contact at the moment $m$. The contact algebra $(\underline{B}_m, C_m)$ contains all $m$-th coordinates of the changing regions.

We denote by $B(\underline{T})$ the set of all dynamic regions. We assume that $B(\underline{T})$ is a Boolean algebra with Boolean constants defined as follows: $1 =< 1_m >_{m\in T}$, $0 =< 0_m >_{m\in T}$, Boolean ordering $a \leq b$ iff $(\forall m \in T)(a_m \leq_m b_m)$ and Boolean operations are defined coordinatewise: $a + b =_{def}< a_m(+_m)b_m >_{m\in T}$, $a \cdot b =_{def}< a_m(\cdot_m)b_m >_{m\in T}$, $a^* =_{def}< a^*_m >_{m\in T}$. We'll call $B(\underline{T})$ *dynamic model of space over the time structure* $(T, \prec)$. The Boolean algebra $B(\underline{T})$ is a actually a subalgebra of the Cartesian product $\prod_{m\in T} B_m$ of the contact algebras $(\underline{B}_m, C_m)$, $m \in T$. A model which coincides with this Cartesian product is called a *full model*. Models that contain all dynamic regions $a$ such that for all $m \in T$ we have $a_m = 0_m$ or $a_m = 1_m$ will be called *rich models*. It's clear that full models are also rich.

Dynamic model of space is a spatio-temporal structure in which one can give explicit definitions of various spatio-temporal relations between dynamic regions. To start off, we'll take a look at the following three basic spatio-temporal relations between dynamic regions mentioned in the abstract definition of DCA: space contact, time contact and precedence relation.

Let $a$ and $b$ be dynamic regions. We'll say that $a$ and $b$ are in *space contact*, denoted by $aC^sb$, iff $(\exists m \in T)(a_m C_m b_m)$. Intuitively, space contact between $a$ and $b$ means that there is a time point in which $a$ and $b$ are in a contact. We'll say that $a$ and $b$ are in *time contact* and write $aC^tb$ iff $(\exists m \in T)(a_m \neq 0_m$ and $b_m \neq 0_m)$. So, two dynamic regions are in time contact if there exists a time point in which both of them exist simultaneously. Finally, we say that *a preceeds* $b$, denoted as $a\mathcal{B}b$, iff

$(\exists m, n \in T)(m \prec n$ and $a_m \neq 0_m$ and $b_n \neq 0_n)$. $\mathcal{B}$ is called *precedence relation.* Colloquially, if $a$ preceeds $b$ then there is a time point in which $a$ exists which is before a time point in which $b$ exists. The following lemma ([7] Part 2, Lemma 1.4) verifies that the relations defined above satisfy the respective axioms of DCA:

**Lemma 1.17.** (i) $C^s$ and $C^t$ are contact relations

(ii) $aC^s b \to aC^t b$

(iii) If the dynamic model of space $B(\underline{T})$ is rich, then $C^t$ satisfies the Efremovich axiom

(iv) $\mathcal{B}$ is a precontact relation.

The following lemma is not from [7] and is a new one. It gives us the possibility to add two new axioms to the abstract DCA definition since they are true in the standard model of DCA.

**Lemma 1.18.** Suppose that the dynamic model of space $B(\underline{T})$ is rich. Then the compositional axioms for $C^t$ and $\mathcal{B}$ are true, that is:

($C^t\mathcal{B}$) If $a\overline{\mathcal{B}}b$, then there exists $c$ such that $a\overline{C^t}c$ and $c^*\overline{B}b$.

($\mathcal{B}C^t$) If $a\overline{\mathcal{B}}b$, then there exists $c$ such that $a\overline{\mathcal{B}}c$ and $c^*\overline{C^t}b$

*Proof.* (i) Suppose $a\overline{\mathcal{B}}b$ and define $c$ coordinate wise:

$$c_k = \begin{cases} 0_k, & \text{if } a_k \neq 0_k \\ 1_k, & \text{if } a_k = 0_k. \end{cases}$$

Since the algebra is rich then $c$ exists. The verification of the conclusions $a\overline{C^t}c$ and $c^*\overline{B}b$ is straightforward.

(ii) In a similar manner, by using the following definition of $c$:

$$c_l = \begin{cases} 0_l, & \text{if } b_l = 0_l \\ 1_l, & \text{if } b_l \neq 0_l. \end{cases}$$

□

**Time conditions.** The structures $(T, \prec)$ that we are basing our intuition on is a fairly abstract structure that strives to describe time. An example of such a structure would be to take the set $T$ to be the set of real numbers and define the $\prec$ relation to coincide with one of the standard ordering relations $<$ or $\leq$ for strict or partial order between numbers. In general, though, the relation $\prec$ may satisfy various abstract properties. The following formulae, called *time conditions* describe some of these properties:

**(RS)** *Right seriality* $(\forall m)(\exists n)(m \prec n)$

**(LS)** *Left seriality* $(\forall m)(\exists n)(n \prec m)$

**(Up Dir)** *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k)$

**(Down Dir)** *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j)$

**(Dens)** *Density* $i \prec j \rightarrow (\exists k)(i \prec k$ and $k \prec j)$

**(Ref)** *Reflexivity* $(\forall m)(m \prec m)$

**(Irr)** *Irreflexivity* $(\forall m)($ not $m \prec m)$

**(Lin)** *Linearity* $(\forall m, n)(m \prec n$ or $n \prec m)$

**(Tri)** *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m)$

**(Tr)** *Transitivity* $(\forall ijk)(i \prec j$ and $j \prec k \rightarrow i \prec k)$

It's worth noting that the above listed conditions for time ordering are not independent. By taking some meaningful subsets of them, we obtain various notions of time order. For instance the subsets $\{(Ref), (Tr), (Lin)\}$, $\{(Irr), (Tr), (Tri), (Dens)\}$, $\{(Irr), (LS), (RS), (Tr), (Tri), (Dens)\}$ are typical for the classical time, while for instance, the subset $\{(Ref), (Tr), (UpDir), (DownDir)\}$ is used to characterize relativistic time.

It turns out that the properties of a time structure $\underline{T} = (T, \prec)$ are in exact correlation with some special conditions of the time contact $C^t$ and precedence relation $\mathcal{B}$. These conditions, called *time axioms*, are given in the following list:

**(RS)** $(\forall m)(\exists n)(m \prec n) \iff$ **(rs)** $a \neq 0 \rightarrow a\mathcal{B}1$

**(LS)** $(\forall m)(\exists n)(n \prec m) \iff$ **(ls)** $a \neq 0 \rightarrow 1\mathcal{B}a$

**(Up Dir)** $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k) \iff$ **(up dir)** $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}p \vee b\mathcal{B}p^*$

**(Down Dir)** $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j) \iff$ **(down dir)** $a \neq 0 \wedge b \neq 0 \rightarrow p\mathcal{B}a \vee p^*\mathcal{B}b$

**(Dens)** $i \prec j \rightarrow (\exists k)(i \prec k \wedge k \prec j) \iff$ **(dens)** $a\mathcal{B}b \rightarrow a\mathcal{B}p$ or $p^*\mathcal{B}b$

**(Ref)** $(\forall m)(m \prec m) \iff$ **(ref)** $aC^tb \rightarrow a\mathcal{B}b$

**(Irr)** $(\forall m)($ not $m \prec m) \iff$ **(irr)** $a\mathcal{B}b \rightarrow (\exists c, d)(c\mathcal{B}d$ and $aC^tc$ and $bC^td$ and $c\overline{C}^td)$

**(Lin)** $(\forall m, n)(m \prec n \vee n \prec m) \iff$ **(lin)** $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}b \vee b\mathcal{B}a$

**(Tri)** $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m) \iff$ **(tri)** $(aC^tc$ and $bC^td$ and $c\overline{C}^td) \rightarrow (a\mathcal{B}b$ or $b\mathcal{B}a)$

**(Tr)** $i \prec j$ and $j \prec k \rightarrow i \prec k \iff$ **(tr)** $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$

The next lemma ([7], Part2, Lemma 2.1) gives more context about these equivalences.

**Lemma 1.19** (Correspondence Lemma)**.** Let $B(\underline{T})$ be a rich model of space over the time structure $(T, \prec)$. Then all the correspondences in the above list are true in the following sense: the left side of a given equivalence is true in $(T, \prec)$ iff the right side is true in $B(\underline{T})$.

**Remark.** Note that Lemma 1.19 remains true if we replace $(tri)$ and $(irr)$ with simpler formulas which define the same time conditions in rich models:

**(tri)** If $a \neq 0$, $b \neq 0$, then $aC^t b$ or $a\mathcal{B}b$ or $b\mathcal{B}a$

**(irr)** If $a\mathcal{B}b$, then $(\exists c \neq 0)(\exists d \neq 0)(c \leq a$ and $d \leq b$ and $c\overline{C}^t d)$

We preserve the old names and further mention of $(tri)$ and $(irr)$ will refer to the simplified conditions. Also, as an example we show the proof of Lemma 1.19 for the case $(tri) \Leftrightarrow (Tri)$ using the new formula for $(tri)$.

*Proof.* $(tri) \Rightarrow (Tri)$. Suppose $(tri)$ and for the sake of contadiction let $(Tri)$ be not true, i.e. for some $i$ and $j$ we have $i \neq j$, $i \not\prec j$ and $j \not\prec i$. Define $a$ and $b$ coordinate wise as follows:

$$a_k = \begin{cases} 1_k, & \text{if } i = k \\ 0_k, & \text{if } i \neq k. \end{cases}, \; b_k = \begin{cases} 1_k, & \text{if } j = k \\ 0_k, & \text{if } j \neq k. \end{cases}$$

Since $B(\underline{T})$ is a rich model of space, then the definition of $a$ and $b$ is correct. It is easy to see that $a \neq 0$, $b \neq 0$, $a\overline{C}^t b$, $a\overline{\mathcal{B}}b$ and $b\overline{\mathcal{B}}a$ which contradicts $(tri)$.
$(Tri) \Rightarrow (tri)$. Suppose $(Tri)$. In order to prove $(tri)$ suppose $a \neq 0$, $b \neq 0$. Then $\exists i, a_i \neq 0_i$ and $\exists j, b_j \neq 0_j$. By $(Tri)$ we have $i = j$ or $i \prec j$ or $j \prec i$. This implies $aC^t b$ or $a\mathcal{B}b$ or $b\mathcal{B}a$ which completes the proof. □

In order to finish the motivation for the abstract definition of DCA we need to take a look at the sets $TR$, $UTR$ and the special element $NOW$. Inspired by phrases like "during the Industrial Revolution", "the epoch of Renaissance" and "the Bronze Age", we can enrich the dynamic model of space by introducing a special set of dynamic regions called *time representatives.* These dynamic regions will exist in a unique point of time and hence, much like the mentioned phrases, will define a concrete unit of time. The formal definition is a follows:

**Definition 1.20** (Time representatives)**.** A dynamic region $c$ in a dynamic model of space is called a *time representative* if there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i$, $c_j = 0_j$. We say also that $c$ is a representative of the time point $i$ and indicate this by writing $c = c(i)$. In the case when $c_i = 1_i$, $c$ is called *universal time representative*. We denote by $TR$ the set of time representatives and by $UTR$ the set of universal time representatives in a given dynamic model of space.

**Lemma 1.21.** Let $B(\underline{T})$ be a rich dynamic model of space over the time structure $(T, \prec)$. Then for each time point $i \in T$ there exist an universal time representative $c(i)$ of $i$. If $a$ is a region such that $a_i \neq 0_i$ and $a_i \neq 1_i$ then $c(i).a$ is a time representative which is not universal.

The above lemma shows that in rich models of space every moment of time is characterized by some universal time representative. This also suggests to enrich the time structure $(T, \prec)$ with a special moment of time denoted by *now*, corresponding to the "present moment of time". We denote by $NOW$ the universal time representative corresponding to *now*.

We are ready to define the standard model, also called "snapshot model", of a dynamic contact algebra.

**Definition 1.22** (Standard DCA)**.** By a *standard dynamic contact algebra* we mean any rich dynamic model of space with time structure $(T, \prec, now)$ with explicit definitions of the relations $C^s$, $C^t$, $\mathcal{B}$, time representatives $TR$, universal time representatives $UTR$ and the universal time representative $NOW$.

The results of [7] Part 2 (Lemma 1.4, Lemma 3.3 and Lemma 3.7) show that standard DCAs satisfy the axioms of the abstract definition of DCA. In fact, the abstract definition of DCA was coined after these properties of standard DCAs. It is shown in [7] Part 3, , Theorem 3.7 that every DCA with a number of additional time axioms is representable as a standard DCA over a time structure satisfying the time conditions determined by the corresponding time axiom.

It is shown in [7] Part 2, Section 3.1 that time representatives, universal time representatives and $NOW$ significantly increase the expressive power of DCA. These notions allow us to express different temporal statement for dynamic regions including talking about the present, past and future.

Some properties of universal time representatives suggest a translation $\tau$ from the first-order language of time structures into the language of DCAs. If $i$ is a variable for a time point then let $c(i)$ be a variable for a UTR. Then replace all atomic formulas $i = j$ and $i \prec j$ with $c(i) = c(j)$ (or, equivalently $c(i)C^tc(j)$) and $c(i)\mathcal{B}c(j)$ respectively. For example, $A = (\forall i)(\exists j)(i \prec j)$ is translated into $\tau(A) = (\forall c(i))(\exists c(j))(c(i)\mathcal{B}c(j))$. Lemma 3.5 from [7] Part 2 asserts the following:

**Lemma 1.23.** Let $B(T)$ be a rich standard DCA with time structure $(T, \prec)$ and let $A$ be a formula among $(RS)$, $(LS)$, $(UpDir)$, $(DownDir)$, $(Dens)$, $(Ref)$, $(Irr)$, $(Lin)$, $(Tri)$, $(Tr)$. Then $A$ is true $(T, \prec)$ iff $\tau(A)$ is true $B(T)$.

## 2 Relational models for DCAs

Lemma 1.18 shows that the snapshot model verifies two additional properties (compositional axioms) for $C^t$ and $\mathcal{B}$ which are not part of the abstract DCA definition. Adding these properties to the definition we obtain the notion of a *strong* DCA.

**Definition 2.1** (Strong DCA)**.** A DCA $\underline{B}$ satisfying the following additional axioms:

($C^t\mathcal{B}$) $a\overline{\mathcal{B}}b \Rightarrow (\exists c)(a\overline{C^t}c$ and $c^*\overline{\mathcal{B}}b)$

($\mathcal{B}C^t$) $a\overline{\mathcal{B}}b \Rightarrow (\exists c)(a\overline{\mathcal{B}}c$ and $c^*\overline{C^t}b)$

is called a *strong dynamic contact algebra* (SDCA).

**Definition 2.2** (Weak DCA)**.** A system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C^s, C^t, \mathcal{B}, TR, UTR, NOW)$ not necessarily satisfying the Efremovich axiom but satisfying all the other axioms of DCA (see Def.1.15) is called a *weak dynamic contact algebra* (WDCA). In particular all DCAs are also WDCAs.

Below we show a few properties of WDCA we'll be interested in further in the paper.

**Lemma 2.3.** The following hold for an arbitrary weak DCA:

(i) If $a \neq 0$, then there exists $c \in UTR$ such that $a \cdot c \neq 0$
(ii) $c \in UTR \Rightarrow (aC^tc$ iff $a.c \neq 0)$
(iii) If $c \in TR$ and $d \in UTR$ then $(c.d \neq 0$ iff $c \leq d)$
(iv) If $c \in UTR$, $d \in TR$ and $c \leq d$ then $c = d$
(v) Let $c, d \in UTR$. Then the following conditions are equivalent:

- $cC^td$
- $c.d \neq 0$
- $c = d$

(vi) If $c \in TR$, then there exists a unique $d \in UTR$ such that $c \leq d$
(vii) If $c \neq 0$, $d \in UTR$ and $c \leq d$, then $c \in TR$
(viii) $c \in TR$ iff $c \neq 0$ and $\exists d \in UTR$ such that $c \leq d$
(ix) $c \in TR$ iff $c \neq 0$ and $(\forall d \in UTR)(c.d \neq 0 \rightarrow c \leq d)$
(x) If $c \in TR$ and $(\forall d \in TR)(c \leq d \rightarrow c = d)$, then $c \in UTR$
(xi) $c \in UTR$ iff $c \in TR$ and $(\forall d \in TR)(c \leq d \rightarrow c = d)$

*Proof.* (i) Let $a \neq 0$. Then $aC^sa$ and by the axiom ($TRC^s$) there exists $c \in UTR$ such that $(a.c)C^sa$ which implies by the contact axioms for $C^s$ that $a.c \neq 0$
(ii) Let $c \in UTR$. ($\Rightarrow$) Suppose $aC^tc$ and for the sake of contradiction that $a \cdot c = 0$. Then $a \leq c^*$ and by $aC^tc$ we get $c^*C^tc$. By axiom (TR2) this contradicts $c \in UTR$. ($\Leftarrow$) Suppose $a \cdot c \neq 0$. Then by the contact axioms for $C^t$ we get $aC^tc$.
(iii) Let $c \in TR$ and $d \in UTR$. ($\Rightarrow$) Suppose $c{\cdot}d \neq 0$ and for the sake of contradiction that $c \not\leq d$. From here we get: $cC^td$, $c \cdot d^* \neq 0$ and $cC^td^*$. Since $c \in TR$, then by axiom (TR1) we get from $cC^td$ and $cC^td^*$ that $dC^td^*$, which contradicts $d \in UTR$.

(⇐) Suppose $c \leq d$. Then $c.d = c \neq 0$ ($c$ is in TR).
(iv) Suppose $c \in UTR$, $d \in TR$ and $c \leq d$. Then $c \cdot d = c \neq 0$ and applying (ii) we get $d \leq c$ and consequently $c = d$.
(v) Follows from (ii) and (iii).
(vi) Suppose $c \in TR$. Then by axiom (TR1) $c \neq 0$ and by (i) there exists $d \in UTR$ such that $c \cdot d \neq 0$. Then by (ii) we get $c \leq d$. For the uniqueness of $d$ suppose that for $d_1, d_2 \in UTR$ we have $c \leq d_1$ and $c \leq d_2$. Then $c \leq d_1.d_2$ and since $c \neq 0$, then $d_1 \cdot d_2 \neq 0$. Then by (v) we get $d_1 = d_2$.
(vii) Suppose $c \neq 0$, $d \in UTR$ and $c \leq d$ and for the sake of contradiction that $c \notin TR$. Then by axiom (TR2) $d \in TR$ and by (TR1) there are $a, b$ such that $aC^tc$, $bC^tc$ and $a\overline{C}^tb$. From here and $c \leq d$ we get $aC^td$, $bC^td$ which, together with $d \in TR$ implies $aC^tb$ - a contradiction.
(viii) This condition follows from (vi) and (vii).
(ix) (⇒) This implication follows by (iii). (⇐) Suppose (1) $c \neq 0$ and (2) $(\forall d \in UTR)(c.d \neq 0 \rightarrow c \leq d)$. From (1) we get by (i) that $c.d \neq 0$ for some $d \in UTR$ and by (2) we obtain that $c \leq d$. Then by (viii) we obtain that $c \in TR$.
(x) Suppose $c \in TR$ and $(\forall d \in TR)(c \leq d \rightarrow c = d)$ and for the sake of contradiction that $c \notin UTR$. Then by axiom (TR2) we get $cC^tc^*$. From $c \in TR$ by (vi) there exists $d \in UTR$ (and hence in TR) such that $c \leq d$. Then by the assumption we get $c = d$ and substituting this in $cC^tc^*$ we obtain $dC^td^*$ which contradicts $d \in UTR$.
(xi) This condition follows from (iv) and (x). □

**Lemma 2.4.** The following statements are universal consequences from the non-universal axioms of WDCA.

(i) If $d \in TR$, $c \neq 0$ and $c \leq d$, then $c \in TR$
(ii) If $c, d \in TR$ and $cC^td$, then $(c + d) \in TR$

*Proof.* (i) The proof of is an easy consequence of axiom (TR1).
(ii) Let $c, d \in TR$ and $cC^td$, then obviously $c + d \neq 0$. To prove that $c + d \in TR$ suppose $aC^t(c + d)$, $bC^t(c + d)$ and proceed to show that $aC^tb$. This will imply by (TR1) that $c + d \in TR$. By the axioms of contact we obtain the following two disjunctions:
(1) $aC^tc$ or (2) $aC^td$,
(1') $bC^tc$ or (2') $bC^td$.
We have to consider four cases. Case (1)(1'): axiom (TR1) implies $aC^tb$ (because $c \in TR$). Similarly, for case (2),(2') (by the assumption that $d \in TR$). Case (1)(2'): $aC^tc$ and the assumption $cC^td$ imply $aC^td$ (because $c \in TR$). Then $bC^td$ and $aC^td$ imply $aC^tb$, since $d \in TR$. In a similar way we reason in the case (2)(1'). □

The proof of the representation theorem from [7] Part 3 (Theorem 3.7) can be done without the use of the Efremovich axiom so the theorem holds for WDCAs as well. In this work, however, we'll focus on proving a new representation theorem based on a kind of relational models which we'll later use for Kripke style semantics of logical system based on DCAs.

## 2.1 Dynamic relational structures

**Definition 2.5** (Dynamic relational structure)**.** Let $\underline{W} = (W, R^s, R^t, \prec, now)$ be a relational structure such that:

(i) $R^t \subseteq W \times W$ is an equivalence relation
(ii) $R^s \subseteq W \times W$ is reflexive and symmetric and $R^s \subseteq R^t$
(iii) $xR^ty,\ y \prec z \Rightarrow x \prec z$
(iv) $x \prec y,\ yR^tz \Rightarrow x \prec z$
(v) $now \in W$

Then $\underline{W}$ is called a *dynamic relational structure* or *dynamic relational space.* The subsystem $(W, R^t, \prec)$ is called the *time substructure* of $\underline{W}$.

Similarly to time conditions shown in the introductory section, we can define time conditions in the context of time substructures of dynamic relational structures (the only difference will be to conditions $(Tri)$ and $(Irr)$). Let $\underline{W}$ be a dynamic relational structure and $(W, R^t, \prec)$ be its time substructure. We'll be interested in the following additional conditions that this time substructure may satisfy:

**(RS)**$_W$ $(\forall x \in W)(\exists y \in W)(x \prec y)$
**(LS)**$_W$ $(\forall x \in W)(\exists y \in W)(y \prec x)$
**(Up Dir)**$_W$ $(\forall x, y \in W)(\exists z \in W)(x \prec z \text{ and } y \prec z)$
**(Down Dir)**$_W$ $(\forall x, y \in W)(\exists z \in W)(z \prec x \text{ and } z \prec y)$
**(Dens)**$_W$ $(\forall x, y \in W)(x \prec y \rightarrow (\exists z \in W)(x \prec z \text{ and } z \prec y)$
**(Ref)**$_W$ $(\forall x \in W)(x \prec x)$
**(Irr)**$_W$ $(\forall x, y \in W)(x \prec y \Rightarrow x\overline{R^t}y)$
**(Lin)**$_W$ $(\forall x, y \in W)(x \prec y \text{ or } y \prec x)$
**(Tri)**$_W$ $(\forall x, y \in W)(xR^ty \text{ or } x \prec y \text{ or } y \prec x)$
**(Tr)**$_W$ $(\forall x, y, z \in W)(x \prec y \text{ and } y \prec z \rightarrow x \prec z)$

With Lemma 1.23 we mentioned a translation $\tau$, studied in [7] Part 2 and 3, from the the language of time structures into the language of DCA.The next lemma is inspired by this translation and, although not defined explicitly, serves as an extension of the translation $\tau$ for time substructures of dynamic relational structures. The proof is the same as [7] part 3, Lemma 1.5 as it does not rely on the Efremovich axiom.

