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Master Thesis

Temporal mereology. Whitehead's epochal theory of time as an extension of mereology

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1 Introduction

1.1 Aim of the paper

Alfred North Whitehead (15 February 1861 − 30 December 1947) is well known as the founder of the contemporary process philosophy. He was an English mathematician and philosopher. The main part of his new philosophical system is related to his view on an integrated theory of space and time which should be put on a relational base, which means that it have to be extracted from natural spatio-temporal relations between real things. His early view on this subject can be found in [4], page 195, where he claims that the theory of space and time should be "point-free" in a sense that neither space points nor time points (instances of time, moments) have to be put on the base of the theory, because they are abstract things with not separate existing in reality. Whitehead's approach to the theory of space is now known as a Region-Based Theory of Space (RBTS) (see [2] for a detailed survey and references). Sometimes RBTS is called also **mereotopology**, because it is can be considered also as an extension of **mereology** by some topological relations like the contact relation (see [1] about mereology). It's a theory describing spatial relations among wholes, parts and parts of parts. Whitehead's theory of time, named in [5, 6] as **Epochal Theory of Time** (ETT), is a quite unusual and interesting theory aiming to explain difficult and old problems concerning the nature of time. However, while one can find in [6] a detailed program how to develop RBTS as a mathematical theory, Whitehead did not present such a program for ETT and developed it in an informal manner and on a quite complicated philosophical terminology. The paper [3] can be considered as an attempt to present a formal explication ETT based on a dynamic version of mereotopology called **dynamic mereotopology**. The temporal component of the theory is based on the relations **temporal contact, precedence** and some special individuals, called **time representatives** to help us define **the present time, the past and the future**. The intuition for those comes the every day phrases like "The time of the First World War", "The epoch of the Renaissance", "During the XXI centuary" and so on. Since ETT does not depend on the topological relations between individuals, we show in this work another formal version of ETT, called **temporal mereology**, based on a simpler system, which is an extension of mereology by the above relations plus **proximity** relation. The proximity property will let us state whenever two regions exist in separate relatevely close epochs. Interesting about this theory is that we can make conclusions about the properties of time independently from the space and it's topology. That is possible because in this paper we have a very simple and basic understanding for the nature of space. We have only the space mereology, i.e. we know the space points, but we don't have the space topology. To have a reasonable sense of what that actualy means we can give a short intuitive example from the geometry. In geometry we know that a set of space points are describing a figure, but no all set of points are describing a valid figure into space. We need to have a boundary points on the exterior of the figure and inner points inside. So in this sense we are making deductions on the time without depending on the space's main feature - topology. We have a translation mechanism to express native time properties with the language of dynamic mereological algebras. Examples of the possible definable notions for expressing our selfs in this model are: There is a region existing at the moment we speak. There will be a region existing after we stop speaking. There will be a region existing shortly after we stop speaking and so on. We can define the notion - **society of contemporaries** to describe all things existing in the same time period, the notion of **near and prior contemporaries**. The main result is a construction of a special standard model of temporal mereology and an abstract axiomatization of the model, justiffed by a corresponding representation theory. We show how based on a given abstract dynamic mereological algebra we can extract a canonical time structure and then define a canonical dynamic mereological algebra and an isomorphic embedding from the initial one to the canonical.

2 What I need to know beforehand?

In this section we will explain basic definitions and lemmas needed for further reading this paper. This includes definitions of Boolean, precontact and contact algebras, review of a relational example of precontact and contact algebras, discrete representation of precontact and contact algebras as well as useful construction for factor Boolean algebras. Attention will be paid to contact algebras satisfying the Efremovich axiom and the notion of clusters.

2.1 Definitions of contact and precontact algebras

Definition 2.1. Partially ordered set. Partially ordered set is a set P on which is defined a binary relation $\leq$ satisfying the following conditions:

(i) $x \leq x$ for all $x \in P$

(ii) $x \leq y, y \leq z$ implies $x \leq z$

(iii) $x \leq y, y \leq x$ implies $x = y$

Definition 2.2. An element a of a partially ordered set is called largest if $(\forall x \in P)(x \leq a)$. If partially ordered set has a largest element then we will always denote this element by the symbol 1. The largest element is always unique.

Definition 2.3. An element a of a partially ordered set is called smallest if $(\forall x \in P)(x \geq a)$. If partially ordered set has a smallest element then we will always denote this element by the symbol 0. The smallest element is always unique.