**Lemma 2.6** (Translation Lemma)**.** Let $\underline{B}$ be a WDCA. The following equivalences hold in the sense that the left part is true in $\underline{B}$ iff the right part is true in $\underline{B}$ (see Section 1.3.2).

(i) $(rs) \longleftrightarrow (\forall a \in UTR)(\exists b \in UTR)(a\mathcal{B}b)$
(ii) $(ls) \longleftrightarrow (\forall a \in UTR)(\exists b \in UTR)(b\mathcal{B}a)$

(iii) $(updir) \longleftrightarrow (\forall a, b \in UTR)(\exists c \in UTR)(a\mathcal{B}c$ and $b\mathcal{B}c)$

(iv) $(downdir) \longleftrightarrow (\forall a, b \in UTR)(\exists c \in UTR)(c\mathcal{B}a$ and $c\mathcal{B}b)$

(v) $(dens) \longleftrightarrow (\forall a, b \in UTR)(a\mathcal{B}b \to (\exists c \in UTR)(a\mathcal{B}c$ and $c\mathcal{B}b))$

(vi) $(ref) \longleftrightarrow (\forall a \in UTR)(a\mathcal{B}a)$

(vii) $(irr) \longleftrightarrow (\forall a, b \in UTR)(a\mathcal{B}b \Rightarrow a\overline{C^t}b)$

(viii) $(lin) \longleftrightarrow (\forall a, b \in UTR)(a\mathcal{B}b$ or $b\mathcal{B}a)$

(ix) $(tri) \longleftrightarrow (\forall a, b \in UTR)(aC^tb$ or $a\mathcal{B}a$ or $b\mathcal{B}a)$

(x) $(tr) \longleftrightarrow (\forall a, b, c \in UTR)(a\mathcal{B}b$ and $b\mathcal{B}c \to a\mathcal{B}c)$

## 2.2 Strong DCAs over dynamic relational structures

Given an arbitrary dynamic relational structure $\underline{W}$, define the structure $\underline{B}(W) = (B_W, \leq, 0, 1, \cdot, +, *, C^s_W, C^t_W, \mathcal{B}_W, TR_W, UTR_W, NOW_W)$ in the following way:

(i) $B_W = 2^W$, $0 = \emptyset$, $1 = W$, $\leq = \subseteq$, $\cdot = \cap$, $+ = \cup$, $*$ = set complement

(ii) $\Gamma C^s_W \Delta \Leftrightarrow \exists a \in \Gamma, \exists b \in \Delta : aR^sb$ for $\Gamma, \Delta \in B_W$

(iii) $\Gamma C^t_W \Delta \Leftrightarrow \exists a \in \Gamma, \exists b \in \Delta : aR^tb$

(iv) $\Gamma \mathcal{B}_W \Delta \Leftrightarrow \exists a \in \Gamma, \exists b \in \Delta : a \prec b$

(v) $\Gamma \in TR_W \Leftrightarrow \Gamma$ is a non-empty subset of an equivalence class of $R^t$

(vi) $\Gamma \in UTR_W \Leftrightarrow \Gamma$ is an equivalence class of $R^t$. $NOW_W = |now|$.

Moving forward, we'll need the following notation for convenience:

$$a \succ b \stackrel{\text{def}}{=} b \prec a$$
$$\langle\prec\rangle\Gamma \stackrel{\text{def}}{=} \{x \mid \exists y \in \Gamma \text{ such that } x \prec y\}$$
$$\langle\succ\rangle\Gamma \stackrel{\text{def}}{=} \{x \mid \exists y \in \Gamma \text{ such that } x \succ y\}$$

**Lemma 2.7.** $\Gamma \cap \langle\prec\rangle\Delta \neq \emptyset \Leftrightarrow \langle\succ\rangle\Gamma \cap \Delta \neq \emptyset$ for arbitrary $\Gamma$ and $\Delta$.

*Proof.* ($\Rightarrow$) Let $\Gamma \cap \langle\prec\rangle\Delta \neq \emptyset$. This means that there is an element, say $x$, such that $x \in \Gamma$ and $x \in \langle\prec\rangle\Delta$, which, by definition, means that $\exists y \in \Delta$ such that $x \prec y$. From $x \prec y$ we get that $y \succ x$ and since $x \in \Gamma$ we have that $y \in \langle\succ\rangle\Gamma$. Since $y \in \Delta$ we naturally get that $\langle\succ\rangle\Gamma \cap \Delta \neq \emptyset$.
($\Leftarrow$) Let $\langle\succ\rangle\Gamma \cap \Delta \neq \emptyset$. Let $x$ be such that $x \in \langle\succ\rangle\Gamma$ and $x \in \Delta$. From here we get that $\exists y \in \Gamma$ such that $x \succ y$. Hence $y \prec x$ and since $x \in \Delta$ we get that $y \in \langle\prec\rangle\Delta$. From here and $y \in \Gamma$ we get that $\Gamma \cap \langle\prec\rangle\Delta \neq \emptyset$. □

Using this notation, we can rewrite the definition of the precedence relation $\mathcal{B}_W$ in the following equivalent way: $\Gamma\mathcal{B}_W\Delta \Leftrightarrow \Gamma \cap \langle\prec\rangle\Delta \neq \emptyset \Leftrightarrow \langle\succ\rangle\Gamma \cap \Delta \neq \emptyset$.

**Lemma 2.8.** $(\exists\Theta \in UTR_W)(\Gamma\mathcal{B}_W\Theta$ and $\Delta\mathcal{B}_W\Theta) \Leftrightarrow \langle\succ\rangle\Gamma \cap \langle\succ\rangle\Delta \neq \emptyset$ for arbitrary $\Gamma$ and $\Delta$ from $B_W$.

*Proof.* ($\Rightarrow$) Let $\Theta \in UTR_W$ be such that $\Gamma \mathcal{B}_W \Theta$ and $\Delta \mathcal{B}_W \Theta$. So, by definition, we have that $\langle \succ \rangle \Gamma \cap \Theta \neq \emptyset$ and $\langle \succ \rangle \Delta \cap \Theta \neq \emptyset$. Let $x$ be such that $x \in \langle \succ \rangle \Gamma$, $x \in \Theta$ and let $y$ be such that $y \in \langle \succ \rangle \Delta$, $y \in \Theta$. Since $\Theta$ is a member of $UTR_W$(hence an equivalence class of $R^t$) and $x \in \Theta, y \in \Theta$ we get that $xR^ty$. From $x \in \langle \succ \rangle \Gamma$ we have that $\exists z \in \Gamma$ such that $x \succ z$ i.e. $z \prec x$. From here, $xR^ty$ and property (iv) of the relational structure (see Def. 2.5) we get that $z \prec y$ i.e. $y \succ z$. We also know that $z \in \Gamma$ so $y \in \langle \succ \rangle \Gamma$. But $y \in \langle \succ \rangle \Delta$ and hence $\langle \succ \rangle \Gamma \cap \langle \succ \rangle \Delta \neq \emptyset$.

($\Leftarrow$) Let $\langle \succ \rangle \Gamma \cap \langle \succ \rangle \Delta \neq \emptyset$ and let $x \in \langle \succ \rangle \Gamma$ and $x \in \langle \succ \rangle \Delta$. Take $\Theta$ to be the equivalence class of $x$ with respect to the $R^t$ relation. It is clear that $\langle \succ \rangle \Gamma \cap \Theta \neq \emptyset$ and $\langle \succ \rangle \Delta \cap \Theta \neq \emptyset$. But, by definition, this means that $\Gamma \mathcal{B}_W \Theta$ and $\Delta \mathcal{B}_W \Theta$ which is what we needed to prove. □

Now we are ready to prove two lemmas that will characterize the structure $\underline{B}(W)$ over the dynamic relational structure $\underline{W}$.

**Lemma 2.9.** $\underline{B}(W)$ is a dynamic contact algebra.

*Proof.* We'll verify that $\underline{B}(W)$ satisfies the axioms of DCA. First of all, let's check that $C^s_W$ is a contact relation. $(C1)$ $\Gamma C^s_W \Delta \Rightarrow \Gamma \neq \emptyset$ and $\Delta \neq \emptyset$? Obvious, since if either $\Gamma$ or $\Delta$ were empty we wouldn't be able to find elements from the two sets that are $R^s$-related, which would contradict the assumption that $\Gamma C^s_W \Delta$. $(C2)$ $\Gamma C^s_W \Delta$ and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta' \Rightarrow \Gamma' C^s_W \Delta'$? From $\Gamma C^s_W \Delta$ we get that there exist $a \in \Gamma$ and $b \in \Delta$ such that $aR^sb$. Since $\Gamma \subseteq \Gamma'$, we get $a \in \Gamma'$ and similarly $b \in \Delta'$. Hence $\Gamma' C^s_W \Delta'$ by definition. $(C3)$ $\Gamma C^s_W (\Delta \cup \Theta) \Rightarrow \Gamma C^s_W \Delta$ or $\Gamma C^s_W \Theta$? From the premise we have that there is $a \in \Gamma$ and $b \in \Delta \cup \Theta$ such that $aR^sb$. Since $b \in \Delta \cup \Theta$ then b must be in at least one of $\Delta$ or $\Theta$. Depending on where b is, we get either $\Gamma C^s_W \Delta$ or $\Gamma C^s_W \Theta$ (or both). $(C4)$ $\Gamma C^s_W \Delta \Rightarrow \Delta C^s_W \Gamma$? Follows directly from the symmetry of the $R^s$ relation. $(C5)$ $\Gamma \neq \emptyset \Rightarrow \Gamma C^s_W \Gamma$? Since $\Gamma$ is not empty then there is an element $a$ such that $a \in \Gamma$. Since $R^s$ is reflexive we have that $aR^sa$ and hence $\Gamma C^s_W \Gamma$.

Proving that $C^t_W$ is a contact and $\mathcal{B}_W$ is a precontact can be done in a similar manner. Let's check $(C^s_W \Rightarrow C^t_W)$. Let $\Gamma C^s_W \Delta$. By definition, $\exists a \in \Gamma, \exists b \in \Delta : aR^sb$. But $R^s \subseteq R^t$ so $aR^tb$ and hence $\Gamma C^t_W \Delta$.

Next, let's make sure that the Efremovich axiom holds for the relation $C^t_W$: $\Gamma \overline{C^t}_W \Delta \Rightarrow \exists \Theta(\Gamma \overline{C^t}_W \Theta$ and $\Theta^* \overline{C^t}_W \Delta)$. Let $\Theta = \{c \in W : \exists s \in \Delta$ such that $cR^ts\}$. So if $c \notin \Theta$ (which means that $c \in \Theta^*$) then $\forall s \in \Delta$ we have $c\overline{R^t}s$. Hence $\Theta^* \overline{C^t}_W \Delta$. Now we have to show that $\Gamma \overline{C^t}_W \Theta$. Towards contradiction, suppose that $\Gamma C^t_W \Theta$. So $\exists a \in \Gamma, \exists c \in \Theta : aR^tc$. Since $c \in \Theta$, by definition, $\exists b \in \Delta : cR^tb$ and by the transitivity of $R^t$ we get that $aR^tb$. This is a contradiction with the premise $\Gamma \overline{C^t}_W \Delta$.

What's left is to check that the axioms for the sets $TR_W$ and $UTR_W$ hold. Firstly, for $(TR1)$ we need to show that $\Theta \in TR_W \Leftrightarrow \Theta \neq 0$ and $\forall \Gamma, \Delta(\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta \Rightarrow \Gamma C^t_W \Delta)$. For the forward direction, let $\Theta \in TR_W$. By definition we have that $\Theta \neq \emptyset$ (first part of what we are trying to prove) and $\Theta$ is contained in an equivalence class of $R^t$. To prove the second part, let $\Gamma$ and $\Delta$ be such that $\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta$ - we want to show that $\Gamma C^t_W \Delta$. From $\Gamma C^t_W \Theta$ we get that

$\exists a \in \Gamma, \exists c \in \Theta : aR^tc$ and similarly from $\Delta C^t_W \Theta$ we have $\exists b \in \Delta, \exists d \in \Theta : bR^td$. Since $\Theta$ is contained in an equivalence class of $R^t$ we have that $cR^td$. From here, the transitivity of $R^t$ and $aR^tc$ we get that $aR^td$. Now taking into account that $bR^td$ and the symmetry and transitivity of $R^t$ we get that $aR^tb$. Hence $\Gamma C^t_W \Delta$ which is what we are trying to prove. For the backward direction, suppose that $\Theta \neq \emptyset$ and $\forall \Gamma, \Delta(\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta \Rightarrow \Gamma C^t_W \Delta)$. We have to show that $\Theta$ is contained in an equivalence class of $R^t$, that is $\forall c, d \in \Theta, cR^td$. Suppose, towards contradiction, that $\exists c, d \in \Theta, c\overline{R^t}d$. Take $\Gamma = \{c\}$, $\Delta = \{d\}$. Obviously, we have that $\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta$ (by the reflexivity of $R^t$) but $\Gamma \overline{C^t}_W \Delta$, which is a contradiction.

For (TR2) we need to show that $\Theta \in UTR_W \Leftrightarrow \Theta \in TR_W$ and $\Theta \overline{C^t}_W \Theta^*$. For the forward direction, suppose that $\Theta \in UTR_W$ and hence, in $TR_W$. We need to show that $\Theta \overline{C^t}_W \Theta^*$. Suppose the contrary, $\Theta C^t_W \Theta^*$ - this would imply that $\exists a \in \Theta, \exists b \in \Theta^*$ such that $aR^tb$. But then $b$ belongs to the same equivalence class as $a$, so $b \in \Theta$. This is a contradiction with $b \in \Theta^*$. For the backward direction let $\Theta \in TR_W$ (so $\Theta$ is not empty and is contained in some equivalence class, say $\Psi$, of $R^t$) and $\Theta \overline{C^t}_W \Theta^*$. In order to prove that $\Theta \in UTR_W$ we need to show that $\Theta = \Psi$. Suppose the contrary, $\Theta \subset \Psi$, so $\exists d \in \Psi$ such that $d \notin \Theta$(so $d \in \Theta^*$). We know that $\Theta \neq \emptyset$, so let's pick an element $c \in \Theta$. Since $\Theta \subset \Psi$ we get that $c \in \Psi$. Since $\Psi$ is an equivalence class of $R^t$ and we have that $c \in \Psi$ and $d \in \Psi$, we get $cR^td$. From here and the fact that $c \in \Theta$ and $d \in \Theta^*$ we get $\Theta C^t_W \Theta^*$ which is a contradiction.

Now, let's check that (TR$C^t$) holds. Let $\Gamma C^t_W \Delta$ - we want to show that $\exists \Theta \in UTR_W$ such that $\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta$. Since $\Gamma C^t_W \Delta$ we have that $\exists a \in \Gamma, \exists b \in \Delta$ such that $aR^tb$. Take $\Theta$ to be equal to the equivalence class of $a$ (obviously $b$ is also in $\Theta$). Then trivially $\Gamma C^t_W \Theta$ and $\Delta C^t_W \Theta$.

For (TR$C^s$) suppose $\Gamma C^s_W \Delta$ and hence $\exists a \in \Gamma, \exists b \in \Delta$ such that $aR^sb$. Take $\Theta$ to be the equivalence class of $a$ with respect to the $R^t$ relation. Clearly $a \in \Gamma \cap \Theta$ and $b \in \Delta$ so $(\Gamma \cap \Theta) C^s_W \Delta$.

For (TR$\mathcal{B}$1) let $\Theta \in TR_W$, $\Theta \mathcal{B}_W \Delta$ and $\Gamma \mathcal{B}_W \Theta$ - we want to show that $\Gamma \mathcal{B}_W \Delta$. From $\Theta \mathcal{B}_W \Delta$ we have that $\exists c \in \Theta, \exists b \in \Delta$ such that $c \prec b$ and from $\Gamma \mathcal{B}_W \Theta$ we get $\exists a \in \Gamma, \exists d \in \Theta$ such that $aR^td$. Since $c \in \Theta$ and $d \in \Theta$ and $\Theta$ is contained in an equivalence class of $R^t$ we get that $dR^tc$. From here, $aR^td$ and the transitivity of $R^t$ we get $aR^tc$. Taking this into account, the fact that $c \prec b$ and Def.2.5(iii) we get that $a \prec b$ and hence $\Gamma \mathcal{B}_W \Delta$. (TR$\mathcal{B}$2) can be proved in the same way using Def.2.5(iv).

For (TR$\mathcal{B}$3) let $\Gamma \mathcal{B}_W \Delta$ and we want to show that $\exists \Theta \in UTR_W$ such that $\Theta \mathcal{B}_W \Delta$ and $\Gamma C^t_W \Theta$. From $\Gamma \mathcal{B}_W \Delta$ we get that $\exists a \in \Gamma, \exists b \in \Delta$ such that $a \prec b$ and take $\Theta$ to be the equivalence class of $a$. We have that $\Gamma C^t_W \Theta$ because $a \in \Theta$, $a \in \Gamma$ and $R^t$ is reflexive. We also have that $\Theta \mathcal{B}_W \Delta$ since $a \in \Theta$, $b \in \Delta$ and $a \prec b$. Axiom (TR$\mathcal{B}$4) can be shown in a similar way.

Lastly, we need to make sure that axioms (UTR$\mathcal{B}$11), (UTR$\mathcal{B}$12), (UTR$\mathcal{B}$21) and (UTR$\mathcal{B}$22) hold. We'll verify axiom (UTR$\mathcal{B}$22) and the others can be proved in a similar way. Let $\Gamma$ and $\Delta$ be members $UTR_W$. For the easier backward direction of axiom (UTR$\mathcal{B}$22) suppose that there is $\Theta \in UTR_W$ such that $\Gamma \mathcal{B}_W \Theta$ and $\Delta \mathcal{B}_W \Theta$.

From here we get that $\exists x \in \Gamma, \exists y \in \Theta$ such that $x \prec y$ and $\exists z \in \Delta, \exists w \in \Theta$ such that $z \prec w$. Since $y$ and $w$ are in $\Theta$ and it is an equivalence class of $R^t$ we have that $yR^tw$. From here and $x \prec y$ we get that $x \prec w$. Now, let $P$ be an arbitrary element of $B_W$ - we want to show that $\Gamma\mathcal{B}_W P$ or $\Delta\mathcal{B}_W P^*$. Suppose $w \in P$ - then obviously $\Gamma\mathcal{B}_W P$ since $x \in \Gamma$, $w \in p$ and $x \prec w$. Alternatively, if $w \notin P$, then $w \in P^*$. But in this case we have that $z \in \Delta$, $w \in P^*$ and $z \prec w$ so $\Delta\mathcal{B}_W P^*$. For the forward direction, let for all $P \in B_W$ be true that $\Gamma\mathcal{B}_W P$ or $\Delta\mathcal{B}_W P^*$ and, toward contradiction, suppose that $\neg(\exists\Theta \in UTR_W)(\Gamma\mathcal{B}_W\Theta$ and $\Delta\mathcal{B}_W\Theta)$. Then, by Lemma 2.8 we get that $\langle\succ\rangle\Gamma \cap \langle\succ\rangle\Delta = \emptyset$. Take $P = \langle\succ\rangle\Delta$ - so, by the premise, we must have that $\Gamma\mathcal{B}_W\langle\succ\rangle\Delta$ or $\Delta\mathcal{B}_W(\langle\succ\rangle\Delta)^*$. Suppose $\Gamma\mathcal{B}_W\langle\succ\rangle\Delta$ - then $\exists x \in \Gamma, \exists y \in \langle\succ\rangle\Delta$ such that $x \prec y$. From here we have that $y \succ x$ and since $x \in \Gamma$ we get that $y \in \langle\succ\rangle\Gamma$. But $y \in \langle\succ\rangle\Delta$ so we have that $\langle\succ\rangle\Gamma \cap \langle\succ\rangle\Delta \neq \emptyset$ which is a contradiction. So it must be the case that $\Delta\mathcal{B}_W(\langle\succ\rangle\Delta)^*$. This means that $\exists x \in \Delta$ and $\exists y \in (\langle\succ\rangle\Delta)^*$ (meaning that $y \notin \langle\succ\rangle\Delta$) such that $x \prec y$. From here we get that $y \succ x$ and since $x \in \Delta$, $y \in \langle\succ\rangle\Delta$ - contradiction. This means that our initial assumption was false which completes the prove of the axiom. □

**Lemma 2.10.** $\underline{B}(W)$ is a strong dynamic contact algebra.

*Proof.* By the previous lemma we know that $\underline{B}(W)$ satisfies the axioms of a DCA. We need to assert that the additional axioms for SDCA hold. Firstly, for $(C^t\mathcal{B})$, suppose $\Gamma\overline{\mathcal{B}}_W\Delta$ - we want to show that $(\exists\Theta)(\Gamma\overline{C^t}_W\Theta$ and $\Theta^*\overline{\mathcal{B}}_W\Delta)$. Take $\Theta = \{y \in W|$ for all $x \in \Gamma$ we have $x\overline{R^t}y\}$(if $\Gamma = \emptyset$, take $\Theta = W$ and the statement follows for trivial reasons). By definition, $\Gamma\overline{C^t}_W\Theta$. To prove that $\Theta^*\overline{\mathcal{B}}_W\Delta$, suppose towards contradiction the opposite - $\Theta^*\mathcal{B}_W\Delta$. By definition, $\exists y \in \Theta^*$, $\exists z \in \Delta$ such that $y \prec z$. Since $y \in \Theta^*$, we have that $y \notin \Theta$ and hence, by the definiton of $\Theta$, $\exists x \in \Gamma$ such that $xR^ty$. From here, $y \prec z$ and Def.2.5(iii) we conclude that $x \prec z$. But since $x \in \Gamma$ and $z \in \Delta$ we get that $\Gamma\mathcal{B}_W\Delta$ which is a contradiction with the premise $\Gamma\overline{\mathcal{B}}_W\Delta$.

For $(\mathcal{B}C^t)$, suppose $\Gamma\overline{\mathcal{B}}_W\Delta$ - we want to show that $(\exists\Theta)(\Gamma\overline{\mathcal{B}}_W\Theta$ and $\Theta^*\overline{C^t}_W\Delta)$. Take $\Theta = \{y \in W|\exists z \in \Delta$ such that $yR^tz\}$. So, by definition, $\Theta^*\overline{C^t}_W\Delta$. Now, towards contradiction, suppose that $\Gamma\mathcal{B}_W\Theta$. This means that $\exists x \in \Gamma$, $\exists y \in \Theta$ such that $x \prec y$. Since $y \in \Theta$ we have that there is $z \in \Delta$ such that $yR^tz$. From here and Def.2.5(iv) we get that $x \prec z$. Since $x \in \Gamma$ and $z \in \Delta$ we conclude that $\Gamma\mathcal{B}_W\Delta$ which is a contradiction. □

The following lemma is analogous to Lemma 1.19 and shows a correspondence between the time axioms in $\underline{B}(W)$ and time conditions in $\underline{W}$:

**Lemma 2.11** (Relational correspondence for time axioms)**.** Let $\alpha$ be any formula from the list of time axioms - $(rs)$, $(ls)$, $(updir)$, $(downdir)$, $(dens)$, $(ref)$, $(irr)$, $(lin)$, $(tri)$, $(tr)$ and let $A$ be the corresponding formula from the list of time conditions - $(LS)_W$, $(RS)_W$, $(UpDir)_W$, $(DownDir)_W$, $(Dens)_W$, $(Ref)_W$, $(Irr)_W$, $(Lin)_W$, $(Tri)_W$, $(Tr)_W$. Then $A$ is true in $\underline{W}$ iff $\alpha$ is true in $\underline{B}(W)$.

*Proof.* Let's verify the claim for the density and irreflexivity conditions. The others can be shown in a similar fashion.
$(Dens)_W \to (dens)$. Suppose that for all $x, y \in W$ we have $x \prec y \to (\exists z)(x \prec z$ and $z \prec y)$ and let $\Gamma \mathcal{B} \Delta$ for some $\Gamma, \Delta \in B_W$ - we'll show that $\Gamma \mathcal{B} \Theta$ or $\Theta^* \mathcal{B} \Delta$, for any $\Theta \in B_W$. By $\Gamma \mathcal{B} \Delta$, there are $x \in \Gamma, y \in \Delta$ such that $x \prec y$ and by the premise we get $x \prec z$ and $z \prec y$, for some $z \in W$. Let $\Theta$ be an arbitrary element of $B_W$. If $z \in \Theta$, then $\Gamma \mathcal{B} \Delta$. If not, then $z \in \Theta^*$ and hence $\Theta^* \mathcal{B} \Delta$ which proves this direction.
$(dens) \to (Dens)_W$. Suppose for all $\Gamma, \Delta, \Theta \in B_W$ we have $\Gamma \mathcal{B} \Delta \Rightarrow \Gamma \mathcal{B} \Theta$ or $\Theta^* \mathcal{B} \Delta$. Towards contradiction, suppose there are $x, y \in W$, $x \prec y$ such that for all $z \in W$, $x \not\prec z$ or $z \not\prec y$. Let $\Gamma = \{x\}$ and $\Delta = \{y\}$. We obviously have that $\Gamma \mathcal{B} \Delta$ and take $\Theta = \{s \in W | x \not\prec s\}$ - clearly $\Gamma \overline{\mathcal{B}} \Theta$. Take a look at $\Theta^* = W \setminus \Theta = \{s : x \prec s\}$. Since $x \prec y$ and $\forall s \in \Theta^*, x \prec s$ by the assumption we must have that $\forall s \in \Theta^*, s \not\prec y$. This means that $\Theta^* \overline{\mathcal{B}} \Delta$ which a contradiction with the premise.
$(Irr)_W \to (irr)$. Let $(\forall x, y)(x \prec y \to x \overline{R}^t y)$. To prove $(irr)$ suppose $\Gamma \mathcal{B} \Delta$. Then there exist $x \in \Gamma$ and $y \in \Delta$ such that $x \prec y$. Define $\Theta = \{x\}$ and $\Omega = \{y\}$. Then obviously $\Theta \neq 0$, $\Omega \neq 0$, $\Theta \leq \Gamma$, $\Omega \leq \Delta$ and $\Theta \overline{C^t} \Omega$.
$(irr) \to (Irr)_W$. Let $\Gamma \mathcal{B} \Delta \to (\exists \Theta, \Omega \neq 0)(\Theta \leq \Gamma$, $\Omega \leq \Delta$, and $\Theta \overline{C^t} \Omega)$. Let $x, y \in W$ be such that $x \prec y$ and take $\Gamma = \{x\}$, $\Delta = \{y\}$. We have that $\Gamma \mathcal{B} \Delta$. Then there exist $\Theta, \Omega \neq 0$ such that $\Theta \subseteq \{x\}$, $\Omega \subseteq \{y\}$ and $\Theta \overline{C^t} \Omega$. From here we get that $\Theta = \{x\}$, $\Omega = \{y\}$ and $\{x\} \overline{C}^t \{y\}$. Hence $x \overline{R^t} y$. □

## 2.3 Canonical constructions over weak DCAs

In this section, given a *weak* DCA $\underline{B}$, we'll construct a dynamic relational space. This will be done by taking a subset of special ultrafilters of $\underline{B}$, called UTR-ultrafilters, and defining several relations between those ultrafilters.

**Definition 2.12** (UTR-ultrafilter)**.** Let $\underline{B}$ be a *weak* DCA and let $U$ be an ultrafilter (see Def.1.8) of $\underline{B}$. We'll call $U$ a *UTR-ultrafilter* if there is an element $c \in UTR$ such that $c \in U$. For convenience, we use the notation $U(c)$, meaning that $U$ is an UTR-ultrafilter containing the element $c \in UTR$. With *UTR-ULT(B)* we'll denote the set of all UTR-ultrafilters of $\underline{B}$.

Let $\underline{B}$ be a WDCA and $F, G$ be arbitrary filters of $B$. We'll define the *canonical relations* $R^s$, $R^t$ and $\prec$ between filters in $\underline{B}$, in the following way:

$$FR^sG \Leftrightarrow \forall a \in F, \forall b \in G \ aC^sb$$
$$FR^tG \Leftrightarrow \forall a \in F, \forall b \in G \ aC^tb$$
$$F \prec G \Leftrightarrow \forall a \in F, \forall b \in G \ a\mathcal{B}b$$

The following lemma will contain some characterizations of the UTR-ultrafilters of weak DCAs with respect to the canonical relations $R^s$, $R^t$ and $\prec$.

**Lemma 2.13.** Let $U(c)$ and $V(d)$ be UTR-ultrafilters. Then:

(i) $U(c)R^tV(d)$ iff $c = d$

(ii) $U(c)R^sV(d)$ iff $c = d$ and $\forall a \in U(c), \forall b \in V(d)\ aC^sb$
(iii) $U(c) \prec V(d)$ iff $c\mathcal{B}d$

*Proof.* (i) ($\Rightarrow$) From $U(c)R^tV(d)$, by definition, we get that $cC^td$. By Lemma 2.3(v) we get $c = d$.
(i) ($\Leftarrow$) Let $c = d$ - we want to show that $U(c)R^tV(c)$, i.e. for any $a \in U(c)$, $b \in V(c)$ we should have $aC^tb$. Since $a \in U(c)$, $c \in U(c)$ and $U(c)$ is an ultrafilter we have that $a \cdot c \neq 0$ and hence $aC^tc$. Similarly $bC^tc$. Since $c \in UTR$ from axiom $(TR1)$ we conclude that $aC^tb$.
(ii) Follows directly by taking into consideration the definition of $R^s$, the fact that $C^s \subseteq C^t$(and hence $R^s \subseteq R^t$) and (i).
(iii) The forward direction is obvious. For the backward direction, let $c\mathcal{B}d$ and $a \in U(c), b \in V(d)$. $a \in U(c)$ means that $a \cdot c \neq 0$ and hence $aC^tc$. Using the fact that $c \in UTR$ and axiom $(TR\mathcal{B}1)$ we get that $a\mathcal{B}d$. Similarly, from $b \in V(d)$ we get $bC^td$. Since $d \in UTR$ by axiom $(TR\mathcal{B}2)$ we get that $a\mathcal{B}b$. Hence $U(c) \prec V(d)$. □

Let $\underline{B}$ be a *weak* DCA. We associate to $\underline{B}$ a system $\underline{W}(B) = (X(B), R^s, R^t, \prec, now)$ where $X(B) =$ *UTR-ULT(B)*, the binary relations $R^s, R^t, \prec$ are the canonical relations between ultrafilters and *now* is the fixed UTR-ultrafilter containing the element $NOW$.

**Lemma 2.14.** $\underline{W}(B)$ is a dynamic relational structure.

*Proof.* Let's verify that $\underline{W}(B)$ satisfies Def.2.5. For the reflexivitty of $R^s$, let $F \in X(B)$ - we want to show that $FR^sF$. Since $F$ is an ultrafilter then we have that $0 \notin F$ (ultrafilters are proper filters). Let $a$ be an arbitrary element of F. Then since $a \neq 0$ and $C^s$ is a contact relation we have that $aC^sa$ and hence $FR^sF$. For the symmetry, let $F, G \in X(B)$ and let $FR^sG$. Let $a, b$ be arbitrary elements in $F$ and $G$ respectively. Since $FR^sG$ we know that $aC^sb$ and by the symmetry of the $C^s$ relation we get that $bC^sa$. Hence $GR^sF$. Reflexivity and symmetry of $R^t$ can be proved in the same way using that $C^t$ is a contact relation. For the transitivity, let $F(c)R^tG(d)$ and $G(d)R^tH(e)$ - we want to show that $F(c)R^tH(e)$. From $F(c)R^tG(d)$ using Lemma 2.13 we get that $c = d$ and from $G(d)R^tH(e)$ - $d = e$. Hence $c = e$ and applying Lemma 2.13 we conclude that $F(c)R^tH(e)$. Proving that $R^s \subseteq R^t$ is trivial since $C^s \subseteq C^t$. Next, let's check point (iii) from Def.2.5. Let $F(c)R^tG(d)$ and $G(d) \prec H(e)$. By Lemma 2.13 we get that $c = d$ and $d\mathcal{B}e$. Hence $c\mathcal{B}e$ and by Lemma 2.13 $F(c) \prec H(e)$. Similarly for (iv), let $F(c) \prec G(d)$ and $G(d)R^tH(e)$. By Lemma 2.13 we ge that $c\mathcal{B}d$ and $d = e$. Hence $c\mathcal{B}e$ and by Lemma 2.13 $F(c) \prec H(e)$. □

The system $\underline{W}(B)$ is called the *canonical dynamic relational structure* over the *weak* DCA $\underline{B}$. Let $\underline{B}(W)$ be the strong DCA over $\underline{W}(B)$ as constructed in the previous section. We'll call $\underline{B}(W)$ the *canonical strong DCA* associated to $\underline{B}$.

**Lemma 2.15.** Let $\underline{B}$ be a *weak* DCA, let $A$ be a formula among $(LS)_W$, $(RS)_W$, $(UpDir)_W$, $(DownDir)_W$, $(Dens)_W$, $(Ref)_W$, $(Irr)_W$, $(Lin)_W$, $(Tri)_W$, $(Tr)_W$ and let $\alpha$ be the corresponding formula from the list of time axioms $(rs)$, $(ls)$, $(updir)$, (*downdir*), (*dens*), (*ref*), (*irr*), (*lin*), (*tri*), (*tr*). Then $\alpha$ is true in $\underline{B}$ iff $A$ is true in $\underline{W}(B)$.

*Proof.* The proof follows by using Lemma 2.6 and Lemma 2.13. To demonstrate, let $\alpha = (tri)$. By Lemma 2.6(ix) we have that $(tri)$ is true in the WDCA $\underline{B}$ iff $(\forall a, b \in UTR)(aC^tb$ or $a\mathcal{B}b$ or $b\mathcal{B}a)$ and let's denote this formula by $(tri)'$. It suffices to show that $(tri)'$ is true in $\underline{B}$ iff $(Tri)$ is true in $\underline{W}(B)$

For ($\Rightarrow$), let $(tri)'$ hold and let $\Gamma(a), \Delta(b) \in X(B)$ be arbitrary UTR-ultrafilters. We have $a \in UTR, b \in UTR$ and by $(tri)'$ we get $aC^tb$ (so $a = b$) or $a\mathcal{B}b$ or $b\mathcal{B}a$. By Lemma 2.13 we have $\Gamma(a)R^t\Delta(b)$ or $\Gamma(a) \prec \Delta(b)$ or $\Delta(b) \prec \Gamma(a)$. For the backward direction let $a, b \in UTR$. By axiom $(TR2)$ we have that $a \neq 0$ and hence there is an UTR-ultrafilter $\Gamma \in X(B)$ such that $\Gamma = \Gamma(a)$. Similarly for $b$ there is an UTR-ultrafilter $\Delta(b) \in X(B)$. By $(Tri)$ we have that $\Gamma(a)R^t\Delta(b)$ or $\Gamma(a) \prec \Delta(b)$ or $\Delta(b) \prec \Gamma(a)$. Thus, by Lemma 2.13 we conclude that $aC^tb$ or $a\mathcal{B}b$ or $b\mathcal{B}a$. □

## 2.4 The relational representation theorem

In this section we'll complete the study of the representation theory for weak DCAs. We'll show that every weak DCA can be isomorphically embedded into the strong DCA associated to it in a way that preserves the time axioms. Let $\underline{B}$ be a *weak* DCA, $\underline{W}(B)$ be the canonical dynamic relational structure over $\underline{B}$ and let $\underline{B}(W)$ be the canonical *strong DCA* associated to $\underline{B}$. We'll define the function $h : B \to B_W$ as follows $h(a) = \{F \in \textit{UTR-ULT}(B), a \in F\}$. Before showing that $h$ is an embedding we will need a couple of lemmas.

**Lemma 2.16.** Let $a, b \in B$. Then the following equivalences hold:

(i) $aC^sb \Leftrightarrow \exists U, V \in Ult(B)$ such that $UR^sV, a \in U, b \in V$

(ii) $aC^tb \Leftrightarrow \exists U, V \in Ult(B)$ such that $UR^tV, a \in U, b \in V$

(iii) $a\mathcal{B}b \Leftrightarrow \exists U, V \in Ult(B)$ such that $U \prec V, a \in U, b \in V$

*Proof.* We'll prove only (i) and the others can be shown in the same way. The backward direction is obvious by definition. For the forward direction, take a look at the filters $[a) = \{c : a \leq c\}$ and $[b) = \{c : b \leq c\}$. Since $aC^sb$ and $C^s$ is a (pre)contact relation, by axiom $(C^s2)$ we get that $[a) \times [b) \subseteq C^s$. By Lemma 1.11 we get that there exist ultrafilters $U$ and $V$, $[a) \subseteq U$, $[b) \subseteq V$ such that $U \times V \subseteq C^s$, or equivalently $UR^sV$. □

We'll be more interested in a stronger version of the previous lemma. We will make heavy use of Lemma 1.16, fully proved in [7] part 3. It's worth noting that we stated this lemma in the introductory section for a slightly stronger class of structures (dynamic contact algebras) but a careful examination of the proof reveals that it can be safely applied for weak DCAs as well, as the proof does not rely on the Efremovich axiom.

**Lemma 2.17.** Let $a, b \in B$. Then the following equivalences hold:

(i) $aC^sb \Leftrightarrow \exists U(c), V(c) \in$ *UTR-ULT*(B) such that $UR^sV, a \in U, b \in V$

(ii) $aC^tb \Leftrightarrow \exists U(c), V(c) \in$ *UTR-ULT*(B) such that $UR^tV, a \in U, b \in V$

(iii) $a\mathcal{B}b \Leftrightarrow \exists U(c), V(d) \in$ *UTR-ULT*(B) such that $U \prec V, a \in U, b \in V$

*Proof.* (i) For the forward direction, let $aC^sb$. By Lemma 1.16 (ii) we have that $\exists c \in UTR$ such that $(a \cdot c)C^s(b \cdot c)$. By the previous lemma we have that there exist ultrafilters $U$ and $V$ such that $a \cdot c \in U$, $b \cdot c \in V$ and $UR^sV$. It follows that $c \in U$ and $c \in V$ and hence $U$ and $V$ are UTR-ultrafilters having the desired properties. The backward direction is obvious. (ii) and (iii) can be proved in the same way using Lemma 1.16 (iii) and (iv). □

**Lemma 2.18** (Embedding Lemma)**.** The function $h$ is an isomorphic embedding of $\underline{B}$ into $\underline{B}(W)$, that is:

(i) $a \leq b \Leftrightarrow h(a) \subseteq h(b)$

(ii) $aC^sb \Leftrightarrow h(a)C^s_Wh(b)$

(iii) $aC^tb \Leftrightarrow h(a)C^t_Wh(b)$

(iv) $a\mathcal{B}b \Leftrightarrow h(a)\mathcal{B}_Wh(b)$

(v) $c \in TR \Leftrightarrow h(c) \in TR_W$

(vi) $c \in UTR \Leftrightarrow h(c) \in UTR_W$

(vii) $h(NOW) = NOW_W$

*Proof. (i)* For the forward direction, let $a \leq b$ and let $\Gamma \in h(a)$ - we want to show that $\Gamma \in h(b)$. By the definition of $h$, we have that $a \in \Gamma$ and $\Gamma$ is a UTR-ultrafilter. From here and $a \leq b$ we get that $b \in \Gamma$ (filter property). Hence, $\Gamma \in h(b)$. For the backward direction, we'll reason by contraposition, that is suppose that $a \nleq b$ - we want to show that $h(a) \nsubseteq h(b)$. Since $a \nleq b$, we know that $a \cdot b^* \neq 0$ by Lemma 1.3. From here and the fact that $C^s$ is a contact relation by $(C5)$ we get that $(a \cdot b^*)C^s(a \cdot b^*)$. From axiom $(TRC^s)$ we get that there is $c \in UTR$ such that $(a \cdot b^* \cdot c)C^s(a \cdot b^*)$ and from axiom $(C1)$ we get that $a \cdot b^* \cdot c \neq 0$. Using the commutativity and associativity of the *meet* operation we get $(a \cdot c) \cdot b^* \neq 0$, which from the properties of $\nleq$ means that $a \cdot c \nleq b$. From Lemma 1.9 we get that there exists an ultrafilter $\Gamma$ such that $a \cdot c \in \Gamma$ and $b \notin \Gamma$. Since $a \cdot c \leq a$, $a \cdot c \leq c$ and $\Gamma$ is a ultrafilter we have that $a \in \Gamma$ and $c \in \Gamma$. Since $c \in UTR$, $\Gamma$ is an UTR-ultrafilter. From $a \in \Gamma$ we get that $\Gamma \in h(a)$ and since $b \notin \Gamma$ we have that $\Gamma \notin h(b)$.

For the forward direction of *(ii)*, let $aC^sb$. By Lemma 2.17(i) we get that $\exists U(c), V(c) \in$ *UTR-ULT*(B) such that $a \in U$, $b \in V$ and $UR^sV$. Hence, we have that $U \in h(a)$ and $V \in h(b)$ and we conclude that $h(a)C^s_Wh(b)$. For the backward direction, let $U \in h(a)$ and $V \in h(b)$ be such that $UR^sV$. We have that $a \in U$ and $b \in V$ and by the definition of $R^s$ we get that $aC^sb$. *(iii)* and *(iv)* can be proved in a similar way using Lemma 2.17.

For the forward direction of *(v)* let $c \in TR$. By axiom $(TR1)$ we get that $c \neq 0$ and it is easy to see that this implies $h(c) \neq 0$. Now we want to show that $h(c)$ is contained in an equivalence class of $R^t$, that is, for any $U, V \in h(c)$ we have that $UR^tV$. Let $U, V$ be arbitrary elements of $h(c)$. Let $a$ and $b$ be arbitrary elements of $U$ and $V$ respectively. Since $U \in h(c)$ we have that $c \in U$. From here and $a \in U$ we get that $a \cdot c \neq 0$ and hence $aC^tc$. Similarly $bC^tc$ and since $c \in TR$ we get $aC^tb$. Since $a, b$ were abritrary we conclude that $UR^tV$ and since $U, V$ were arbitrary then $h(c)$ is contained in an equivalence class of $R^t$. Hence $h(c) \in TR_W$. For the backward direction, let $h(c) \in TR_W$. We want to show that $c \in TR$, i.e. $c$ fulfils axiom $(TR1)$. It is clear that $c \neq 0$ - otherwise there wouldn't have been an UTR-ultrafilter which contains it (ultrafilters are proper filters). Now, let $aC^tc$ and $bC^tc$ - we want to show that $aC^tb$. Using (iii) we get that $h(a)C^t_Wh(c)$ and $h(b)C^t_Wh(c)$. Taking into account that $h(c) \in TR_W$ and $\underline{B}(W)$ is an SDCA by $(TR1)$ we get that $h(a)C^t_Wh(b)$ and applying *(iii)* again we conclude that $aC^tb$.

For *(vi)*($\Rightarrow$), let $c \in UTR$ - so $c \in TR$ and $c\overline{C^t}c^*$. By *(v)* we know that $h(c) \in TR_W$, i.e. $h(c)$ is a subset of an equivalence class of $R^t$. Let $X$ be the equivalence class of $R^t$ such that $h(c) \subseteq X$. In order to show that $h(c) \in UTR_W$ we have to make sure that $h(c) = X$. Suppose that this is not the case, that is $h(c) \subset X$, so $X \setminus h(c) \neq 0$. Let $U \in h(c)$ and $V \in X \setminus h(c)$ - $U \in h(c)$ means that $c \in U$ and $V \in X \setminus h(c)$ implies that $c^* \in V$. Since X is an equivalence class we have that $UR^tV$ and hence $cC^tc^*$ which is a contradiction. For ($\Leftarrow$) Let $h(c) \in UTR_W$. Since $\underline{B}(W)$ is an SDCA we have that $h(c) \in TR_W$ and $h(c)\overline{C^t}_Wh(c)^*$. By *(v)* we get that $c \in TR$ - we just have to show that $c\overline{C^t}c^*$. Since $h(c)$ is a set of ultrafilters it is easy to see that $h(c)^* = h(c^*)$. Hence $h(c)\overline{C^t}_Wh(c^*)$ and using *(iii)* we arrive at $c\overline{C^t}c^*$. □

**Theorem 2.19** (Relational representation theorem for weak DCAs)**.** Let $\underline{B}$ be a *weak* dynamic contact algebra. Then there exists a *strong* dynamic contact algebra $\widehat{\underline{B}}$ and an isomorphic embedding of $\underline{B}$ into $\widehat{\underline{B}}$. Additionally, if $\alpha$ is a formula among the list of time axioms $(rs)$, $(ls)$, $(updir)$, $(downdir)$, $(dens)$, $(ref)$, $(irr)$, $(lin)$, $(tri)$, $(tr)$, then $\alpha$ is true in $\underline{B}$ iff $\alpha$ is true in $\widehat{\underline{B}}$.