Definition 2.4. Supremum. Let L be a partially ordered set. An element z is called a least upper bound (supremum) of the elements x and y, denoted by $z = x + y$ if:

(i) $x \leq z, y \leq z$

(ii) $x \leq u, y \leq u$ implies $z \leq u$

Definition 2.5. Infimum. Let L be a partially ordered set. An element z is called a greatest lower bound (infimum) of the elements x and y, denoted by $z = x \cdot y$ if:

(i) $x \geq z, y \geq z$

(ii) $x \geq u, y \geq u$ implies $z \geq u$

Definition 2.6. Lattice. A lattice is a partially ordered set is which every two elements have a least upper bound and greatest lower bound.

Definition 2.7. Distributive lattice. A lattice L is called distributive if one of the following laws holds:

(i) $x \cdot (y + z) = x \cdot y + x \cdot z$

(ii) $x + (y \cdot z) = (x + y) \cdot (x + z)$

Definition 2.8. A lattice L with a 0 and 1 is called **complemented** if for every element $x \in L$, there exists an element $x^* \in L$ such that $x + x^* = 1$ *and* $x \cdot x^* = 0$.
x^* is called the complement of x.

Definition 2.9. Boolean algebra is a distributive complemented lattice with 0 and 1.

Definition 2.10. Contact algebra. Let $(B, 0, 1, \leq, +, \cdot, *)$ be a non-degenerate $(0 \neq 1)$ Boolean algebra and C be a binary relation in B. C is called a **contact** relation in B if the following axioms are satisfied:

(C1) If aCb then $a \neq 0$ and $b \neq 0$

(C2) If aCb and $a \leq a'$ and $b \leq b'$ then $a'Cb'$

(C3) If $aC(b + c)$ then aCb or aCc

(C4) If $(a + b)Cc$ then aCc or bCc

(C5) If aCb then bCa

(C6) If $a \cdot b \neq 0$ then aCb

If C is a contact relation in B, then the pair (B, C) is called a contact algebra. We write $\overline{C}$ for the complement of C.

Let us mention that on the base of (C5) only the one of the axioms (C3) and (C4) is needed. Note that (C6) is equivalent to the following more simple axiom:

(C7) If $a \neq 0$ then aCa

From (C7) and (C1) it follows that $a \neq 0$ iff aCa

In the present context we treat the Boolean part of the contact algebra as its mereological component. In our treating of mereology we consider the zero element 0 as a non-existing region and this can be used to define the ontological predicate of existence in the following way: a ontologically exists iff $a \neq 0$. For simplicity, instead of "ontologically exists" we will say simply "exists" and from the context it will be clear that this is not the existential quantifier. The predicate of existence, will be very important in the theory of dynamic mereological algebras, to indicate the fact that a dynamic region may have moments in his life history in which it does not exist and moments in which it exists.

The definitions of mereological relations "part-of" and "overlap" are the following:

- a is part of b iff $a \leq b$, i.e. part-of is just a Boolean ordering
- a overlaps b (in symbols aOb) iff there exists a region $c \neq 0$ such that $c \leq a$ and $c \leq b$ iff $a \cdot b \neq 0$

Definition 2.11. Precontact algebra. Let $(B, 0, 1, \leq, +, \cdot, *)$ be a non-degenerate Boolean algebra and C is a binary relation in B. C is called a **precontact** relation in B if it satisfies the axioms (C1), (C2), (C3) and (C4). If C is a precontact relation in B then the pair (B, C) is called a precontact algebra. Precontact relation will be used later on in the formalization of some temporal relations between changing regions in dynamic mereological algebras.

We will be interested later on contact and precontact algebras satisfying the following additional axiom:
(CE) *If* $a\overline{C}b$ *then* $(\exists c)(a\overline{C}c$ *and* $c^*\overline{C}b)$. This axiom is called sometimes Efremovich axiom, because it is used in the definition of Efremovich proximity spaces.

Definition 2.12. Filter. A subset F of a Boolean algebra B is a filter if the following conditions hold:

(i) $1 \in F$.

(ii) $x \leq y, x \in F$ implies that $y \in F$.

(iii) $x, y \in F$ impies $x \cdot y \in F$.

Definition 2.13. Proper filter. Proper filter is a filter satisfying the following condition:

$0 \notin F$.

Definition 2.14. Ultrafilter. Ultrafilter is a proper filter satisfying the following condition:

$x + y \in F$ implies that $x \in F$ *or* $y \in F$.

Definition 2.15. Ideal. An ideal of a Boolean algebra B is a subset $I \subseteq B$, satisfying the following conditions:

(i) $0 \in I$.

(ii) $x \in I, t \leq x, t \in B$ implies $t \in I$.

(iii) $x \in I, y \in I$ implies $x + y \in I$.