*Proof.* Let $\widehat{\underline{B}}$ be the canonical *strong* DCA associated to $\underline{B}$ and let $h$ be defined as above. The Embedding Lemma shows that $h$ is an isomorphic embedding of $\underline{B}$ into $\widehat{\underline{B}}$. The claim about time axioms follows directly from the construction by combining Lemma 2.15 and Lemma 2.11. □

**Corollary 2.20.** Every DCA can be isomorphically embedded into a strong DCA.

# 3 Basic dynamic contact algebras

In this section we'll introduce the notion of a *basic* dynamic contact algebra (BDCA) - a new type of DCAs that is a generalization of weak DCAs. The BDCA definition will contain the universal axioms of DCA and a couple of universal consequences from some the remaining axioms of weak DCA (see Lemma 2.4). Inspired by Lemma 2.3(vi), the signature of BDCA will contain an additional function, which for elements $c \in TR$ gives the unique $d \in UTR$ such that $c \leq d$. Our ultimate goal will be to show that the universal first-order theory of BDCA concides with that of WDCA, DCA and SDCA.

## 3.1 Abstract definition and basic properties

**Definition 3.1** (Basic dynamic contact algebra)**.** A *basic dynamic contact algebra*(BDCA) is any system $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C^s, C^t, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$, where $(B, \leq, 0, 1, \cdot, +, *)$ is a nondegenerate Boolean algebra and the following properties hold:

(i) $C^s$ is a contact relation on $B$, which is called *space contact*

(ii) $C^t$ is a contact relation on $B$, called *time contact* which satisfies axiom

$(C^s \Rightarrow C^t)$ $aC^sb \Rightarrow aC^tb$

(iii) $\mathcal{B}$ is a precontact relation on $B$ called the *precendence relation*

(iv) $TR$ and $UTR$ are subsets of $B$ called *time representatives* and *universal time representatives* respectively, satisfying the following axioms:

$(TR1)$ $c \in TR \Rightarrow c \neq 0$ and $(\forall a, b \in B)(aC^tc$ and $bC^tc \Rightarrow aC^tb)$

$(TR2)$ $c \in UTR \Leftrightarrow c \in TR$ and $c\overline{C^t}c^*$

$(TR\mathcal{B}1)$ $c \in TR, c\mathcal{B}b$ and $aC^tc \Rightarrow a\mathcal{B}b$

$(TR\mathcal{B}2)$ $d \in TR, a\mathcal{B}d$ and $bC^td \Rightarrow a\mathcal{B}b$

$(TR \leq)$ $c \in TR$, $d \leq c$ and $d \neq 0 \Rightarrow d \in TR$,

$(TR\cup)$ $c, d \in TR$ and $cC^td \Rightarrow (c + d) \in TR$

(v) $\mathcal{U}tr$ is a function satisfying the following axioms:

$(TR\mathcal{U}tr1)$ $c \in TR \Rightarrow \mathcal{U}tr(c) \in UTR$ and $c \leq \mathcal{U}tr(c)$

$(TR\mathcal{U}tr2)$ $a \notin TR \Rightarrow \mathcal{U}tr(a) = 0$

The function $\mathcal{U}tr$ will give us the unique UTR element (*UTR-witness*) corresponding to a specific time representative. The purpose of axiom $(TR\mathcal{U}tr2)$ is both to make the function $\mathcal{U}tr$ total and give us a convenient method to check if something is a time representative. Note that properties (ii),(iii),(iv) and (v) from Lemma 2.3 also hold for BDCAs since the proofs use only the universal axioms for DCA. By $\Sigma_{basic}$ we'll denote the class of all BDCAs. Let $\Theta$ be a set of the so-called time axioms. Then $\Sigma^{\Theta}_{basic}$ is the class of all BDCAs satisfying the time axioms from $\Theta$.

**Lemma 3.2.** The following properties hold for any basic dynamic contact algebra:

(i) $c \in UTR \Rightarrow \mathfrak{U}tr(c) = c$
(ii) $c \in TR \Rightarrow \mathfrak{U}tr(\mathfrak{U}tr(c)) = \mathfrak{U}tr(c)$
(iii) If $c, d \in TR$ and ($cC^t d$ or $c \leq d$ or $c + d \in TR$) then $\mathfrak{U}tr(c) = \mathfrak{U}tr(d)$
(iv) If $\{c_1, \cdots, c_n\} \subseteq TR$ and for all $i, j \in \{1, \cdots, n\}$ we have $c_i C^t c_j$, then $c_1 + \cdots + c_n \in TR$
(v) If $d = c_1 + \cdots + c_n \in TR$, and for all $i \in \{1, \cdots, n\}$ we have $c_i \neq 0$, then $\{c_1, \cdots, c_n\} \subseteq TR$ and for all $i, j \in \{1, \cdots, n\}$ we have $c_i C^t c_j$. Moreover, $\mathfrak{U}tr(d) = \mathfrak{U}tr(c_1) = \ldots = \mathfrak{U}tr(c_n)$
(vi) If $c, d \in UTR$ and $c \neq d$, then $c.d = 0$
(vii) If $a \cdot b \in TR$ then $a \cdot \mathfrak{U}tr(a \cdot b) \in TR$ and $\mathfrak{U}tr(a \cdot \mathfrak{U}tr(a \cdot b)) = \mathfrak{U}tr(a \cdot b)$
(viii) If $a_1 ... a_n \in TR$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, then $a_{i_1} ... a_{i_n}.\mathfrak{U}tr(a_1 ... a_n) \in TR$ and $\mathfrak{U}tr(a_{i_1} ... a_{i_n}.\mathfrak{U}tr(a_1 ... a_n)) = \mathfrak{U}tr(a_1 ... a_n)$
(ix) If $c_1, \ldots, c_n, d \in UTR$, $c_1^* ... c_n^*.a \neq 0$ and $c_1^* ... c_n^*.a \leq d$, then $d \notin \{c_1, \ldots, c_n\}$
(x) If $c_1, \ldots, c_n \in UTR$ and $c_1 + \ldots + c_n = 1$, then $UTR = \{c_1, \ldots, c_n\}$

*Proof.* (i) Let $d = \mathfrak{U}tr(c)$. By ($TR\mathfrak{U}tr1$) we have that $c \leq d$, $d \in UTR$ and by Lemma 2.3(iv) $c = d$.
(ii) Follows directly from (i).
(iii) Let $c, d \in TR$. Suppose $cC^t d$. From here, $c \leq \mathfrak{U}tr(c)$ and $d \leq \mathfrak{U}tr(d)$, by the contact axioms we have that $\mathfrak{U}tr(c) C^t \mathfrak{U}tr(d)$ and from Lemma 2.3(v) we have $\mathfrak{U}tr(c) = \mathfrak{U}tr(d)$. Now suppose, $c \leq d$. Since $c, d \in TR$ we have that $c, d \neq 0$ and considering $c \leq d$ we have that $c \cdot d \neq 0$ and then $cC^t d$ which brings us to the previous case. For the last case, let $c + d \in TR$. Since $c \leq c + d$, $d \leq c + d$, reasoning the same way as in the second case, we get $cC^t(c + d)$ and $dC^t(c + d)$ and since $c + d \in TR$ we get $cC^t d$ which reduces this case to the first one.
(iv) This follows easily from axiom ($TR\cup$) for BDCA.
(v) Let's take a look at arbitrary elements $c_i$, $c_j$. Since $c_i \neq 0$, $c_i \leq d$, $c_j \neq 0$, $c_j \leq d$ and $d \in TR$ by BDCA axiom ($TR \leq$) we get that $c_i, c_j \in TR$. Since $c_i, c_j \neq 0$ and $c_i \leq d, c_j \leq d, d \in TR$ we have $c_i C^t c_j$. From (iii) we get that $\mathfrak{U}tr(c_i) = \mathfrak{U}tr(c_j)$.
(vi) This is clear by Lemma 2.3 (v).
(vii). By axiom (TR$\mathfrak{U}tr$1) we have $\mathfrak{U}tr(a \cdot b) \in UTR$ (and also $\mathfrak{U}tr(a \cdot b) \in TR$) and $a \cdot b \leq \mathfrak{U}tr(a \cdot b)$. Since $a \cdot b \leq a$ we get $a \cdot b \leq (\mathfrak{U}tr(a \cdot b)) \cdot a$ and because $a \cdot b \neq 0$ we get $(\mathfrak{U}tr(a \cdot b)) \cdot a \neq 0$. We also have that $\mathfrak{U}tr(a \cdot b) \cdot a \leq \mathfrak{U}tr(a \cdot b)$. But $\mathfrak{U}tr(a \cdot b) \in TR$, $(\mathfrak{U}tr(a \cdot b)) \cdot a \neq 0$ and $(\mathfrak{U}tr(a \cdot b)) \cdot a \leq \mathfrak{U}tr(a \cdot b)$ imply by axiom (TR$\leq$) that $(\mathfrak{U}tr(a \cdot b)) \cdot a \in TR$. Using the facts that $a \cdot b \leq (\mathfrak{U}tr(a \cdot b)) \cdot a$, $a \cdot b \in TR$ and $(\mathfrak{U}tr(a \cdot b)) \cdot a \in TR$ by (iii) we get that $\mathfrak{U}tr(a \cdot \mathfrak{U}tr(a \cdot b)) = \mathfrak{U}tr(a \cdot b)$.
(viii) The proof is analogous to that of (vii).
(ix) Suppose $d = c_i$, $1 \leq i \leq n$. Then $c_1^* ... c_n^*.a \leq c_i$ and multiplying both sides of the inequality by $c_i^*$ we get $c_1^* ... c_n^*.a \leq 0$ - a contradiction.
(x) Let $d \in UTR$, then $d = d.1 = d.(c_1 + \ldots + c_n) \neq 0$. So there exists $1 \leq i \leq n$ such that $d.c_i \neq 0$ which by (vii) implies that $d = c_i$ and that $UTR = \{c_1, \ldots, c_n\}$. ☐

## 3.2 UTR-finite basic DCAs

Lemma 3.2 (x) suggests to introduce the following definition.

**Definition 3.3** (UTR-finite basic DCA)**.** Let $\underline{B}$ be a basic DCA. $\underline{B}$ is called a *UTR-finite basic dynamic contact algebra* if there is a finite subset $\{u_1, \ldots, u_n\} \subseteq UTR(B)$ such that $u_1 + \ldots + u_n = 1$.

**Lemma 3.4.** Let $\underline{B}$ be a UTR-finite basic DCA and $c \in B$ be such that $c \neq 0$. Then there exists $u \in UTR$ such that $c.u \neq 0$

*Proof.* Let $\underline{B}$ be a UTR-finite basic DCA. By definition, there exists a finite subset $\{u_1, \ldots, u_n\}$ of different elements of $UTR$ such that $u_1+\ldots+u_n = 1$. By Lemma 3.2 (x) $UTR = \{u_1, \ldots, u_n\}$, meaning that each $UTR$ member is one of $u_i$, $i = 1, \ldots, n$. Let $c \neq 0$. We have $c = c.1 = c.(u_1 + \ldots + u_n) = c.u_1 + \ldots + c.u_n$, so there exists $i = 1, ...n$ such that $c.u_i \neq 0$. ☐

**Lemma 3.5.** Every UTR-finite basic DCA is a weak DCA.

*Proof.* Let $\underline{B}$ be a UTR-finite basic DCA and let $\{u_1, \ldots, u_n\}$ be a finite subset of different elements of $UTR$ such that $u_1 + \ldots + u_n = 1$. We have that each $UTR$ member is one of $u_i$, $i = 1, \ldots, n$. Since $\underline{B}$ is a basic DCA we know that it satisfies the universal axioms of WDCA. We shall verify that $\underline{B}$ satisfies the non-universal axioms of weak DCA as well.

Firstly, for the backward direction of $(TR1)$, let $c \neq 0$ and let for all $a, b \in B$: $aC^tc$ and $bC^tc$ impy $aC^tb$. We'll show that $c \in TR$. This would indeed be the case if there exists $u_i \in UTR$ such that $c \leq u_i$. Suppose the contrary, i.e. for all $u_i \in UTR$ we have $c \not\leq u_i$ and hence $c \cdot u_i^* \neq 0$ and $u_i^*C^tc$. Since $c \neq 0$, by Lemma 3.4 we obtain that there exists $u_j \in UTR$ such that $c \cdot u_j \neq 0$ and hence $u_jC^tc$. We also have $u_j^*C^tc$ and by the premise of the claim we obtain $u_jC^tu_j^*$ which contradicts the fact that $u_j \in UTR$.

Next, for $(TRC^t)$ let $aC^tb$ - we want to show that there exists $u \in UTR$ such that $aC^tu$ and $bC^tu$. Let's rewrite the premise a bit - $aC^tb$ iff $(a \cdot 1)C^t(b \cdot 1)$ iff $a \cdot (u_1 + \ldots + u_n)C^t(b \cdot (u_1 + \ldots + u_n)$ iff $(a \cdot u_1 + \ldots + a \cdot u_n)C^t(b \cdot u_1 + \ldots + b \cdot u_n)$ iff there exists $i, j$, $1 \leq i \leq j \leq n$ such that $a \cdot u_iC^tb \cdot u_j$. This implies $u_iC^tu_j$ and by Lemma 2.3(v) we get $u_i = u_j$ and $i = j$. Thus, there exists $i : 1 \leq i \leq n$ such that $a \cdot u_iC^tb \cdot u_i$. This, by the axioms of contact, means that $aC^tu_i$ and $bC^tu_i$. Axioms $(TRC^s)$can be verified in a similar way.

Finally, for $(TR\mathcal{B}3)$, let $a\mathcal{B}b$ - we want to show that there is $u \in UTR$ such that $u\mathcal{B}b$ and $uC^ta$. Again, $a\mathcal{B}b$ iff $(a \cdot 1)\mathcal{B}(b \cdot 1)$ iff $a \cdot (u_1 + \ldots + u_n)\mathcal{B}(b \cdot (u_1 + \ldots + u_n)$ iff $(a \cdot u_1 + \ldots + a \cdot u_n)\mathcal{B}(b \cdot u_1 + \ldots + b \cdot u_n)$ iff there exists $i, j$, $1 \leq i \leq j \leq n$ such that $a \cdot u_i\mathcal{B}b \cdot u_j$. Since $a \cdot u_i \leq u_i$ and $b \cdot u_j \leq b$ we have $u_i\mathcal{B}b$. Also by precontact axioms we have $a \cdot u_i \neq 0$ and hence $aC^tu_i$. Axiom (TR$\mathcal{B}$4) can be shown in a similar manner. ☐

## 3.3 Finite generation lemma

**Lemma 3.6** (Finite Generation Lemma)**.** Let $\underline{B} = (B, C^t, C^s, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ be a basic DCA and let $A = \{a_1, \dots, a_n\}$ be a finite subset of $B$ containing $NOW$. Then there exists a finite subalgebra $B_0$ of $\underline{B}$ containing $A$.

*Proof.* In order to make the proof easy to follow we will prove the statement for the following special representative case: let $A = \{u, v, c, d\}$ where $u, v$ are two different elements of UTR one of which is $NOW$ and $c, d$ are two different elements of $B$ which are different from 0 and 1 and are not from $UTR$. Since $u \neq v, u, v \in UTR$ by Lemma 3.2(vi) we have $u \cdot v = 0$ and hence $u.v^* \neq 0$ and $u^*.v \neq 0$. The case $u^* \cdot v^* = 0$ implies that $u + v = 1$ which by Lemma 3.2 (x) shows that the only UTR elements of $\underline{B}$ are $u$ and $v$. In this case take the Boolean subalgebra $B_0$ generated by the set $A$ and consider it with the same contacts $C^t, C^s$ and the precontact $\mathcal{B}$. Define $UTR_{B_0} = \{u, v\} = UTR_B$, $TR_{B_0} = \{a \in B_0 : a \in UTR_B\}$ and $\mathcal{U}tr_{B_0}$ to be the restriction of $\mathcal{U}tr_B$ to $B_0$. Obviously $\mathcal{U}tr_{B_0}$ is defined for the elements of $TR_{B_0}$ and takes values in $UTR_{B_0}$, so $\underline{B_0}$ is a basic DCA which is a subalgebra of $\underline{B}$.

Let's now consider the case $u^*.v^* \neq 0$. Take a look at the following 16 elements of $B$ grouped in the following 4 groups:

(I) $u \cdot v \cdot c \cdot d$, $u \cdot v \cdot c \cdot d^*$, $u \cdot v \cdot c^* \cdot d$, $u \cdot v \cdot c^* \cdot d^*$,

(II) $u \cdot v^* \cdot c \cdot d$, $u \cdot v^* \cdot c \cdot d^*$, $u \cdot v^* \cdot c^* \cdot d$, $u \cdot v^* \cdot c^* \cdot d^*$,

(III) $u^* \cdot v \cdot c \cdot d$, $u^* \cdot v \cdot c \cdot d^*$, $u^* \cdot v \cdot c^* \cdot d$, $u^* \cdot v \cdot c^* \cdot d^*$,

(IV) $u^* \cdot v^* \cdot c \cdot d$, $u^* \cdot v^* \cdot c \cdot d^*$, $u^* \cdot v^* \cdot c^* \cdot d$, $u^* \cdot v^* \cdot c^* \cdot d^*$

Note that all elements from the group (I) are 0 because $u.v = 0$ ($u$ and $v$ are two different elements of UTR, see Lemma 3.2(vi)). We claim that it is not possible for all elements from group (II) to be equal to 0. Suppose that this is so, then we get the following: $0 = u \cdot v^* \cdot c \cdot d + u \cdot v^* \cdot c \cdot d^* + u \cdot v^* \cdot c^* \cdot d + u \cdot v^* \cdot c^* \cdot d^* = u \cdot v^* \cdot (c \cdot d + c \cdot d^* + c^* \cdot d + c^* \cdot d^*) = u \cdot v^* \cdot 1$, hence $u.v^* = 0$ which is not true. In a similar way we show that not all members from the groups (III) and (IV) are equal to 0 (for (III) we use the fact that $u^*.v \neq 0$ and for (IV) that $u^*.v^* \neq 0$).

Now, consider all possible sums of the members of the above groups. In particular, some of these sums are equal to the elements $u, v, c, d$ and the sum of the members of all groups gives the element 1 (these are basic facts from the theory of Boolean algebras). They form a Boolean subalgebra of $\underline{B}$ which may not be closed with respect to the operation $\mathcal{U}tr$, which is different from 0 only on members which are from the set $TR$. We claim that all non-zero elements from groups (II) and (III) (and such exist) are members of TR. Let's take a look, for instance, at the first member of (II) $u \cdot v^* \cdot c \cdot d \neq 0$. We have that $u \cdot v^* \cdot c \cdot d \leq u \in UTR$ which implies by axioms (TR$\leq$) and (TR2) that $u \cdot v^* \cdot c \cdot d \in TR$. For this member we have $\mathcal{U}tr(u \cdot v^* \cdot c \cdot d) = u$ and similarly for the other members of groups (II) and (III).

Other candidates for $TR$ from the groups above are the non-zero members of group (IV) (such elements exist) and we look in the algebra $\underline{B}$ for UTR-witnesses of those elements. Let, for simplicity, all members from group (IV) be members of $TR$.

Applying to them the function $\mathcal{U}tr$ we find four elements $w_1, w_2, w_3, w_4$ from UTR such that the following holds:
(#) $u^* \cdot v^* \cdot c \cdot d \leq w_1$, $u^* \cdot v^* \cdot c \cdot d^* \leq w_2$, $u^* \cdot v^* \cdot c^* \cdot d \leq w_3$, $u^* \cdot v^* \cdot c^* \cdot d^* \leq w_4$.

We'll show that $w_1, w_2, w_3, w_4$ are different from $u$ and $v$. Suppose, for example, that $w_1 = u$ - then we have $u^* \cdot v^* \cdot c \cdot d \leq u$. Multiplying both sides of this inequality with $u^*$ we obtain $u^* \cdot v^* \cdot c \cdot d = 0$ which is impossible, because $u^* \cdot v^* \cdot c \cdot d \in TR$. We arrive at the same conclusion if $w_1 = v$. So, $w_1, w_2, w_3, w_4$ are new members which we should include in the subalgebra we are looking for. For that purpose we consider the group (V) of the following elements:

(1) $w_1 \cdot w_2^* \cdot w_3^* \cdot w_4^*$, (2) $w_1^* \cdot w_2 \cdot w_3^* \cdot w_4^*$, (3) $w_1^* \cdot w_2^* \cdot w_3 \cdot w_4^*$,
(4) $w_1^* \cdot w_2^* \cdot w_3^* \cdot w_4$, (5) $w_1^* \cdot w_2^* \cdot w_3^* \cdot w_4^*$

Next, form the meets (multiplications) of each element from the groups (I) - (IV) with each element from the set (V) and then consider all possible joins (sums) between these newly formed elements. They generate a new finite Boolean subalgebra of $\underline{B}$, denoted by $B_0$, containing the elements $u, v, c, d$ and $w_1, w_2, w_3, w_4$. We are interested if this subalgebra is closed under the operation $\mathcal{U}tr$ applied to members of $B_0$ which are members of $TR$. In order to verify this, let's inspect the members of $TR$ in this subalgebra and if their UTR-witnesses are in the set $u, v, w_1, w_2, w_3, w_4$. Note that all multiplications of the members from groups (II) and (III) with elements (1), (2), (3) and (4) from the group (V) are equal to 0, because they contain two different elements from $UTR$. So the only possible non-zero multiplications from these groups are with element (5) $w_1^* \cdot w_2^* \cdot w_3^* \cdot w_4^*$. For instance, for the first member of (II) the result is $u \cdot v^* \cdot c \cdot d \cdot w_1^* \cdot w_2^* \cdot w_3^* \cdot w_4^* \leq u$. If it is non-zero then it is a member of $TR$ with UTR-witness $u$. The other possible members of $TR$ from these multiplications will have as UTR-witness either $u$ or $v$.

Now, let's consider possible multiplications of the members from group (IV) with the elements from the group (V). The member $u^* \cdot v^* \cdot c \cdot d$ can have possible non-zero multiplication only with element (1) $w_1 \cdot w_2^* \cdot w_3^* \cdot w_4^*$ and as a result we get $u^* \cdot v^* \cdot c \cdot d \cdot w_1 \cdot w_2^* \cdot w_3^* \cdot w_4^* \leq w_1$. If it is non-zero then it is a member of TR with UTR-witness $w_1$. Why do the other combinations give zero multiplication? Consider, for instance, the multiplication of $u^* \cdot v^* \cdot c \cdot d$ with (2) - the result is $u^* \cdot v^* \cdot c \cdot d \cdot w_1^* \cdot w_2 \cdot w_3^* \cdot w_4^*$. This element is $\leq w_1$ and $\leq w_1^*$ which implies that it is equal to 0. We obtain the same result by multiplying $u^* \cdot v^* \cdot c \cdot d$ with the elements (3), (4) and (5). This shows that the possible TR-members from the multiplications of the group (IV) and (V) have UTR-witnesses from the set $w_1, w_2, w_3, w_4$.

Lastly, we need to check if the sums of the newly formed elements can form new TR elements and what their UTR-witnesses would be. If a sum $d = a_1 + \ldots + a_k$ of non-zero members of the above considered groups is a member of TR, then by Lemma 3.2 (v) we may conclude that all $a_i$ are also members of TR and the UTR-witness of the sum $d$ and all $a_i$ are equal. So it is possible to have new TR members but their UTR-witnesses are from the set $u, v, w_1, w_2, w_3, w_4$ which are contained in the finite Boolean subalgebra generated by $u, v, c, d$ plus $w_1, w_2, w_3, w_4$. This shows that this finite Boolean subalgebra is also closed with respect to the function $\mathcal{U}tr$

applied for the members of $B_0$ which are members of TR.

We consider $C^t_{B_0}$, $C^s_{B_0}$, $\mathcal{B}_{B_0}$, $TR_{B_0}$ and $UTR_{B_0}$ to be the restrictions of the corresponding relations from $B$ in the set $B_0$, then this makes $B_0$ a finite basic DCA which is a subalgebra of $\underline{B}$. Let us note that the proof of the general case can go in the same way. □

Let us note that we may consider basic DCAs satisfying some of the time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr)*. However, we can not state that the Translation Lemma (see Lemma 2.6) which holds for weak DCAs is true for basic DCAs. The proof of this lemma for weak DCA essentially uses the non-universal axioms which are excluded from the definition of basic DCA. Note, however, that all time axioms except *(irr)* and *(tr)* are universal statements. Since universal statements are preserved under subalgebras, we can obtain the following version of Lemma 3.6 as a simple corollary.

**Corollary 3.7.** Let $\underline{B} = (B, C^t, C^s, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ be a basic DCA and let $A = \{a_1, \ldots, a_n\}$ be a finite subset of $B$ containing *NOW*. Suppose, in addition, that $\underline{B}$ satisfies a set $\Theta$ of universal time axioms. Then there exists a finite subalgebra $\underline{B_0}$ of $\underline{B}$ containing $A$ and satisfying the axioms from $\Theta$.

## 3.4 Relational models for basic dynamic contact algebras

In this section we will introduce a generalization of relational dynamic spaces which were introduced with Definition 2.5.

**Definition 3.8** (Basic dynamic relational space)**.** By a *basic dynamic relational structure* or *basic dynamic relational space* we mean any relational system $\underline{W} = (W, W^0, R^t, R^s, \prec, now)$ such that $W \neq \varnothing$, $W^0$ is a subset of $W$ containing *now* and the following additional conditions are satisfied:

(i) $R^t$ is a symmetric and reflexive relation in $W$

(ii) $R^t$ is an equivalence relation in $W^0$

(iii) If $x \in W^0$ and $xR^ty$, then $y \in W^0$

(iv) $R^s$ is a reflexive and symmetric relation included in $R^t$

(v) If $xR^ty$, $y \in W^0$ and $y \prec z$, then $x \prec z$

(vi) If $x \prec y$, $y \in W^0$ and $yR^tz$, then $x \prec z$

The subsystem $(W, W^0, R^t, \prec, now)$ is called the time substructure of the basic dynamic relational space.

We'll denote the class of all basic dynamic relational spaces by $\Delta_{basic}$. Let $\Omega$ be a subset of time conditions (special conditions on the relation $\prec$, as shown in section 2.1). Then $\Delta^{\Omega}_{basic}$ denote the class of all basic dynamic relational spaces satisfying the conditions from $\Omega$.

Obviously, $W^0$ with the restriction of all relations to $W^0$ is a dynamic relational space, so if we add the additional condition that $W^0 = W$ then the system coincides with the system of dynamic relational spaces. This shows that, indeed, Definition 3.8 is more general than Definition 2.5 and that all dynamic relational spaces are basic dynamic relational spaces, so $\Delta_{rel} \subseteq \Delta_{basic}$.

Let $\underline{W}$ be a basic dynamic relational space. We associate a structure $\underline{B}(W)$ to $\underline{W}$ via the following constructions. Define the contact relations $C^t$, $C^s$ and the precontact $\mathcal{B}$ as this is done for dynamic relational spaces in Section 2.2. Define $UTR(W)$ to be the set of equivalence classes of $W^0$ under the relation $R^t$ and $TR(W)$ to be the set of nonempty subsets of the equivalence classes in $W^0$. Define $NOW$ to be the equivalence class containing *now*. Finally, for $a \in TR(W)$ define $\mathcal{U}tr(a)$ to be the unique equivalence class containing $a$.

**Lemma 3.9.** $\underline{B}(W)$ is a basic DCA.

*Proof.* The BDCA axioms can be verified easily using the properties of the basic dynamic relational structure. For demonstration, suppose $c$ is an equivalence class - we want to show that it satisfies $c\overline{C}c^*$. Suppose the contrary, i.e. that $cC^tc^*$. Then there exist $x \in c$ and $y \notin c$ such that $xR^ty$, Since $c \subseteq W_0$ then $x \in W^0$. Then $xR^ty$ implies, by Definition 3.8 (iii), that $y \in W^0$ and since $c$ is equivalence class, that $y \in c$ - a contradiction. □

The analog of Lemma 2.11 has the same formulation and the same proof.

**Lemma 3.10.** Let $\alpha$ be any formula from the list of time axioms - $(rs)$, $(ls)$, $(updir)$, $(downdir)$, $(dens)$, $(ref)$, $(irr)$, $(lin)$, $(tri)$, $(tr)$ and let $A$ be the corresponding formula from the list of time conditions - $(LS)_W$, $(RS)_W$, $(UpDir)_W$, $(DownDir)_W$, $(Dens)_W$, $(Ref)_W$, $(Irr)_W$, $(Lin)_W$, $(Tri)_W$, $(Tr)_W$ (see Section 2.1). Then $A$ is true in $\underline{W}$ if $\alpha$ is true in $\underline{B}(W)$.

## 3.5 P-morphisms between basic relational dynamic spaces

In this section we will study p-morphisms between basic dynamic relational spaces. We'll use the following fact: if $f$ is a p-morphism from a a basic dynamic relational space $\underline{W}_1$ onto a basic dynamic relational space $\underline{W}_2$ then $f^{-1}$ is an isomorphic embedding of the basic DCA over $\underline{W}_2$ into the basic DCA over $\underline{W}_1$. Using p-morphisms we will prove that every basic DCA over a basic dynamic relational space can be embedded into a strong DCA.

**Definition 3.11.** Let $\underline{W_1} = (W_1, W_1^0, R_1^s, R_1^t, \prec_1, now_1)$ and $\underline{W_2} = (W_2, W_2^0, R_2^s, R_2^t, \prec_2, now_2)$ be basic dynamic relational structures. A surjection $f : W_1 \twoheadrightarrow W_2$ is called a *p-morphism* from $\underline{W_1}$ to $\underline{W_2}$ if for any $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ the following conditions are satisfied:

(i) if $x_1R_1^ty_1$ then $f(x_1)R_2^tf(y_1)$
(ii) if $x_2R_2^ty_2$ then $(\exists x_1, y_1 \in W_1)(x_2 = f(x_1), y_2 = f(y_1), x_1R_1^ty_1)$
(iii) if $x_1R_1^sy_1$ then $f(x_1)R_2^sf(y_1)$
(iv) if $x_2R_2^sy_2$ then $(\exists x_1, y_1 \in W_1)(x_2 = f(x_1), y_2 = f(y_1), x_1R_1^sy_1)$

(v) if $x_1 \prec_1 y_1$ then $f(x_1) \prec_2 f(y_1)$
(vi) if $x_2 \prec_2 y_2$ then $(\exists x_1, y_1 \in W_1)(x_2 = f(x_1), y_2 = f(y_1), x_1 \prec_1 y_1)$
(vii) Let $\underline{W}_i^0$ be the restriction of the system $\underline{W}_i$ to the set $W_i^0$, $i = 1, 2$. Then $f$ is an isomorphism from $\underline{W}_1^0$ onto $\underline{W}_2^0$. In particular we have $f(now_1)=now_2$

The system $\underline{W}_2$ is called a p-morphic image of $\underline{W}_1$ and $\underline{W}_1$ is called a p-morphic preimage of $\underline{W}_2$.

Let $f$ be a p-morphism from $\underline{W_1}$ to $\underline{W_2}$. Define $g : W_2 \to 2^{W_1}$ as follows: $g(x) = \{y | y \in W_1 \text{ and } f(y) = x\}$. Since $f$ is a surjection then $g$ is a total function such that $g(x) \neq \emptyset$ for all $x$. Let $\underline{B_2}$ be the basic DCA over $\underline{W_2}$ and $\underline{B_1}$ be the basic DCA over $\underline{W_1}$. Define $h_f : B_2 \to B_1$ in the following way: $h_f(a) = f^{-1}(a) = \bigcup_{x \in a} g(x)$. Then the following holds:

**Lemma 3.12.** Let $a, b \in B_2$. Then:

(i) $a \subseteq b$ iff $h(a) \subseteq h(b)$
(ii) $aC_2^s b$ iff $h(a)C_1^s h(b)$
(iii) $aC_2^t b$ iff $h(a)C_1^t h(b)$
(iv) $a\mathcal{B}_2 b$ iff $h(a)\mathcal{B}_1 h(b)$

*Proof.* *(i)* The forward direction is obvious since $h$ is union of $g(x)$, $x \in a \subseteq b$. For the backward direction reason by contraposition. Let $a \not\subseteq b$ so $\exists x \in a, x \notin b$. Since $g(x) \neq \emptyset$ pick $y \in g(x) \subseteq h(a)$. Suppose that $y \in h(b)$. Then $y \in g(z)$ for some $z \in b$. By the definition of $g$, $z = f(y) = x$. So $x \in b$ - contradiction.
*(ii)* ($\Rightarrow$) Let $a, b \in B_2$ be such that $aC_2^s b$. By definition, $\exists x_2 \in a, \exists y_2 \in b$ such that $x_2 R_2^s y_2$. Since $f$ is a p-morphism between $\underline{W_1}$ and $\underline{W_2}$ we have that $\exists x_1, y_1 \in W_1$ such that $f(x_1) = x_2$, $f(y_1) = y_2$ and $x_1 R_1^s y_1$. Since $x_2 \in a$ we have that $g(x_2) \subseteq h(a)$. From here and the fact that $f(x_1) = x_2$ we get that $x_1 \in h(a)$. Similarly $y_1 \in h(b)$. Since $x_1 R_1^s y_1$ we conclude that $h(a)C_1^s h(b)$.
($\Leftarrow$) Let $h(a)C_1^s h(b)$. Then $\exists x_1 \in h(a), \exists y_1 \in h(b)$ such that $x_1 R_1^s y_1$. By the p-morphism definition we get that $f(x_1)R_2^s f(y_1)$. Since $x_1 \in h(a)$ then $x_1 \in g(x_2)$ for some $x_2 \in a$ and hence $f(x_1) = x_2$. Similarly $f(y_1) = y_2$ for some $y_2 \in b$. Therefore $aC_2^s b$. The rest follows in a similar way. □

**Lemma 3.13** (P-morphism Lemma)**.** Let $\underline{W_i} = (W_i, W_i^0, R_i^t, R_i^s, \prec_i, \mathbf{now}_i)$, $i = 1, 2$ be two basic dynamic relational spaces and let $f : W_1 \to W_2$ be a p-morphism from $W_1$ onto $W_2$. Let $\underline{B}(W_i)$ be the basic DCA over the space $\underline{W_i}$, $i = 1, 2$. Then $h_f$ is an isomorphic embedding of $\underline{B}(W_2)$ into $\underline{B}(W_1)$.

*Proof.* To show that $h_f$ preserves Boolean and precontact relations we can reason as in Lemma 3.12. In order to assert that $h_f$ preserves TR and UTR sets, $NOW$ and the function $\mathcal{U}tr$ we just have to use condition (vii) from the p-morphism definition. □

In Section 3.5.1 we will show that each basic dynamic relational space is a p-morphic image of a basic dynamic relational space with $R^t$ being equivalence relation on the whole set $W$. In Section 3.5.2 we will show that each basic dynamic relational space with $R^t$ an equivalence relation is a p-morphic image of a dynamic relational space. This will imply what we need, namely that every basic DCA over a basic dynamic relational space can be embedded into a strong DCA.

### 3.5.1 The first p-morphism

**Lemma 3.14.** Let $\underline{W_1} = (W_1, W_1^0, R_1^t, R_1^s, \prec_1, now_1)$ be a basic dynamic relational space. Then there exist a basic dynamic relational space $\underline{W_2} = (W_2, W_2^0, R_2^t, R_2^s, \prec_2, now_2)$ with $R_2^t$ being an equivalence relation and a p-morphism $f_1$ from $\underline{W_2}$ onto $\underline{W_1}$.

*Proof.* Let $W_2^0 = W_1^0$, $W_2 = W_1^0 \cup \{(x, \alpha) : x \in \alpha \text{ and } \alpha = \{u, v\}, uR^t v, \alpha \cap W_1^0 = \varnothing\}$. Define $R_2^t$ in $W_2$ by cases as follows:

1. $x, y \in W_1^0$: $xR_2^t y$ iff $xR_1^t y$
2. $(x, \alpha)R_2^t(y, \beta)$ iff $\alpha = \beta$
3. $x \in W_1^0$: $x\overline{R}_2^t(y, \beta)$, $(y, \beta)\overline{R}_2^t x$

Definition of $R_2^s$:

1. $x, y \in W_1^0$: $xR_2^s y$ iff $xR_1^s y$
2. $(x, \alpha)R_2^s(y, \beta)$ iff $xR_1^s y$ and $\alpha = \beta$
3. $x \in W_1^0$: $x\overline{R}_2^s(y, \beta)$, $(y, \beta)\overline{R}_2^s x$

Definition of $\prec_2$:

1. $x, y \in W_1^0$: $x \prec_2 y$ iff $x \prec_1 y$
2. $(x, \alpha) \prec_2 (y, \beta)$ iff $x \prec_1 y$
3. $x \in W_1^0$: $x \prec_2 (y, \beta)$ iff $x \prec_1 y$, $(y, \beta) \prec_1 x$ iff $y \prec_1 x$

$now_2 =_{def} now_1$.

The first p-morphism, denoted by $f_1$, is defined as follows:

$$f_1(x) = x, \text{ for } x \in W_2^0$$
$$f_1((x, \alpha)) = x, \text{ for } (x, \alpha) \in W_2 \setminus W_2^0$$

Verifying the six conditions from the basic dynamic relational space definition (see Def. 3.8) is straightforward and confirms that $\underline{W_2}$ is indeed a basic dynamic relational space. $R_2^t$ is, obviously, an equivalence relation by the given definition. Let's check that $f_1$ is indeed a p-morphism, by following Definition 3.11:

(i) Suppose $aR_2^t b$ - we want to show that $f_1(a)R_1^t f_1(b)$. In case, $a, b \in W_2^0$ we have this by the definition of $R_2^t$. If $a, b \in W_2 \setminus W_2^0$, then $a = (x, \alpha)$ and $b = (y, \beta)$ - we want to show that $xR_1^t y$. Since $aR^t b$ we have $\alpha = \beta$ and by the definition of $W_2$ we have that $x \in \alpha$, $y \in \beta$ and $xR^t y$.

(ii) Suppose $xR_1^ty$ - we want to show that there are elements $a, b \in W_2$ such that $f_1(a) = x$, $f_1(b) = y$ and $aR_2^tb$. Consider the following three cases:

**Case 1:** $x, y \in W_1^0$. Take $a = x$, $b = y$.

**Case 2:** $x, y \notin W_1^0$. Take $\alpha = \{x, y\}$, $a = (x, \alpha)$, $b = (y, \alpha)$.

**Case 3:** $x \in W_1^0$ and $y \notin W_1^0$, or $x \notin W_1^0$ and $y \in W_1^0$. This case is impossible because if $xR_1^ty$ and $x \in W_1^0$, then $y \in W_1^0$ (by the definition of basic dynamic relation space) and similarly for the second case.

(iii) This is obvious by the given definition of $R_2^s$.

(iv) In the same way as (ii) using the fact that $R_1^s$ is included in $R_1^t$.

(v) Follows directly from the given definition of $\prec_2$.

(vi) Suppose $x \prec_1 y$ and consider the three cases for $x$ and $y$ as in (ii). Case 1 is obvious. For case 2 take $\alpha = \{x\}$ and $\beta = \{y\}$ (by reflexivity we have $xR_1^tx$, so $(x, \alpha)$ is correctly defined and similarly for $(y, \beta)$). Then obviously $(x, \alpha) \prec_2 (y, \beta)$ and $f_1((x, \alpha)) = x$ and $f_1((y, \beta)) = y$. We reason in a similar way for case 3.

(vii) This is obvious because $f_1$ acts as the identity function on $W_2^0$ and $W_2^0 = W_1^0$ and $now_2 = now_1$ by definition. □

### 3.5.2 The second p-morphism

**Lemma 3.15.** Let $\underline{W_1} = (W_1, W_1^0, R_1^t, R_1^s, \prec_1, now_1)$ be a basic dynamic relational space such that $R_1^t$ is an equivalence relation. Then there exist a dynamic relational space $\underline{W_2} = (W_2, W_2^0, R_2^t, R_2^s, \prec_2, now_2)$ and a p-morphism $f_2$ from $\underline{W_2}$ onto $\underline{W_1}$.

*Proof.* Let $W_2^0 = W_1^0$ and $W_2 = W_1^0 \cup \{(x, i) : x \notin W_1^0 \text{ and } i \in \{1, 2\}\}$.
Define $R_2^t$ in $W_2$ by cases as follows:

1. $x, y \in W_1^0$: $xR_2^ty$ iff $xR_1^ty$
2. $(x, i)R_2^t(y, j)$ iff $xR_1^ty$ and ($i = j = 1$ or $i = j = 2$ and $x = y$)
3. $x \in W_1^0$: $x\overline{R}_2^t(y, j)$, $(y, j)\overline{R}_2^tx$

Definition of $R_2^s$:

1. $x, y \in W_1^0$: $xR_2^sy$ iff $xR_1^sy$
2. $(x, i)R_2^s(y, j)$ iff $xR_1^sy$ and ($i = j = 1$ or $i = j = 2$ and $x = y$)
3. $x \in W_1^0$: $x\overline{R}_2^s(y, j)$, $(y, j)\overline{R}_2^sx$

Definition of $\prec_2$:

1. $x, y \in W_1^0$: $x \prec_2 y$ iff $x \prec_1 y$
2. $(x, i) \prec_2 (y, j)$ iff $x \prec_1 y$ and $i = j = 2$
3. $x \in W_1^0$: $x \prec_2 (y, j)$ iff $x \prec_1 y$ and $j = 2$, $(y, j) \prec_1 x$ iff $y \prec_1 x$ and $j = 2$

Also define $now_2 =_{def} now_1$. The second p-morphism, denoted by $f_2$, is defined as follows:

$f_2(x) = x$, for $x \in W^0$
$f_2((x, i)) = x$, for $(x, i) \in W_2 \setminus W_2^0$

Verifying that $\underline{W_2}$ is a dynamic relational space is straightforward and should be fairly obvious from the way elements are structured in $W_2$. The verification of p-morphism conditions for $f_2$ can be done in the same way as for $f_1$. □

As a consequence of Lemma 3.14 and Lemma 3.15 we obtain the following corollary.

**Corollary 3.16.** Every basic relational dynamic space is a p-morphic image of a relational dynamic space.

*Proof.* Let $\underline{W_1}$ be a basic dynamic relational space. By Lemma 3.14 there exists a basic dynamic relational space $\underline{W_2}$ in which the relation $R^t_2$ is an equivalence relation and a p-morphism $f_1$ from $\underline{W_2}$ onto $\underline{W_1}$. By Lemma 3.15 there exist a dynamic relational space $\underline{W_3}$ and a p-morphism $f_2$ from $\underline{W_3}$ onto $\underline{W_2}$. Then the composition $f = f_2 \circ f_1$ of the two p-morphisms is a p-morphism from $\underline{W_3}$ onto $\underline{W_1}$. □

**Lemma 3.17.** Let $\underline{W}$ be a basic dynamic relational structure and let $\underline{B}(W)$ be the basic DCA over $\underline{W}$. Then there exists a strong DCA $\underline{B}$ and an isomorphic embedding of $\underline{B}(W)$ into $\underline{B}$.

*Proof.* Let $\underline{W}$ be a basic dynamic relational space and let $\underline{B}(W)$ be the basic DCA over $\underline{W}$. By Corollary 3.16 there exists a dynamic relational space $\underline{W}'$ and a p-morphism $f$ from $\underline{W}'$ onto $\underline{W}$. Let $\underline{B}(W')$ be the strong DCA over $\underline{W}'$. Then the mapping $h_f$ (see Lemma 3.13) is an embedding from the basic DCA $\underline{B}(W)$ into the strong DCA $\underline{B}(W')$. □

**Definition 3.18.** Let $\underline{W_1}$ and $\underline{W_2}$ be basic dynamic relational spaces, $f$ be a p-morphism from $\underline{W_1}$ onto $\underline{W_2}$ and $A$ be a time condition from the list $(LS)_W$, $(RS)_W$, $(UpDir)_W$, $(DownDir)_W$, $(Dens)_W$, $(Ref)_W$, $(Irr)_W$, $(Lin)_W$, $(Tri)_W$, $(Tr)_W$ (see Section 2.1). We say that $f$ preserves $A$ if the following holds: $\underline{W_2}$ satisfies $A$ whenever $\underline{W_1}$ satisfies $A$.

Most of the time conditions state that every element of the structure has some kind of a preceding or a succeeding element with respect to the $\prec$ relation. Unfortunately, the second p-morphism makes the elements of the first copy of $W \setminus W_0$ have no $\prec$-related element, thus not preserving those time conditions. The lemma below lists the conditions that are actually preserved through the two p-morphisms. The verification of this lemma is trivial.

**Lemma 3.19.** The first and second p-morphisms from Lemma 3.14 and Lemma 3.15 preserve time conditions $(Dens)_W$, $(Irr)_W$ and $(Tr)_W$.

**Corollary 3.20.** Let $\underline{W}$ be a basic dynamic relational space satisfying some (or all) of the time conditions $(Dens)_W$, $(Irr)_W$ and $(Tr)_W$ and let $\underline{B}(W)$ be the basic DCA over $\underline{W}$. Then there exists a strong DCA $\underline{B}$ satisfying the corresponding time axioms and an isomorphic embedding of $\underline{B}(W)$ into $\underline{B}$.

*Proof.* The proof follows by a modification of the proof of Lemma 3.17 using Lemma 3.19 and Lemma 3.10. □

## 3.6 Relational representation theory for finite basic DCAs

In this section we will focus on proving a representation theorem for finite basic DCAs asserting that every finite basic DCA is isomorphic with a basic DCA over a finite basic dynamic relational space. We do not know, unfortunately, if such a representation theorem holds for arbitrary basic DCAs.