Definition 2.16. I is called **proper** if $1 \notin I$

Relational examples of precontact and contact algebras. Let X be a nonempty set, whose elements are considered as points and R be a reflexive and symmetric relation in X. Such reflexive and symmetric relations are sometimes called **adjacency relations** and in such intuitive interpretation xRy means that x is adjacent to y. Pairs (X, R) with reflexive and symmetric R are called by Galton **adjacency spaces**. The following is a natural example of adjacency space: points are the squares in chess board and two squares are adjacent if they are identical or if they touch each other. One can construct a contact algebra from an adjacency space as follows: take a class B of subsets of X which form a Boolean algebra under the set theoretical operations of union $a \cup b$, intersection $a \cap b$ and complement $\overline{a} = X \setminus a$ and define contact C_R between two members of B as follows: aC_Rb iff there exist $x \in a$ *and* $y \in b$ such that xRy. So, a and b are in a contact if there is a point in a which is adjacent to a point in b. We use the notation C_R for the contact relation just to indicate that it depends on the adjacency relation R. It can easily be verified that all axioms of contact are satisfied. Symmetry of R is used to verify the axiom $(C5)$ and reflexivity is used to verify the axiom $(C6)$. Let us note that there are more general adjacency spaces in which neither reflexivity nor symmetry for the relation R are assumed. In the chessboard example such a relation is, for instance, square b to be the left adjacent of the square a. Obviously this relation is neither reflexive nor symmetric. We reserve the name adjacency space for such more general spaces and for the special case where R is a reflexive and symmetric relation we will say adjacency spaces in the sense of Galton. If we repeat the above construction then the axioms $(C1)$, $(C2)$, $(C3)$ and $(C4)$ will be true (but in general the axioms $(C5)$ and $(C6)$ will not be satisfied) and in this way we obtain examples of pre-contact algebras which are not contact algebras. The following lemma will be of later use:

Lemma 2.17. [3] ***Characterization of reflexivity, symmetry and transitivity.*** *Let (X, R) be an adjacency space and $(B(X), C_R)$ be a precontact algebra over all subsets of X. Then the following conditions hold:*

(i) R is reflexive relation in X iff $(B(X), C_R)$ satisfies the axiom (C6) If $a \cdot b \neq 0$ then aCb

(ii) R is symmetric relation in X iff $(B(X), C_R)$ satisfies the axiom (C5) If aC_Rb then bC_Ra

(iii) R is transitive relation in X iff $(B(X), C_R)$ satisfies the axiom (CE) If $a\overline{C}b$ then $(\exists c)(a\overline{C}b$ and $c^*\overline{C}b)$

2.2 Discrete (relational) representation of contact and precontact algebras.

Discrete representation theory. One way to obtain a representation theory of precontact algebras with relational representation of contact is to consider as points of a given precontact algebra (B, C) the ultrafilters of (B, C) (as in the Stone representation theory of Boolean algebras). It remains to show how to define a relation R in the set of ultrafilters $Ult(B)$ of (B, C). Let $U, V \in Ult(B)$ and define $URV \leftrightarrow_{def} (\forall a, b)($ *If* $a \in U$ *and* $b \in V$ *then* $aCb)$. The relational system $(Ult(B), R)$ with just defined R is called a canonical adjacency space over (B, C) and R is called the canonical adjacency relation in (B, C). Note that the definition of the canonical relation R is meaningfull for arbitrary filters and the following technical lemma is useful in the representation theory of precontact algebras.

Lemma 2.18. [10] ***R-extension Lemma.*** *Let U_0 and V_0 be filters in a precontact algebra (B, C) and let U_0RV_0. Then there exists ultrafilters U and V such that $U_0 \subseteq U$, $V_0 \subseteq V$ and URV.*

Then, as in the representation theorem for Boolean algebras, define the Stone embedding $h(a) = \{U \in Ult(B) : a \in U\}$. From the representation theory of Boolean algebras we have that h is an isomorphic embedding of B into the Boolean algebra of all subsets of $Ult(B)$. Using the theory of precontact algebras one can prove the following technical lemmas:

Lemma 2.19.

(i) aCb iff there exists ultrafilters U, V such that $URV, a \in U$ and $b \in V$

(ii) aCb iff $h(a)C_Rh(b)$

Idea for the proof.
First: Define filters generated by a and b: $[a) = \{c : a \leq c\}$ and $[b) = \{c : b \leq c\}$.
Second: aCb implies $[a)R[b)$ and then apply the R-extension Lemma 2.18.

Lemma 2.20. [10] *Let $Ult(B)$ be the set of ultrafilters of (B, C). Then:*

(i) R is a symmetric relation in $Ult(B)$ iff (B, C) satisfies the axiom $(C4)$.