### 3.6.1 Canonical basic dynamic relational space over a finite basic DCA

Let $\underline{B}$ be a finite basic DCA. Since $\underline{B}$ is a finite Boolean algebra by Lemma 1.6 we have that it is atomic. Let $At(B)$ be the set of atoms of $\underline{B}$. Define a relational system $\underline{W}(B) = (W, W^0, R^t, R^s, \prec, now)$ associated with $\underline{B}$ as follows: $W = At(B)$, $W^0 = \{a \in At(B) : a \in TR(B)\}$, for $a, b \in At(B)$ define $aR^tb$ iff $aC^tb$, $aR^sb$ iff $aC^sb$ and $a \prec b$ iff $a\mathcal{B}b$. To define $now$ consider the region $NOW$. Since $NOW \neq \varnothing$ and $\underline{B}$ is atomic, then there is at laest one atom $a \in At(B)$ such that $a \leq NOW$ and and let $now$ be one of them. By axiom (TR$\leq$) $now \in TR(B)$ and hence $now \in W^0$.

**Lemma 3.21.** $\underline{W}(B) = (W, W^0, R^t, R^s, \prec, now)$ is a basic dynamic relational space.

*Proof.* The only nontrivial part of the proof is to verify that condition (iii) of Definition 3.8 holds, that is, if $a \in W^0$, $b \in W$ and $aR^tb$, then $b \in W^0$. From $a \in W^0$ we get $a \in TR(B)$. Let $c = \mathcal{U}tr(a)$ so $c \in UTR$ and $a \leq c$. By definition $aR^tb$ means $aC^tb$ and by $a \leq c$ we obtain $cC^tb$. By Lemma 2.3 (ii) we get $c.b \neq 0$. Then there exists an atom $d$ such that $d \leq (c.b)$. From here we get $d \leq b$ and since $d$ and $b$ are atoms, then $d = b$, hence $b \leq (c.b) \leq c$. But $c \in UTR$, so $c \in TR$ and $b \leq c$. Since $b$ is an atom, then $b \neq 0$ which together with $b \leq c$ imply (by axiom TR $\leq$) that $b \in TR(B)$, hence $b \in W^0$. □

The relational system $\underline{W}(B)$, as defined above, is called the *canonical basic dynamic relational space* over the finite basic DCA $\underline{B}$.

**Lemma 3.22.** Let $\underline{B}$ be a finite basic DCA and let $\underline{W}(B) = (W, W^0, R^t, R^s, \prec, now)$ be the canonical basic dynamic relational space over $\underline{B}$. Let $\alpha$ be any formula from the list of *time axioms* (*rs*), (*ls*), (*updir*), (*downdir*), (*dens*), (*ref*), (*irr*), (*lin*), (*tri*), (*tr*) and $A$ be its corresponding formula from the list of *time conditions* $(LS)_W$, $(RS)_W$, $(UpDir)_W$, $(DownDir)_W$, $(Dens)_W$, $(Ref)_W$, $(Irr)_W$, $(Lin)_W$, $(Tri)_W$, $(Tr)_W$. Then $A$ is true in $\underline{W}(B)$ iff $\alpha$ is true in $\underline{B}$.

*Proof.* By the canonical construction we have that $W$ is the set $At(B)$ of the atoms of $B$ and let $At(B) = \{a_1, \ldots, a_n\}$. We will illustrate the proof by considering the case $(Dens)_W \Leftrightarrow (dens)$. All other cases can be proved in a similar way working with atoms.

($\Rightarrow$) Suppose that $(Dens)_W$ is true. In order to prove (*dens*) suppose $a\mathcal{B}b$. We have to show that for all $p$ we have $a\mathcal{B}p$ or $p^*\mathcal{B}b$. Let us assume that $a = a_{i_1} + \ldots + a_{i_k}$ and $b = a_{j_1} + \ldots + a_{j_l}$ (since $\underline{B}$ is atomic). Then by the distribution axioms of precontact relation we obtain from $a\mathcal{B}b$ that $a_{i_s}\mathcal{B}a_{j_t}$ for some $s \leq k$ and $t \leq l$ (we have $a_{i_s} \leq a$ and $a_{j_t} \leq b$). This shows that $a_{i_s} \prec a_{j_t}$ in $\underline{W}(B)$. By $(Dens)_W$ there exists an atom $a_m$ such that $a_{i_s} \prec a_m \prec a_{j_t}$ i.e. $a_{i_s}\mathcal{B}a_m$ and $a_m\mathcal{B}a_{j_t}$. Let $p$ be an arbitrary element of $B$. There are two cases for the atom $a_m$: $a_m \leq p$ or $a_m \leq p^*$.

Case 1: $a_m \leq p$. Then from $a_{i_s} \leq a$, $a_{i_s}\mathcal{B}a_m$, by precontact axioms, we get $a\mathcal{B}p$.

Case 2: $a_m \leq p^*$. From this and $a_m\mathcal{B}a_{j_t}$, $a_{j_t} \leq b$ we obtain $p^*\mathcal{B}b$.

($\Leftarrow$) Suppose that $(dens)$ is true. Let $a_k$ and $a_l$ be two atoms and suppose $a_k \prec a_l$ (i.e. $a_k\mathcal{B}a_l$). We have to show that there exists an atom $a_m$ such that $a_k \prec a_m \prec a_l$ i.e. $a_k\mathcal{B}a_m$ and $a_m\mathcal{B}a_l$. Suppose the contrary, namely

($\sharp$) for all $a_m$: either $a_k\overline{\mathcal{B}}a_m$ or $a_m\overline{\mathcal{B}}a_l$.

Since $a_k \neq 0$ and $a_l \neq 0$, then by $(dens)$ we have that the following holds:

($\natural$) For all $p \in B$: either $a_k\mathcal{B}p$ or $p^*\mathcal{B}a_l$.

Let $P$ be the set of all atoms $a_m$ such that $a_k\overline{\mathcal{B}}a_m$ and let $p$ be their sum. Then by the distributivity axioms of precontact we get $a_k\overline{\mathcal{B}}p$ and by ($\natural$) we get that $p^*\mathcal{B}a_l$. Obviously $p^*$ will be the sum of all elements from the complement of $P$ for which we have: $a_k\mathcal{B}a_m$. But by ($\sharp$) we obtain that for these elements we have $a_m\overline{\mathcal{B}}a_l$ and for their sum $p^*$ that $p^*\overline{\mathcal{B}}a_l$ which contradicts $p^*\mathcal{B}a_l$. □

### 3.6.2 The isomorphism theorem for finite basic DCAs

**Lemma 3.23** (Relational representation lemma for finite basic DCAs)**.** Let $\underline{B}$ be a finite basic DCA and let $\underline{W}(B)$ be the canonical basic dynamic relational space over $\underline{B}$. Denote by $\underline{B}(W)$ the basic DCA over $\underline{W}(B)$. Then:

(i) $\underline{B}$ is isomorphic with $\underline{B}(W)$

(ii) If $\underline{B}$ satisfies some of the time axioms then $\underline{B}(W)$ satisfies the same axioms

*Proof.* (i) Because $\underline{B}$ is a finite Boolean algebra then there is a Boolean isomorphism $h$ of $B$ with the Boolean algebra of subsets of $\underline{W}(B)$ (remember that the elements of $\underline{W}(B)$ are the atoms of $\underline{B}$), namely $h(a) = \{c \in At(B) : c \leq a\}$. Let $h(a) = \{c_1, \cdots, c_k\}$. We have that $a = c_1 + \cdots + c_k$. Using this, it can be easily shown that $h$ preserves the relations $C^t, C^s, \mathcal{B}$, the sets $TR(B)$ and $UTR(B)$, and that $h(NOW(B)) = NOW(B(W))$. As an example, let's verify that $a \in UTR(B)$ iff $h(a) \in UTR(B(W))$.

($\Rightarrow$) Suppose that $a \in UTR(B)$ and let $a = c_1 + \cdots + c_k$ where $h(a) = \{c_1, \cdots, c_k\}$. Then we have $c_1 + \cdots + c_k \leq a$ and also $a \in TR$. By Lemma 3.2 (v) we have $c_iC^tc_j$ (hence $c_iR^tc_j$ ) for all $i, j \leq k$ and $c_i \in TR(B)$ for all $i \leq k$. We will show that the set $\{c_1, \cdots, c_k\}$ is an $R^t$-equivalence class. Suppose that $b \in W$ and that $c_1R^tb$. Then $c_1C^tb$. Since $c_1 \leq a$ we have $aC^tb$. Since $c_1 \in W^0$ and $c_1R^tb$ then $b \in W^0$, so $b \in TR(B)$. We will show that $b \leq a$. Suppose not, i.e. $b \not\leq a$. Then $a^*.b \neq 0$ and $a^*C^tb$. Then from $a^*C^tb$, $aC^tb$ and $b \in TR(B)$ we get that $aC^ta^*$ which contradicts $a \in UTR$. So we have that $b \leq a = c_1 + \cdots + c_k$. This implies that there exists $i \leq k$ such that $b = c_i$ (since $b$ is an atom) which completes the proof that $h(a)$ is an equivalence class with respect to $R^t$ and hence $h(a) \in UTR(B(W))$.

($\Leftarrow$) Suppose that $h(a) \in UTR(B(W))$ i.e. that (by definition) $h(a) = \{c_1, \cdots, c_k\}$ is an equivalence class with respect to $R^t$. First we show that $a \in TR$. We have that for all $i, j \leq k$ $c_i R^t c_j$. Then by Lemma 3.2 (iv) $a = c_1 + \cdots + c_k \in TR$. It remains to show that $a\overline{C}^t a^*$. Suppose for the sake of contradiction that $aC^t a^*$. So $a^* \neq 0$. Let $h(a^*) = \{d_1, \ldots, d_l\}$. Then $h(a) \cap h(a^*) = \varnothing$ ($h$ is a Boolean isomorphism) and consequently $d_j \notin \{c_1, \ldots, c_k\}$, $j \leq l$. However, $aC^t a^*$ implies that for some $i, j : i \leq k$ and $j \leq l$ we have that $c_i C^t d_j$, i.e. $c_i R^t d_j$ and since $h(a)$ is an equivalence class, then $d_j \in h(a)$ - a contradiction.

(ii) Let $\underline{B}$ satisfy some of the time axioms. Then by Lemma 3.22 the canonical space $\underline{W}(\underline{B})$ satisfies the corresponding time conditions. Applying Lemma 3.10 we get that $\underline{B}(W)$ satisfies the considered time axioms. □

**Theorem 3.24.** Every finite basic DCA $\underline{B}$ can be isomorphically embedded into a strong DCA $\widehat{\underline{B}}$. Furthermore, if $\underline{B}$ satisfies some (or all) of the axioms (*dens*), (*irr*) and (*tr*), then $\widehat{\underline{B}}$ can be chosen to satisfy the same axioms.

*Proof.* The theorem follows directly from Lemma 3.23 and Lemma 3.17. □

# 4 Quantifier-free logics for space and time

In this work we've considered several classes of DCAs - dynamic contact algebras (Def. 1.15), basic DCAs (Def. 3.1), weak DCAs (Def. 2.2) and strong DCAs (Def. 2.1). All these types of DCAs are based on the same first-order language except the language of basic DCA which contains the additional function $\mathcal{U}tr$. This function is, however, definable in the other kinds of DCAs so we may assume that all four types of DCAs are based on one and the same language.

In this section we'll present minimal quantifier-free logics for the four studied classes of DCAs. We'll denote the logics in the following way, $\mathbb{L}^{min}_{basic}$ for the logic of basic DCAs, $\mathbb{L}^{min}_{weak}$ for weak DCAs, $\mathbb{L}^{min}_{DCA}$ for DCAs and $\mathbb{L}^{min}_{strong}$ for strong DCAs. We assert the completeness of these logics in their respective classes of DCAs and use the completeness results along with the results from previous sections to conclude some interesting metalogical properties of the proposed systems. This section closely follows Section 3 from [2]. A lot of the statements will be similar to those in [2] and hence their proofs will be either shortly mentioned or skipped whatsoever as the proof ideas remain the same.

## 4.1 Language and notation

We consider a first-order language $\mathbb{L}$ without quantifiers containing the following symbols:

(i) a denumerable set $Var$ of *Boolean variables*
(ii) *constants* - 0, 1 and $NOW$
(iii) *functional symbols* - $+$, $\cdot$, $*$, $\mathcal{U}tr$
(iv) *predicate symbols* - $\leq$, $C^s$, $C^t$, $\mathcal{B}$, $TR$, $UTR$
(v) *connectives* - $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
(vi) *brackets* - ) and (

The notions of *term* and *formula* are standard:

**Definition 4.1** (Term)**.** The terms in our language are defined from Boolean variables and constants using functional symbols as follows:

(i) every Boolean variable $v \in Var$ is a term
(ii) the constants 0, 1 and $NOW$ are terms
(iii) let $a$ and $b$ be terms. Then $a + b$, $a \cdot b$, $a^*$ and $\mathcal{U}tr(a)$ are also terms.

**Definition 4.2** (Atomic formula)**.** The atomic formulae of our language are formulae of the following types: $a \leq b, aC^sb, aC^tb, a\mathcal{B}b$, $TR(a)$, $UTR(a)$ where $a$ and $b$ are terms.

**Definition 4.3** (Formula)**.** The formulae in the language are defined in the following way:

(i) atomic formulae are formulae

(ii) if $A$ is a formula then $\neg A$ is also a formula

(iii) if $A$ and $B$ are formulae then $A \Rightarrow B$, $A \Leftrightarrow B$ , $A \vee B$ and $A \wedge B$ are also formulae

We adopt the standard rules in first-order logic for omission of brackets. Additionally, we'll use the following abbreviations for convenience:

(i) $a = b \stackrel{\text{def}}{=} (a \leq b) \wedge (b \leq a)$

(ii) $a \neq b \stackrel{\text{def}}{=} \neg(a = b)$

(iii) $a \nleq b \stackrel{\text{def}}{=} \neg(a \leq b)$

(iv) $a\overline{C^s}b \stackrel{\text{def}}{=} \neg aC^s b$

(v) $a\overline{C^t}b \stackrel{\text{def}}{=} \neg aC^t b$

(vi) $a\overline{\mathcal{B}}b \stackrel{\text{def}}{=} \neg a\mathcal{B}b$

(vii) $\bot \stackrel{\text{def}}{=} (a \nleq a)$

## 4.2 Semantics

In this section we'll explore a couple of ways for interpreting the statements of our language into different semantic structures.

### 4.2.1 Algebraic semantics

First, we introduce algebraic semantics for the language $\mathbb{L}$. Let $\underline{B}$ be a DCA of one of the four types. We define a mapping (valuation) $v : Var \to B$ which is extended for terms in the following way:

$v(a + b) = v(a) + v(b)$

$v(a \cdot b) = v(a) \cdot v(b)$

$v(a^*) = v(a)^*$

$v(\mathcal{U}tr(a)) = \mathcal{U}tr(v(a))$

$v(0) = 0$

$v(1) = 1$

$v(NOW) = NOW(B)$

We'll call the pair $\mathcal{M} = (\underline{B}, v)$ an *algebraic model* (or simply a *model*). The truth of a formula $\alpha$ in $\mathcal{M} = (\underline{B}, v)$ is denoted by $v(\alpha) = 1$ or $\mathcal{M} \models \alpha$. Similarly, the falsehood of a formula will be denoted by $v(\alpha) = 0$ or $\mathcal{M} \not\models \alpha$. We'll use the following conditions to determine the truth of an atomic formulae of $\mathbb{L}$:

$v(a \leq b) = 1$ if and only if $v(a) \leq v(b)$

$v(aC^s b) = 1$ if and only if $v(a)C^s v(b)$

$v(aC^tb) = 1$ if and only if $v(a)C^tv(b)$

$v(a\mathcal{B}b) = 1$ if and only if $v(a)\mathcal{B}v(b)$

$v(TR(a)) = 1$ if and only if $v(a) \in TR$

$v(UTR(a)) = 1$ if and only if $v(a) \in UTR$

For complex formula, the definition is extended in the standard way:

$v(\neg\alpha) = 1$ if and only if $v(\alpha) = 0$

$v(\alpha \wedge \beta) = 1$ if and only if $v(\alpha) = 1$ and $v(\beta) = 1$

$v(\alpha \vee \beta) = 1$ if and only if $v(\alpha) = 1$ or $v(\beta) = 1$

$v(\alpha \Rightarrow \beta) = 1$ if and only if $v(\alpha) = 0$ or $v(\beta) = 1$

$v(\alpha \Leftrightarrow \beta) = 1$ if and only if $v(\alpha \Rightarrow \beta) = 1$ and $v(\beta \Rightarrow \alpha) = 1$

We say that $\mathcal{M}$ *is a model* of a formula $\alpha$ (or $\mathcal{M}$ *models* $\alpha$) if $\mathcal{M} \models \alpha$. We say that a formula $\alpha$ is true in a dynamic contact algebra $\underline{B}$ if for every structure $\mathcal{M} = (\underline{B}, v)$ we have that $\mathcal{M} \models \alpha$. If $\Sigma$ is a class of dynamic contact algebras we say that $\alpha$ *is true in* $\Sigma$ if it is true in all DCAs from $\Sigma$. By $\mathcal{L}(\Sigma)$ we'll denote the set of formulae which are true in $\Sigma$ and we will call this set the *logic* of $\Sigma$.

If $\Sigma$ is a class of DCAs, denote by $\Sigma^{fin}$ the set of finite members of $\Sigma$. We use the following notation for the different classes of DCAs: $\Sigma_{basic}$ for basic DCAs, $\Sigma_{weak}$ for weak DCAs, $\Sigma_{DCA}$ for DCAs and $\Sigma_{strong}$ for strong DCAs. We have the following inclusions: $\Sigma_{basic} \supseteq \Sigma_{weak} \supseteq \Sigma_{DCA} \supseteq \Sigma_{strong}$. Let $\Theta$ be a set of time axioms - we denote by $\Sigma^{\Theta}$ the class of all members of $\Sigma$ satisfying the axioms of $\Theta$. We have also the following inclusions: $\Sigma^{\Theta}_{basic} \supseteq \Sigma^{\Theta}_{weak} \supseteq \Sigma^{\Theta}_{DCA} \supseteq \Sigma^{\Theta}_{strong}$. The following lemma is obvious:

**Lemma 4.4.** Let $\Sigma_1$ and $\Sigma_2$ be two classes of dynamic contact algebras and $\Sigma_1 \subseteq \Sigma_2$. Then $\mathcal{L}(\Sigma_2) \subseteq \mathcal{L}(\Sigma_1)$.

**Proposition 4.5.** Let $\Theta$ be a set of time axioms which are universal sentences, i.e. $\Theta$ does not contain *irr* and *tr*. Then $\mathcal{L}(\Sigma^{\Theta}_{basic}) = \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$.

*Proof.* ($\subseteq$) Since $\Sigma^{fin,\Theta}_{basic} \subseteq \Sigma^{\Theta}_{basic}$ by Lemma 4.4 we get $\mathcal{L}(\Sigma^{\Theta}_{basic}) \subseteq \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$.
($\supseteq$) Suppose, towards contradiction that $\mathcal{L}(\Sigma^{fin,\Theta}_{basic}) \not\subseteq \mathcal{L}(\Sigma^{\Theta}_{basic})$. There there is a formula $A \in \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$ such that $A \notin \mathcal{L}(\Sigma^{\Theta}_{basic})$. This means that there is a basic DCA $\underline{B}$ and a valuation $v$ such that $(\underline{B}, v) \not\models A$. Let $a_1 \dots a_n$ be the Boolean variables which occur in the formula $A$. Take a look at the set $C = \{v(a_1), v(a_2) \dots v(a_n), NOW(B)\}$. By Corollary 3.7 there is a finite subalgebra $\underline{B_0}$ of $\underline{B}$ containing $C$ and satisfying the axioms from $\Theta$, i.e. $\underline{B_0} \in \Sigma^{fin,\Theta}_{basic}$. Let $v'$ be some modification of $v$ on the set $B_0$ which preserves the values for the variables $a_1 \dots a_n$. Then obviously $(\underline{B_0}, v') \not\models A$ which is a contradiction with $A \in \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$. □

**Proposition 4.6.** $\mathcal{L}(\Sigma^{fin}_{basic}) = \mathcal{L}(\Sigma_{strong})$

*Proof.* ($\subseteq$) By Lemma 4.4 we have that $\mathcal{L}(\Sigma_{basic}) \subseteq \mathcal{L}(\Sigma_{strong})$ and by Proposition 4.5 we have $\mathcal{L}(\Sigma^{fin}_{basic}) \subseteq \mathcal{L}(\Sigma_{strong})$.
($\supseteq$) Towards contradiction, suppose that the converse inclusion does not hold. Then there is a formula $A \in \mathcal{L}(\Sigma_{strong})$ and an algebra $\underline{B} \in \Sigma^{fin}_{basic}$ and a model $(\underline{B}, v)$ such that $(\underline{B}, v) \not\models A$. By Theorem 3.24 there is a strong DCA $\widehat{\underline{B}}$ and an isomorphic embedding $h$ of $\underline{B}$ into $\widehat{\underline{B}}$. Let $v' = h \circ v$ be the composition of $h$ and $v$. Then obviously $(\widehat{B}, v') \not\models A$ contrary to the fact that $A \in \mathcal{L}(\Sigma_{strong})$. □

**Proposition 4.7.** $\mathcal{L}(\Sigma_{basic}) = \mathcal{L}(\Sigma_{strong})$

*Proof.* Follows from Proposition 4.5 and Proposition 4.6. □

**Theorem 4.8.** The logics $\mathcal{L}(\Sigma_{basic})$, $\mathcal{L}(\Sigma_{weak})$, $\mathcal{L}(\Sigma_{DCA})$ and $\mathcal{L}(\Sigma_{strong})$ are equal.

*Proof.* By Proposition 4.4 and Proposition 4.7 we have $\mathcal{L}(\Sigma_{basic}) \subseteq \mathcal{L}(\Sigma_{weak}) \subseteq \mathcal{L}(\Sigma_{DCA}) \subseteq \mathcal{L}(\Sigma_{strong}) = \mathcal{L}(\Sigma_{basic})$, which implies the required equality. □

**Theorem 4.9.** Let $\Theta$ be a set of time axioms. Then the logics $\mathcal{L}(\Sigma^{\Theta}_{weak})$, $\mathcal{L}(\Sigma^{\Theta}_{DCA})$, $\mathcal{L}(\Sigma^{\Theta}_{strong})$ are equal.

*Proof.* We have $\Sigma^{\Theta}_{strong} \subseteq \Sigma^{\Theta}_{DCA} \subseteq \Sigma^{\Theta}_{weak}$. By Lemma 4.4 we obtain $\mathcal{L}(\Sigma^{\Theta}_{weak}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{DCA}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{strong})$. By Theorem 2.19 we obtain $\mathcal{L}(\Sigma^{\Theta}_{strong}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{weak})$ which combined with the previous inclusions implies the equality of the three logics. □

A stronger form of Proposition 4.7 is the following.

**Proposition 4.10.** Let $\Theta$ be a set consisting of some (or of all) of the time axioms (*dens*), (*irr*) and (*tr*). Then $\mathcal{L}(\Sigma^{\Theta}_{basic}) = \mathcal{L}(\Sigma^{\Theta}_{strong})$.

*Proof.* The proof follows from Lemma 4.4, Theorem 3.24 and Proposition 4.5. □

**Corollary 4.11.** Let $\Theta$ be a set consisting of some (or all) of the time axioms (*dens*), (*irr*) and (*tr*). Then the logics $\mathcal{L}(\Sigma^{\Theta}_{basic})$, $\mathcal{L}(\Sigma^{\Theta}_{weak})$, $\mathcal{L}(\Sigma^{\Theta}_{DCA})$, $\mathcal{L}(\Sigma^{\Theta}_{strong})$ are equal.

### 4.2.2 Relational semantics

Let $\Delta_{basic}$ be the class of all basic dynamic relational spaces and $\Delta_{rel}$ be the class of all dynamic relational spaces. Note that $\Delta_{rel} \subseteq \Delta_{basic}$. Relational (Kripke style) semantics for $\mathbb{L}$ can be defined as follows. Let $\underline{W}$ be a basic dynamic relational space and $\underline{B}(W)$ be the DCA over $\underline{W}$. Let $v$ be a function associating to each variable $a$ a subset $v(a) \subseteq W$. The pair $(\underline{W}, v)$ is called a *relational model* or *Kripke model.* We say that a formula $A$ is true in the relational model $(\underline{W}, v)$ if it is true in the algebraic model $(\underline{B}(W), v)$, and similarly for the notions "true in a space $\underline{W}$" and "true in a class of spaces". Let $\Delta$ be a class of basic dynamic relational spaces and denote by $\mathcal{L}(\Delta)$ the set of all formulas true in $\Delta$ - call this set the logic of $\Delta$. If $\Delta$ is a class of basic dynamic relational spaces we denote by $\Sigma(\Delta)$ the class of all DCAs over the members of $\Delta$. Obviously, we have $\mathcal{L}(\Delta) = \mathcal{L}(\Sigma(\Delta))$. Thus, all notions related to Kripke semantics can be reduced to corresponding notions related to algebraic semantics. It is easy to see that interesting statements concerning logics of some classes of algebraic models can be easily transformed into statements about logics of some classes of dynamic relational spaces. Because of this, further in this paper we'll be focusing on algebraic semantics.

## 4.3 Axiomatization

In this section we'll take a look at the axiomatizations of the minimal logics for the four DCA classes and their extensions with time axioms and rules.

### 4.3.1 Axiomatization of the minimal logic for basic DCAs

The axiomatic system for $\mathbb{L}^{min}_{basic}$ will be based on Modus Ponens. We'll take as axioms the complete set of axioms for classical propositional logic, all first-order axioms for Boolean algebra and all axioms for basic DCAs plus an additonal rule to handle the $\mathcal{U}tr$ operation. Note that all of these are universal statements. A more detailed list is given below.

**Axioms.**

(i) the complete set of axiom schemes for classical propositional logic

(ii) the full set of axioms of Boolean algebra, e.g. $a \leq a$ (poset), $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (distributive lattice), $a + a^* = 1$ (boolean algebra) etc.

(iii) axioms for $C^s$ and $C^t$

$(C^s1)$ $aC^sb \Rightarrow (a \neq 0) \wedge (b \neq 0)$

$(C^s2)$ $(aC^sb \wedge a \leq a' \wedge b \leq b') \Rightarrow a'C^sb'$

$(C^s3)$ $aC^s(b + c) \Rightarrow aC^sb \vee aC^sc$

$(C^s3')$ $(a + b)C^sc \Rightarrow aC^sc \vee bC^sc$

$(C^s4)$ $aC^sb \Rightarrow bC^sa$

$(C^s5)$ $a \cdot b \neq 0 \Rightarrow aC^sb$

$(C^s5')$ $a \neq 0 \Rightarrow aC^sa$

$(C^t1)$ $aC^tb \Rightarrow (a \neq 0) \wedge (b \neq 0)$

$(C^t2)$ $(aC^tb \wedge a \leq a' \wedge b \leq b') \Rightarrow a'C^tb'$

$(C^t3)$ $aC^t(b + c) \Rightarrow aC^tb \vee aC^tc$

$(C^t3')$ $(a + b)C^tc \Rightarrow aC^tc \vee bC^tc$

$(C^t4)$ $aC^tb \Rightarrow bC^ta$

$(C^t5)$ $a \cdot b \neq 0 \Rightarrow aC^tb$

$(C^t5')$ $a \neq 0 \Rightarrow aC^ta$

$(C^s \Rightarrow C^t)$ $aC^sb \Rightarrow aC^tb$

(iv) axioms for $\mathcal{B}$

$(\mathcal{B}1)$ $a\mathcal{B}b \Rightarrow (a \neq 0) \wedge (b \neq 0)$

$(\mathcal{B}2)$ $(a\mathcal{B}b \wedge a \leq a' \wedge b \leq b') \Rightarrow a'\mathcal{B}b'$

$(\mathcal{B}3)$ $a\mathcal{B}(b + c) \Rightarrow a\mathcal{B}b \vee a\mathcal{B}c$

$(\mathcal{B}3')$ $(a + b)\mathcal{B}c \Rightarrow a\mathcal{B}c \vee b\mathcal{B}c$

(v) axioms for $TR$ and $UTR$

(TR1) $TR(c) \Rightarrow c \neq 0 \wedge (aC^t c \wedge bC^t c \Rightarrow aC^t b)$

(TR2) $UTR(c) \Leftrightarrow TR(c) \wedge c\overline{C}^t c*$

(TR$\mathcal{B}$1) $TR(c) \wedge c\mathcal{B}b \wedge aC^t c \Rightarrow a\mathcal{B}b$

(TR$\mathcal{B}$2) $TR(d) \wedge a\mathcal{B}d \wedge bC^t d \Rightarrow a\mathcal{B}b$

(TR$\leq$) $TR(c) \wedge d \leq c \wedge d \neq 0 \Rightarrow TR(d)$

(TR$\cup$) $TR(c) \wedge TR(d) \wedge cC^t d \Rightarrow TR((c+d))$

(UTRNOW) $UTR(NOW)$

(vi) axioms for $\mathcal{U}tr$

($TR\mathcal{U}tr1$) $TR(c) \Rightarrow UTR(\mathcal{U}tr(c)) \wedge c \leq \mathcal{U}tr(c)$

($TR\mathcal{U}tr2$) $\neg TR(c) \Rightarrow \mathcal{U}tr(c) = 0$

($\mathcal{U}tr$-Replacement) $a = b \Rightarrow \mathcal{U}tr(a) = \mathcal{U}tr(b)$

**Rules of inference.**

The only rule of inference of the minimal logic for basic DCA will be Modus Ponens:

$$\frac{A, (A \Rightarrow B)}{B} \ (MP)$$

### 4.3.2 Non-standard rules of inference

The minimal logic $\mathbb{L}^{min}_{weak}$ for weak DCAs can be obtained as an extension of the logic $\mathbb{L}^{min}_{basic}$. Inspecting the definitions of basic DCAs and weak DCAs notice that the additional axioms for weak DCAs are all non-universal statements. Also note, that all of these statements can be tranformed in the following special form:

(✠) $(\forall b_1, \ldots, b_m)A(a_1, \ldots, a_n, b_1, \ldots, b_m) \Rightarrow B(a_1, \ldots, a_n)$

where $a_1, \ldots, a_n$ are terms, $b_1, \ldots, b_m$ are Boolean variables which are not included in the formula $B(a_1, \ldots, a_n)$, and the terms $a_1, \ldots, a_n$. Also, the notation $A(a_1, ..., a_n, b_1, ..., b_m)$ means that $a_1, \ldots, a_n, b_1, \ldots, b_m$ are the only terms included in $A$ (respectively the same for $B(a_1, \ldots, a_n)$). We transform a formula of type (✠) into the following quantifier-free rule of inference

$$\frac{C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)}{C \Rightarrow B(a_1, \ldots, a_n)} \ (✠R)$$

which is subject to the following constraints: $a_1, \ldots, a_n$ are terms, $b_1, \ldots, b_m$ are Boolean variables which are not included in the formulas $C$, $B(a_1, \ldots, a_n)$, and consequently in the terms $a_1, \ldots, a_n$. The formula $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is called the *premise* of the rule and the formula $C \Rightarrow B(a_1, \ldots, a_n)$ is called the *conclusion* of the rule. We'll call rules of type (✠$R$) *non-standard rules of inference.* Non-standard rules of inference are studied in [1] and [2].

As already mentioned, all non-universal axioms for weak DCAs can be tranformed into formulae of type ✠. As an example, we will do this for the forward direction of $(UTR\mathcal{B}11)$ (see Definition 1.15). The forward direction of $(UTR\mathcal{B}11)$ can be written as the following first-order formula:
(1) $(UTR(c) \wedge UTR(d) \wedge (\forall p)(p\mathcal{B}c \vee p^*\mathcal{B}d)) \Rightarrow ((\exists e)(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d))$
Rewritting the implications and moving the negations through the brackets we obtain:
(2) $\neg(UTR(c) \wedge UTR(d)) \vee (\exists p)\neg(p\mathcal{B}c \vee p^*\mathcal{B}d) \vee (\exists e)(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d)$
which is the same as
(3) $(\exists p)\neg(p\mathcal{B}c \vee p^*\mathcal{B}d) \vee (\exists e)(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d) \vee \neg(UTR(c) \wedge UTR(d))$
This can be transformed into the following implication:
(4) $(\forall p)(p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge (\forall e)\neg(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d) \Rightarrow \neg(UTR(c) \wedge UTR(d))$
Finally, we can move the quantifier $(\forall e)$ trough $(p\mathcal{B}c \vee p^*\mathcal{B}d)$ and obtain a formula which is in the shape of (✠):
(5) $(\forall p)(\forall e)((p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge \neg(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d)) \Rightarrow \neg(UTR(c) \wedge UTR(d))$.
The corresponding rule is the following:
$\dfrac{C \Rightarrow (p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge \neg(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d)}{C \Rightarrow \neg(UTR(c) \wedge UTR(d))}\ (UTR\mathcal{B}11R)$, where $p$ and $e$ are variables not occurring in the terms $c$, $d$ and the formula $C$.

Thus, the axiomatization of the minal logic $\mathbb{L}^{min}_{weak}$ for weak DCAs can be obtained by adding the following non-standard rules of inference to $\mathbb{L}^{min}_{basic}$:

**Rules for *TR*.**

$$\frac{C \Rightarrow \neg UTR(p) \vee a\overline{C^t}p \vee b\overline{C^t}p}{C \Rightarrow a\overline{C^t}b}\ (TRC^tR)$$

$$\frac{C \Rightarrow \neg UTR(p) \vee (a \cdot p)\overline{C^s}b}{C \Rightarrow a\overline{C^s}b}\ (TRC^sR)$$

$$\frac{C \Rightarrow \neg UTR(p) \vee p\overline{\mathcal{B}}b \vee a\overline{C^t}p}{C \Rightarrow a\overline{\mathcal{B}}b}\ (TR\mathcal{B}3R)$$

$$\frac{C \Rightarrow \neg UTR(p) \vee a\overline{\mathcal{B}}p \vee b\overline{C^t}p}{C \Rightarrow a\overline{\mathcal{B}}b}\ (TR\mathcal{B}4R)$$

The $TR$ rules above have the following constraint - $p$ is a Boolean variable which does not occur in $C, a$ and $b$.

From DCA axiom $(TR1)$ we get the following rule, where $p$ and $q$ are Boolean variables that do not occur in $C$ and $c$:

$$\frac{C \Rightarrow c \neq 0 \wedge (pC^tc \wedge qC^tc \Rightarrow pC^tq)}{C \Rightarrow TR(c)}\ (TR1R)$$

**Rules for *UTR*.**

$$\frac{C \Rightarrow (p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge \neg(UTR(e) \wedge e\mathcal{B}c \wedge e\mathcal{B}d)}{C \Rightarrow \neg(UTR(c) \wedge UTR(d))} \; (UTR\mathcal{B}11R)$$

$$\frac{C \Rightarrow (p\mathcal{B}c \vee d\mathcal{B}p^*) \wedge \neg(UTR(e) \wedge e\mathcal{B}c \wedge d\mathcal{B}e)}{C \Rightarrow \neg(UTR(c) \wedge UTR(d))} \; (UTR\mathcal{B}12R)$$

$$\frac{C \Rightarrow (c\mathcal{B}p \vee p^*\mathcal{B}d) \wedge \neg(UTR(e) \wedge c\mathcal{B}e \wedge e\mathcal{B}d)}{C \Rightarrow \neg(UTR(c) \wedge UTR(d))} \; (UTR\mathcal{B}21R)$$

$$\frac{C \Rightarrow (c\mathcal{B}p \vee d\mathcal{B}p^*) \wedge \neg(UTR(e) \wedge c\mathcal{B}e \wedge d\mathcal{B}e)}{C \Rightarrow \neg(UTR(c) \wedge UTR(d))} \; (UTR\mathcal{B}22R)$$

In all of the $UTR$ rules above $p$ and $e$ are Boolean variables that do not occur in $C$, $a$ and $b$.

Similarly, the minimal logic $\mathbb{L}^{min}_{DCA}$ for DCAs can be obtained as an extension of $\mathbb{L}^{min}_{weak}$ with the following non-standard rule of inference which corresponds to the Efremovich axiom:

$$\frac{C \Rightarrow aC^t p \vee p^* C^t b}{C \Rightarrow aC^t b} \; (C^t ER)$$, where $p$ does not occur in $C, a$ and $b$

Finally, the minimal logic $\mathbb{L}^{min}_{strong}$ for the class of strong DCA can be obtained as an extension of $\mathbb{L}^{min}_{DCA}$ with the following non-standard rules of inference, corresponding to conditions $(C^t\mathcal{B})$ and $(\mathcal{B}C^t)$ (see Def. 2.1) respectively:

$$\frac{C \Rightarrow aC^t p \vee p^*\mathcal{B}b}{C \Rightarrow a\mathcal{B}b} \; (C^t\mathcal{B}R)$$, where $p$ does not occur in $C, a$ and $b$

$$\frac{C \Rightarrow a\mathcal{B}p \vee p^* C^t b}{C \Rightarrow a\mathcal{B}b} \; (\mathcal{B}C^t R)$$, where $p$ does not occur in $C, a$ and $b$

### 4.3.3 Extensions with time axioms and rules

In the context of logics *time axioms* will be called the conditions $(rs)$, $(ls)$, $(updir)$, $(downdir)$, $(dens)$, $(ref)$, $(lin)$, $(tri)$ from Section 1.3.2 but written in the language $\mathbb{L}$. The remaining two conditions $(irr)$ and $(tr)$ are non-universal statements:

$(irr)$ $a\mathcal{B}b \to (\exists c, d)(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)$,

$(tr)$ $a\overline{\mathcal{B}}b \to (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

These two formulas can easily be transformed in the form of (✠):

$(irr')$ $(\forall c, d)\neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d) \Rightarrow a\overline{\mathcal{B}}b$,

$(tr')$ $(\forall c)(a\mathcal{B}c \vee c^*\mathcal{B}b) \Rightarrow a\mathcal{B}b$.

Thus, we obtain the following two non-standard rules of inference corresponding to $(irr)$ and $(tr)$ respectively:

$$\frac{C \Rightarrow \neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)}{C \Rightarrow a\overline{\mathcal{B}}b} \; (irrR)$$, where $c, d$ are variables not

occurring in the terms $a, b$ and the formula $C$.

$\frac{C \Rightarrow a\mathcal{B}c \vee c^*\mathcal{B}b}{C \Rightarrow a\mathcal{B}b}$ $(trR)$, where $c$ does not occur in $a, b$ and the formula $C$.
The above two rules replace the time axioms $(irr)$ and $(tr)$ and will, henceforth, be called *time rules*. We may consider extensions of the minimal logics with some time axioms and some of the rules $(irrR)$ and $(trR)$.

## 4.4 Soundness

**Definition 4.12** (Proof)**.** A finite sequence $P_1, P_2, ..., P_n$ of formulae such that every $P_i$ is either an axiom or is obtained by applying a rule of inference on one or more elements with indices less than $i$ is called a *proof.*

**Definition 4.13** (Theorem)**.** A formula $A$ is called a *theorem* if it is the last formula of some proof.

**Lemma 4.14.** Let $\mathbb{L}$ be any of the logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$ or $\mathbb{L}^{min}_{strong}$ possibly extended with some time axioms. Then the axioms of $\mathbb{L}$ are true in the respective class of DCAs satisfying the corresponding time axioms.

*Proof.* This is easy to see since the axioms of any of the mentioned logics and any additional time axioms are just the respective DCA or time axioms rewritten in the language of our logic. □

**Lemma 4.15.** Every non-standard rule of inference of the form $(\maltese R)$ preserves the validity in any class of DCAs satisfying the non-universal axiom $(\maltese)$ corresponding to the rule.

*Proof.* Consider the non-standard rule in the form
$\frac{C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)}{C \Rightarrow B(a_1, \ldots, a_n)}$ $(\maltese R)$, where $a_1, \ldots, a_n$ are terms and $b_1, \ldots, b_m$ are Boolean variables which are not included in the formulas $C$, $B(a_1, \ldots, a_n)$, and consequently in the terms $a_1, \ldots, a_n$. Let
$(\maltese)$ $(\forall b_1, \ldots, b_m)A(a_1, \ldots, a_n, b_1, \ldots, b_m) \Rightarrow B(a_1, \ldots, a_n)$
be the formula corresponding to this rule. Let $\Sigma$ be a class of DCAs which satisfies the condition $(\maltese)$. We'll show that whenever the premise $C \Rightarrow A(a_1 \ldots a_n, b_1 \ldots b_m)$ is true in $\Sigma$, then the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$ is also true in $\Sigma$. Suppose that this is not so. Then the premise $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is true in $\Sigma$ and there is an algebra $\underline{B} \in \Sigma$ and a model $(\underline{B}, v)$ such that $(\underline{B}, v) \not\models C \Rightarrow B(a_1, \ldots, a_n)$. This means that $(\underline{B}, v) \models C$ and $(\underline{B}, v) \not\models B(a_1, \ldots, a_n)$, so $B(v(a_1), \ldots, v(a_n))$(let's denote this way the interpretation of the formula $B$ in $\underline{B}$ under $v$) is not true $\underline{B}$. Since $\underline{B}$ satisfies condition $(\maltese)$, there are $c_1 \ldots c_m \in \underline{B}$ such that $A(v(a_1), \ldots, v(a_n), c_1 \ldots c_m)$ is not true in $\underline{B}$. Define $v'$ for the variables $b_1, \ldots, b_n$ as follows $v'(b_1) = c_1, \ldots, v'(b_m) = c_m$. Let $v'$ act as $v$ for the variables in $C$ and $a_1, \ldots, a_n$. By the constraints on $b_1, \ldots, b_m$ we obtain that $v'(a_1) = v(a_1)$,... $v'(a_n) = v(a_n)$ and $(\underline{B}, v') \models C$. Substituting in

$A$ we get: $A(v'(a_1),\ldots,v'(a_n),v'(b_1),\ldots,v'(b_m))$ is not true in $B$. Since $(\underline{B},v') \models C$ we obtain that $(\underline{B},v') \not\models C \Rightarrow A(a_1,\ldots,a_n,b_1,\ldots,b_m)$, contrary to the assumption that $C \Rightarrow A(a_1,\ldots,a_n,b_1,\ldots,b_m)$ is true in $\Sigma$. □

**Lemma 4.16.** Let $\mathbb{L}$ be any of the logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$ or $\mathbb{L}^{min}_{strong}$ possibly extended with some time axioms and rules. Then the rules of inference of $\mathbb{L}$ preserve the validity in the respective class $\Sigma$ of DCAs, in the sense that whenever the premise of the rule is true in $\Sigma$ then so is the conclusion of that rule.

*Proof.* Follows from the fact that Modus Ponens preserves the validity and Lemma 4.15. □

**Theorem 4.17** (Soundness theorem)**.** Let $\mathbb{L}$ be any of the logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$ or $\mathbb{L}^{min}_{strong}$ possibly extended with some time axioms and time rules. Then all theorems of $\mathbb{L}$ are true in the respective class of DCAs $\Sigma$ satisfying the corresponding time axioms.

*Proof.* Let $A$ be a theorem of $\mathbb{L}$ and let $B_1, B_2, ..., B_n$, where $B_n = A$, be the proof of $A$. By induction on $i = 1, ..., n$ we'll show that $B_i$ is true in $\Sigma$. For $i = 1$, the first member of the proof $B_1$ must be an axiom. From Lemma 4.14 it follows that $B_1$ is a true in $\Sigma$. Suppose that for $i = 1...k, k < n$ the statement is true. Let's check for $k + 1 \leq n$:

Case 1: $B_{k+1}$ is an axiom. Then the statement follows from Lemma 4.14.

Case 2: $B_{k+1}$ is obtained by using a rule of inference on some formulae of the proof with indices $< k + 1$. By the induction hypothesis these formulae are true in $\Sigma$ and by Lemma 4.16 so is $B_{k+1}$.

□

## 4.5 Completeness

In this section we'll prove the completeness theorems with respect to the algebraic semantics of the minimal logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$, $\mathbb{L}^{min}_{strong}$ and their extensions with time axioms and rules. The method is based on a version of the canonical model construction which is a modification of Henkin's completeness proof for classical first-order logic. In the context of logics for region-based theories of space this method was applied for the first time in [1] for relational and topological models and in [2] for algebraic semantics. This section closely follows [1] and [2] and will use slightly modified constructions and lemmas suitable for our purposes. The proofs will be easy modifications of the ones in [1] and [2] and as such will either be briefly mentioned or entirely skipped.

### 4.5.1 Canonical models

Let $\mathbb{L}$ be any of the minimal logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$, $\mathbb{L}^{min}_{strong}$ possibly extended with some new time axioms and rules. A pair $T = (T_1, T_2)$ is called an $\mathbb{L}$-theory (or simply a theory) if $T_1$ is a set of variables and $T_2$ is a set of formulae satisfying the following conditions:

(i) All theorems of $\mathbb{L}$ belong to $T_2$

(ii) If $A$ and $A \Rightarrow B$ belong to $T_2$ then $B$ belongs to $T_2$

(iii) Let (✠$R$) be any of the non-standard rules of inference of $\mathbb{L}$ and suppose that the premise $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ belongs to $T_2$ for some variables $b_1, \ldots, b_m$ not belonging to $T_1$ and to the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$. Then the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$ also belongs to $T_2$

The variables in $T_1$ are called *free variables* of $T$ and the members of $T_2$ are called formulae of $T$. We say that a formula $A$ *belongs to* $T$ and write $A \in T$ if $A \in T_2$. We say that $T$ is included in $T'$ if $T_1 \subseteq T_1^{'}$ amd $T_2 \subseteq T_2^{'}$. We say that $T$ is a *consistent theory* if $\bot \notin T_2$. If $T$ is not consistent then it is called *inconsistent*. A set of formulae is consistent if it is contained in a consistent theory. A theory $T$ is called a *complete* theory if it is a consistent theory and for any formula $A$ either $A \in T_2$ or $\neg A \in T_2$. A theory $T$ is called a *good* theory if out of $T_1$ there are infinitely many Boolean variables. $T$ is called a *rich theory* if for any non-standard rule of the logic (say (✠$R$)) the following holds: if the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$ does not belong to $T_2$, then the premise $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ does not belong to $T_2$ for some variables $b_1, \ldots, b_m$ not included in $a_1, \ldots, a_n$.