(ii) R is a reflexive relation in $Ult(B)$ iff (B, C) satisfies the axiom $(C5)$.

(iii) R is transitive relation in $Ult(B)$ iff (B, C) satisfies the axiom (CE).

Theorem 2.21. [10] ***Relational representation theorem for precontact and contact algebras*** *Let (B, C) be a precontact algebra, $(Ult(B), R)$ be the canonical adjacency space over (B, C) and h be the stone embedding. Then:*

(i) h is an embedding of (B, C) into the precontact algebra over the canonical adjacency space $(Ult(B), R)$.

(ii) If (B, C) is a contact algebra then the precontact algebra over the canonical adjacency space over (B, C) is a contact algebra.

Definition 2.22. [11] **Definition of clan.** Let $\underline{B} = (B, C)$ be a contact algebra. A subset $\Gamma \subseteq B$ is called a **clan** in (B, C) if it satisfies the following conditions:

(i) $1 \in \Gamma$ and $0 \notin \Gamma$

(ii) If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$

(iii) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$

(iv) If $a, b \in \Gamma$ then aCb

Γ is a **maximal clan** if it is a maximal set under the set iclusion. We denote by $Ult(\Gamma)$ the set of all ultrafilters contained in Γ and $Clans(B)$ - the set of all clans of (B, C).

The above definition is an algebraic abstraction from an analogous notion in the proximity theory. Let us note that ultrafilters are clans, but there are other clans and they can be obtained otherwise.

Definition 2.23. R-clique. Let $\sum$ be a nonempty set of ultrafilters of (B, C) such that if $U, V \in \sum$ then URV, where R is the canonical adjacency relation in the set of ultrafilters of (B, C). Such sets of ultrafilters are called R-cliques. An R-clique is maximal, if it is a maximal set under set-inclusion.

By the axiom of choice every R-clique is contained in a maximal R-clique. Let Γ be the union of all ultrafilters from $\sum$. Then it can be verified that Γ is a clan. More over, every clan can be obtained by this construction from an R-clique and there is a correspondence between maximal cliques and maximal clans. All these facts about clans are contained in the following technical lemma:

Lemma 2.24. [11]

(i) Every clan is contained in a maximal clan (by the axiom of choice)

(ii) Let $\sum$ *be an R-clique and* $\Gamma(\sum) = \bigcup_{\Gamma \in \sum} \Gamma$*. Then* $\Gamma(\sum)$ *is a clan.*

(iii) If $U, V \in Ult(\Gamma)$ *then* URV*, so* $Ult(\Gamma)$ *is an R-clique.*

(iv) If Γ *is a clan and* $a \in \Gamma$ *then there is an ultrafilter* $U \in Ult(\Gamma)$ *such that* $a \in U$ *(by the axiom of choice).*

(v) Let Γ *be a clan and* $\sum$ *be the R-clique* $Ult(\Gamma)$*. Then* $\Gamma = \Gamma(\sum)$*, so every clan can be defined by an R-clique as in (ii).*

(vi) If $\sum$ *is a maximal R-clique the* $\Gamma(\sum)$ *is a maximal clan.*

(vii) If Γ *is a maximal clan then* $Ult(\Gamma)$ *is a maximal R-clique.*

(viii) For all ultrafilters $U, V : URV$ *if there exists a (maximal) clan* Γ *such that* $U, V \in Ult(\Gamma)$*.*

(ix) For all $a, b \in B : aCb$ *iff there exists a (maximal) clan* Γ *such that* $a, b \in \Gamma$*.*

(x) For all $a, b \in B : a \not\leq b$ *iff there exists an ultrafilter (clan)* Γ *such that* $a \in \Gamma$ *and* $b \notin \Gamma$*.*

(xi) $a \neq 0$ *iff there exists a clan* Γ *containing* a*.*

2.3 Factor Boolean algebras determined by a set of ultrafilters.

In the representation theory of dynamic mereological algebras we will need a construction of a mereological algebra from a given set of ultrafilters. The construction is taken from [7].

Let Δ be an ideal in a Boolean algebra $\underline{B}$. It is known from the theory of Boolean algebras that the relation $a \equiv_\Delta b$ iff $a \cdot b^* + a^* \cdot b \in \Delta$ is a congruence relation in $\underline{B}$ and the factor algebra $\underline{B}_{\equiv_\Delta}$ under this congruence (called also factor algebra under Δ and denoted by $\underline{B}_\Delta$) is a Boolean algebra. Denote the congruence class determined by an element a of B by $|a|_\Delta$ (or simply by $|a|$). Boolean operations in $\underline{B}_\Delta$ are defined as follows: $|a| + |b| = |a + b|$, $|a| \cdot |b| = |a \cdot b|$, $|a|^* = |a^*|$, $0 = |0|$, $1 = |1|$. Recall that Boolean ordering in $\underline{B}_\Delta$ is defined by $|a| \leq |b|$ iff $a \cdot b^* \in \Delta$ (see [8] for details).