**Lemma 4.18** (Lindenbaum lemma)**.** Every good consistent $\mathbb{L}$-theory $T = (T_1, T_2)$ can be extended into a complete rich $\mathbb{L}$-theory $T' = (T_1^{'}, T_2^{'})$.

*Proof.* The proof is similar to the proof of Lemma 7.10 from [1]. □

**Lemma 4.19** (Conservativeness Lemma)**.** Every consistent $\mathbb{L}$-theory can be extended into a good consistent $\mathbb{L}$-theory by a possible extensions of the language with a countably infinite set of new Boolean variables.

*Proof.* The proof is similar to the proof of Lemma 7.11 from [1]. □

Let $T = (V, S)$ be a complete rich $\mathbb{L}$-theory. Define the following relation between the terms in our language:

$$a \equiv b \Leftrightarrow a = b \in S$$

Since $\equiv$ is an equivalence relation depending on $S$, let's consider equivalence classes of Boolean terms $|a| = \{b : a \equiv b\}$. Define the structure $B_s = (B, \leq, 0, 1, \cdot, +, *, C^s, C^t, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ over $S$ like follows:

- $B$ = equiv. classes of $\equiv$
- $0 = |0|$
- $1 = |1|$
- $|a| \cdot |b| = |a \cdot b|$
- $|a| + |b| = |a + b|$
- $|a|^* = |a^*|$
- $\mathcal{U}tr(|a|) = |\mathcal{U}tr(a)|$
- $|a| \leq |b| \Leftrightarrow a \leq b \in S$
- $|a|C^s|b| \Leftrightarrow aC^sb \in S$
- $|a|C^t|b| \Leftrightarrow aC^tb \in S$
- $|a|\mathcal{B}|b| \Leftrightarrow a\mathcal{B}b \in S$
- $|a| \in TR \Leftrightarrow TR(a) \in S$
- $|a| \in UTR \Leftrightarrow UTR(a) \in S$
- $NOW = |NOW|$

Let's also define the valuation $v_s$ for Boolean variables in the following way: $v_s(p) = |p|$, extended for terms in a standard way: $v_s(a) = |a|$. We'll call the pair $\mathcal{M}_s = (\mathcal{B}_s, v_s)$ the *canonical model* over $S$.

**Lemma 4.20.** $\mathcal{B}_s$ is a Boolean algebra.

*Proof.* This is easy to see since the axioms of Boolean algebra are part of the axiomatization of the logic $\mathbb{L}$. For example, look at $|a|$, $|b|$ and $|c|$ in $B$. We have that $a, b, c$ are terms and we that the Boolean algebra axiom $a \cdot (b + c) \leq a \cdot b + a \cdot c$ is in S. By the definition if $\leq$ we get that $|a \cdot (b + c)| \leq |a \cdot b + a \cdot c|$. This can be rewritten as $|a| \cdot (|b + c|) \leq |a \cdot b| + |a \cdot c|$. Finally, rewritting as $|a| \cdot (|b| + |c|) \leq |a| \cdot |b| + |a| \cdot |c|$ we get the distributive lattice rule in $\mathcal{B}_s$. □

**Lemma 4.21.** $\mathcal{B}_s$ is a basic dynamic contact algebra.

*Proof.* By Lemma 4.20 we have that $\mathcal{B}_s$ is a Boolean algebra. We have to show that $C^s$ and $C^t$ are contact relations and $\mathcal{B}$ is a precontact relation. We also have to show that the $TR$ axioms for basic DCA hold as well as the axioms for $\mathcal{U}tr$.

We'll only assert that $C^s$ is a contact relation (in the same way we can check that $C^t$ is a contact relation and $\mathcal{B}$ is a precontact relation). We'll follow the points of the *contact* definition (see Def. 1.12). (i) Suppose $|a|C^s|b|$ but $|a| = |0|$ or $|b| = |0|$ (where $=$ in $\mathcal{B}_s$ is defined in a standard way). Without loss of generality suppose that $|a| = |0|$. So we have that $|a| \leq |0|$ and $|0| \leq |a|$, or equivalently $a \leq 0 \in S$ and $b \leq 0 \in S$. Hence $a = 0 \in S$. To reach a contradiction let's look at $|a|C^s|b|$. By definition, we have that $aC^sb \in S$ and by the $\mathbb{L}$ axiom $aC^sb \Rightarrow (a \neq 0) \wedge (b \neq 0)$ we have $a \neq 0 \in S$. This is a contradiction. (ii) Let $|a|C^s|b|$ and $|a| \leq |a'|$ and $|b| \leq |b'|$ - we need to show that $|a'| \leq |b'|$. From the premises we get that $aC^sb \in S$, $a \leq a' \in S$ and $b \leq b' \in S$ and hence the formula $aC^sb \wedge a \leq a' \wedge b \leq b' \in S$. From here and the $\mathbb{L}$ axiom $(aC^sb \wedge a \leq a' \wedge b \leq b') \Rightarrow a'C^sb'$ we conclude that $a'C^sb' \in S$. Thus we get $|a'| \leq |b'|$. (iii) Let $|a|C^s(|b| + |c|)$ - we have to show that $|a|C^s|b|$ or $|a|C^s|c|$. We can rewrite $|a|C^s(|b| + |c|)$ as $|a|C^s(|b + c|)$ which implies that $aC^s(b + c) \in S$. From here and the $\mathbb{L}$ axiom $aC^s(b + c) \Rightarrow aC^sb \vee aC^sc$ we get that $aC^sb \vee aC^sc \in S$. This means that $aC^sb \in S$ or $aC^sc \in S$ (as a more general statement. for a complete $\mathbb{L}$-theory S if

$A \vee B \in S$ then at least one of $A \in S$ or $B \in S$ should be true - assuming the contrary quickly produces a contradiction with the consistency of S). But from here we get that $|a|C^s|b|$ or $|a|C^s|c|$ which is what we are trying to prove. (iv) Let $|a|C^s|b|$ - we'll show that $|b|C^s|a|$. This is easy to see by considering the $\mathbb{L}$ axiom $aC^sb \Rightarrow bC^sa$ and the fact that $aC^sb \in S$. (v) Assume that $|a| \cdot |b| \neq |0|$ - we have to show that $|a|C^s|b|$. From the assumption we get that $|a \cdot b| \neq |0|$ which is the case only when $\neg(|a \cdot b| \leq |0|)$ or $\neg(|0| \leq |a \cdot b|)$. Without loss of generality, assume that $\neg(|a \cdot b| \leq |0|)$. This means that $a \cdot b \leq 0 \notin S$ and by the completess of S we have that $\neg(a \cdot b \leq 0) \in S$. By this and the propositional axiom $\neg(a \cdot b \leq 0) \Rightarrow \neg(a \cdot b \leq 0) \vee \neg(0 \leq a \cdot b)$ $(A \Rightarrow A \vee B)$ we get that $\neg(a \cdot b \leq 0) \vee \neg(0 \leq a \cdot b) \in S$. But this is simply $a \cdot b \neq 0$ and from the $\mathbb{L}$ axiom $a \cdot b \neq 0 \Rightarrow aC^sb$ we conclude that $aC^sb \in S$. Therefore $|a|C^s|b|$. Also, let's see that axiom $(C^s \Rightarrow C^t)$, establishing the connection between space and time contact, does indeed hold in the structure $\mathcal{B}_s$. Let $|a|C^s|b|$. But this means that $aC^sb \in S$ and from the $\mathbb{L}$ axiom $aC^sb \Rightarrow aC^tb$ we have that $aC^tb \in S$. Thus $|a|C^t|b|$ and hence the axiom holds.

For axiom (TR1) let $|c| \in TR$ - we want to show that $|c| \neq |0|$ and for any $|a|$ and $|b|$, $|a|C^t|c|$ and $|b|C^t|c|$ implies $|a|C^t|b|$. From $|c| \in TR$ we get that $TR(c) \in S$ and combining this with the $\mathbb{L}$ axiom $TR(c) \Rightarrow c \neq 0 \wedge (aC^tc \wedge bC^tc \Rightarrow aC^tb)$ we get that $c \neq 0 \in S$ and $aC^tc \wedge bC^tc \Rightarrow aC^tb \in S$ which can easily be translated into what we are trying to prove. The same kind of reasoning can be applied for the other TR axioms as well as the $\mathcal{U}tr$ axioms of basic DCAs. □

**Lemma 4.22.** If $\mathbb{L}$ contains a non-standard rule of inference then $B_s$ satisfies the non-universal axiom corresponding to the rule.

*Proof.* Consider, for example, the rule

$$\frac{C \Rightarrow \neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)}{C \Rightarrow a\overline{\mathcal{B}}b} \ (irrR)$$

, where $c, d$ are variables not occurring in the terms $a, b$ and the formula $C$. The rule corresponds to the time axiom

$(irr)$ $a\mathcal{B}b \rightarrow (\exists c, d)(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)$

Suppose $|a|\mathcal{B}|b|$, then $a\mathcal{B}b \in S$. Since $S$ is a complete theory this is equivalent to $a\overline{\mathcal{B}}b \notin S$. Since $S$ is a rich theory, then for some variables $c, d$ we have the following:

$\neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d) \notin S$

Again by the completeness of $S$ this is equivalent to the following:

$c \neq 0 \in S$ and $d \neq 0 \in S$ and $c \leq a \in S$ and $d \leq b \in S$ and $c\overline{C}^t d \in S$

This, by the definitions of the canonical relations is equivalent to: there are $|c|, |d| \in B$ such that $|c|, |d| \neq |0|$, $|c| \leq |a|$, $|d| \leq |b|$ and $|c|\overline{C}^t|d|$ which shows that axiom $(irr)$ holds in $B_s$. The statement can be shown for the other non-standard rules in the same way using the completeness and richness of $S$. □

**Lemma 4.23** (Canonical structure)**.** Let $T = (V, S)$ be a complete rich $\mathbb{L}$-theory and let $\mathcal{M}_s = (\mathcal{B}_s, v_s)$ be the canonical model over S. Then $\mathcal{B}_s$ is dynamic contact algebra of the type $\mathbb{L}$ corresponds to. If $\mathbb{L}$ contains some of the time axioms or rules then $\mathcal{B}_s$ also satisfies the corresponding time axioms.

*Proof.* Follows from Lemma 4.21 and Lemma 4.22. □

**Lemma 4.24** (Truth lemma)**.** Let $T = (V, S)$ be a complete rich $\mathbb{L}$-theory and $\mathcal{M}_s = (\mathcal{B}_s, v_s)$ be the canonical model over $S$. Then for any formula $\alpha$ we have that $(\mathcal{B}_s, v_s) \models \alpha \Leftrightarrow \alpha \in S$.

*Proof.* ($\Leftarrow$) By induction on the complexity of the formula $\alpha$. For the base case, consider the case when $\alpha \in S$ is an atomic formula, that is a formula of one the following forms $a \leq b$, $aC^s b$, $aC^t b$, $a\mathcal{B}b$, $TR(a)$ or $UTR(a)$ where $a$ and $b$ are terms. We need to show that $(\mathcal{B}_s, v_s) \models \alpha$, or equivalently, $v_s(\alpha) = 1$. This is immediate by the definition of the canonical model, the definition of the canonical valuation $v_s$ and the way it interprets truthfulness in the structure.

Let $A$ and $B$ be formulae for which the statement is true. We need to show that for complex formulae the statement holds. Let's review the complex formulae in our language and prove the statement separately for each:

1. Let $\neg A \in S$. Since S is complete we have that $A \notin S$ and hence by the induction hypothesis we have that $v_s(A) = 0$. By the defintion of $v_s$ we have that $v_s(\neg A) = 1$.

2. Let $A \wedge B \in S$. Since S is an $\mathbb{L}$-theory it contains the propositional axioms $A \wedge B \Rightarrow A$ and $A \wedge B \Rightarrow B$ and hence $A \in S$ and $B \in S$. By the induction hypothesis we have that $v_s(A) = 1$ and $v_s(B) = 1$ and by the definition of $v_s$ we have that $v_s(A \wedge B) = 1$.

3. Let $A \vee B \in S$. Suppose that $A \notin S$ and $B \notin S$. Since S is complete we have that $\neg A \in S$. Also note that $A \vee B$ can be rewritten as $\neg A \Rightarrow B \in S$. Since S is an $\mathbb{L}$-theory we have that $B \in S$ which contradicts that assumption. So it must be the case that at least one of $A$ or $B$ is in S. Without loss of generality suppose that $A \in S$. Then by the induction hypothesis we get that $v_s(A) = 1$ and hence $v_s(A \vee B) = 1$ by the definition of $v_s$.

4. Let $A \Rightarrow B \in S$. This can be rewritten as $\neg A \vee B \in S$ and by (3) we get that $v_s(\neg A \vee B) = 1$. This, by the defintion of $v_s$, means that $v_s(\neg A) = 1$ or $v_s(B) = 1$ and further, $v_s(A) = 0$ or $v_s(B) = 1$. But this is exactly when $v_s(A \Rightarrow B) = 1$.

($\Rightarrow$) Can be proven again by induction using similar arguments. □

**Lemma 4.25.** The following conditions are equivalent for any formula $A$:

(i) $A$ is a theorem of $\mathbb{L}$
(ii) $A$ is true in all canonical models $\mathcal{M}_s$ of $\mathbb{L}$

*Proof.* $(i) \rightarrow (ii)$. Let $A$ be a theorem of $\mathbb{L}$ and let $\mathcal{M}_s$ be a canonical model over some complete rich $\mathbb{L}$-theory $T = (V, S)$. Since $T$ is an $\mathbb{L}$-theory we have that $A \in S$ and by the Truth Lemma we have that $\mathcal{M}_s \models A$.

*(ii)* → *(i)* We'll prove the contraposition. Suppose that $A$ is not a theorem of $\mathbb{L}$. Take the minimal $\mathbb{L}$-theory $T_0 = (\emptyset, \Gamma_0)$ where $\Gamma_0$ is the set of all theorems of $\mathbb{L}$. We have that $A \notin \Gamma_0$ (since $A$ is not a theorem). This means extending $T_0$ with $\neg A$ produces the good consistent $\mathbb{L}$-theory $T_1$. By the Lindembaum Lemma $T_1$ can be extended into a complete rich DCA-theory $T = (V, S)$. Since T is complete and $\neg A \in S$ we have that $A \notin S$. By the Truth Lemma we get that $A$ is falsified in the canonical model over S. □

### 4.5.2 Completeness theorems and their implications

In this section we'll look at the weak and strong completeness theorems for the minimal logics $\mathbb{L}^{min}_{basic}$, $\mathbb{L}^{min}_{weak}$, $\mathbb{L}^{min}_{DCA}$, $\mathbb{L}^{min}_{strong}$ and their extensions with time axioms and rules. If $\Theta$ is a set of some time axioms and rules and $\mathbb{L}$ is any of the minimal logics, then $\mathbb{L}^{\Theta}$ will denote the extension of $\mathbb{L}$ with the axioms and rules from $\Theta$.

**Theorem 4.26** (Weak completeness for the minimal logics)**.** Let $\mathbb{L}$ be any of the minimal logics and let $\Sigma$ be the corresponding class of DCAs for $\mathbb{L}$. Then the following conditions are equivalent for any formula $A$:

(i) $A$ is a theorem of $\mathbb{L}$
(ii) $A$ is true in the class $\Sigma$ of DCAs corresponding to $\mathbb{L}$

*Proof.* *(i)* → *(ii)* This is the Soundness Theorem (Theorem 4.17).

*(ii)* → *(i)* Let $A$ be true in the class $\Sigma$. By definition, this means that $A$ is true in all models $\mathcal{M} = (\underline{B}, v)$ where $\underline{B}$ is a DCA of the corresponding type and, in particular, all canonical models of $\mathbb{L}$. By the Lemma 4.25 we have that $A$ is a theorem of $\mathbb{L}$. □

**Corollary 4.27.** The completeness theorem for the minimal logics yields the following results:

(i) All four minimal logics have equal sets of theorems which coincide with the set of theorems of $\mathbb{L}^{min}_{basic}$.
(ii) Theorems of the minimal logics do not depend on the non-standard rules of inference, that is, the non-standard rules of inference are *admissible.*
(iii) The set of theorems of the minimal logics is decidable.

*Proof.* (i) Follows from Theorem 4.26 and Theorem 4.8.
(ii) Since $\mathbb{L}^{min}_{basic}$ does not have non-standard rules of inference the statement follows from (i).
(iii) By Proposition 4.5 we have $\mathcal{L}(\Sigma^{fin}_{basic}) = \mathcal{L}(\Sigma_{basic})$, which together with the completeness theorem implies that the set of theorems of $\mathbb{L}^{min}_{basic}$ (and hence for the other minimal logics) is decidable. □

**Theorem 4.28** (Weak completeness for extensions with time axioms and rules)**.** Let $\mathbb{L}^\Theta$ be any of the minimal logics extended with a set $\Theta$ of additional time axioms and rules and let $\Sigma^\Theta$ be the corresponding class of DCAs satisfying the respective time axioms. Then the following conditions are equivalent for any formula $A$:

(i) $A$ is a theorem of $\mathbb{L}^\Theta$
(ii) $A$ is true in $\Sigma^\Theta$

*Proof.* The proof is similar to the one of Theorem 4.26. □

**Corollary 4.29.** The completeness theorem for extensions of the minimal logics with time axioms and rules yields the following results:

(i) Let $\Theta$ be a set of time axioms. Then the logic $\mathbb{L}^\Theta_{basic}$ is decidable.
(ii) Let $\Theta$ be a set of time axioms and rules. Then the sets of theorems of $\mathbb{L}^\Theta_{weak}$, $\mathbb{L}^\Theta_{DCA}$ and $\mathbb{L}^\Theta_{strong}$ coincide.
(iii) Let $\Theta$ be a set consisting of some (or all) of the time axiom $(dens)$ and time rules $(irrR)$ and $(trR)$. Then the logics $\mathbb{L}^\Theta_{basic}$, $\mathbb{L}^\Theta_{weak}$, $\mathbb{L}^\Theta_{DCA}$, $\mathbb{L}^\Theta_{strong}$ have equal sets of theorems.

*Proof.* (i) By Theorem 4.28 the set of theorems of $\mathbb{L}^\Theta_{basic}$ coincides with $\mathcal{L}(\Sigma^\Theta_{basic})$. By Proposition, 4.5 $\mathcal{L}(\Sigma^\Theta_{basic}) = \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$, which implies the decidability of $\mathbb{L}^\Theta_{basic}$.
(ii) The statement follows from Theorem 4.28 and Theorem 4.9.
(iii) The proof follows from Theorem 4.28 and Corollary 4.11. □

Let $\Psi$ be a set of formulae and $\Sigma$ be a class of DCAs. We say that $\Psi$ has a model in $\Sigma$ if there is a model $(\underline{B}, v)$ such that $\underline{B} \in \Sigma$ and for any $A \in \Psi$ we have $(\underline{B}, v) \models A$. In such a case we write $(\underline{B}, v) \models \Psi$.

**Theorem 4.30** (Strong completeness)**.** Let $\mathbb{L}^\Theta$ be any of the the minimal logics extended with a set $\Theta$ of additional time axioms and rules and let $\Sigma^\Theta$ be the corresponding class of DCAs. Then the following conditions are equivalent for any set of formulas $\Psi$:

(i) $\Psi$ is a consistent set of formulae
(ii) $\Psi$ has a model in $\Sigma^\Theta$

*Proof.* $(i) \to (ii)$. Let $\Psi$ be a consistent set of formulae. Then by Lemma 4.19 and Lemma 4.18 $\Psi$ can be extended into a complete and rich theory $T = (V, S)$ in a possible extension of the language with new variables. Then the canonical model based on $T$ is a model for $\Psi$ by Lemma 4.24.
$(ii) \to (i)$ Let $\Psi$ have a model $(\underline{B}, v)$ in $\Sigma^\Theta$. Let $\Gamma$ be the set of all formulae $A$ such that $(\underline{B}, v) \models A$. Obviously $\Psi$ is included in $\Gamma$ and $T = (\varnothing, \Gamma)$ is a consistent theory, so $\Psi$ is a consistent set of formulae. □

## 5 Conclusion and open problems

With regards to open problems, firstly, we would like an extension of Corollary 3.7 to be true for basic DCAs satisfying the non-universal axioms *irr* and *tr*. Another thing would be to show coincidence of the extensions of the minimal logics with arbitrary time axioms and rules (or at least for some interesting combinations), i.e. to obtain various extensions of Corollary 4.11. This corollary depends essentially on the properties of the p-morphisms developed in Section 3.5.1 and Section 3.5.2 and, more precisely, on which of time conditions are preserved by these p-morphisms. Studying modifications of these p-morphisms that preserve important sets of time conditions would be one of the possible advancements on the topic.

## References

[1] Balbiani, Ph., Tinchev, T. and Vakarelov, D. *Modal Logics for Region-based Theories of Space.* Fundamenta Informaticae 81, 2007, 29–82

[2] Vakarelov, D. *Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics.* In: Dov Gabbay et al. (Eds.) Mathematical Problems from Applied Logics. New Logics for the XXIst Century. II. Springer, 2007, 267-348.

[3] Dimov G. and Vakarelov, D. *Contact Algebras and Region-based Theory of Space. A Proximity Approach.* I and II. Fundamenta Informaticae, 74(2-3):209-249, 251-282, 2006.

[4] Düntsch I. and Vakarelov, D. *Region–based theory of discrete spaces. A proximity approach.* In: Nadif, M., Napoli, A., SanJuan, E., and Sigayret, A. EDS, Proceedings of Fourth International Conference Journées de l'informatique Messine, 123-129, Metz, France, 2003. Journal version in: Annals of Mathematics and Artificial Intelligence, 49(1-4):5-14, 2007.

[5] Whitehead A.N. *The Organization of Thought.* London, William and Norgate, 1917

[6] Givant S. and Halmos P. *Introduction to Boolean Algebras.* Springer, 2009, ISBN: 978-0-387-40293-2, ISSN: 0172-6056

[7] Vakarelov D. *Dynamic mereotopology. III. Whiteheadean type of integrated point-free theories of space and time.* Part I, Algebra and Logic, vol. 53, No 3, 2014, 300-322. Part II, Algebra and Logic, vol. 55, No 1, 2016, 14-36. Part III, Algebra and Logic, vol. 55, No 3, 2016, 273-299.