Let $\underline{B}$ be a Boolean algebra and $\alpha \subseteq Ult(\underline{B}), \alpha \neq \emptyset$. Now we will define a construction of a Boolean algebra B_α corresponding to α. Define $I(\alpha) = \{a \in B : \alpha \cap g(a) = \emptyset\}$. It is easy to see that $I(\alpha)$ is a proper ideal in B, i.e. $1 \notin I(\alpha)$. The congruence defined by $I(\alpha)$ is denoted by $\equiv_\alpha$. So we have $a \equiv_\alpha b$ *iff* $a^* \cdot b + a \cdot b^* \in I(\alpha)$ *iff* $a^* \cdot b \in I(\alpha)$ *and* $a \cdot b^* \in I(\alpha)$. Now define $B_\alpha = \{|a|_\alpha : a \in B\}$, where $g(a) = \{U \in Ult(B) : a \in U\}$.

2.4 Contact algebras satisfying Efremovich axiom (CE). Clusters.

We will show in this section that in contact algebras satisfying the Efremovich axiom (CE) we can introduce a new kind of abstract points, called clusters. Our definition is an algebraic abstraction of the analogous notion used in the compactification theory of proximity spaces (see for instance [9]). Clusters will be used later on to define time points in dynamic mereological algebras.

Definition 2.25. [11] **Clusters.** Let (B, C) be a contact algebra. A subset $\Gamma \subseteq B$ is called a cluster in (B, C) if it is a clan satisfying the following condition:

(Cluster) *If* $a \notin \Gamma$ then there exists $b \in \Gamma$ such that $a\overline{C}b$.

The set of clusters of (B, C) is denoted by $Clusters(B)$.

Lemma 2.26. [3] *Let (B, C) be a contact algebra satisfying the Efremovich axiom (CE). Then Γ is a cluster in (B, C) iff Γ is a maximal clan in (B, C).*

Lemma 2.27. *Let (B, C) be a contact algebra satisfying the Efremovich axiom (CE). Then for any $a, b \in B : aCb$ iff there is a cluster Γ containing a and b.*

Proof. aCb iff (by Lemma 2.24) there exists a maximal clan Γ containing a *and* b. By Lemma 2.26 Γ is a cluster. □

Lemma 2.28. *Let (B, C) be a contact algebra satisfying the Efremovich axiom and let Γ and Δ be clusters. Then the following conditions are equivalent:*

(i) $\Gamma \neq \Delta$

(ii) there exists $a \in \Gamma$ *and* $b \in \Delta$ *such that* $a\overline{C}b$

(iii) there exists $c \in B$ *such that* $c \notin \Gamma$ *and* $c^* \notin \Delta$

Proof. $(i) \Rightarrow (ii)$ Suppose $\Gamma \neq \Delta$, then since they are maximal clans, there exists $a \in \Gamma$ *and* $a \notin \Delta$. Consequently, there exists $b \in \Delta$ such that $a\overline{C}b$, so (ii) is fulfilled.
$(ii) \Rightarrow (iii)$ Suppose that there exist $a \in \Gamma$ *and* $b \in \Delta$ such that $a\overline{C}b$. From $a\overline{C}b$ we obtain by the Efremovich axiom that there exists c such that $a\overline{C}c$ *and* $c^*\overline{C}b$. Conditions $a \in \Gamma$ *and* $a\overline{C}c$ imply $c \notin \Gamma$. Similarly $b \in \Delta$ *and* $c^*\overline{C}b$ imply $c^* \notin \Delta$.
$(iii) \Rightarrow (i)$ Suppose that there exists $c \in B$ such that $c \notin \Gamma$ *and* $c^* \notin \Delta$ and for the sake of contradiction that $\Gamma = \Delta$. Since $c + c^* = 1$ then either $c \in \Gamma$ *or* $c^* \in \Delta$ - a contadiction. □

Remark 2.29. In this paper we will consider Boolean algebras with several contact and precontact relations with axioms relating the different relations. Such systems are the dynamic mereological algebras to be introduced later on. The canonical structure over ultrafilters of such algebras will have a canonical binary relation R for each precontact or contact relation C defined as in the discrete representation theory of precontact algebras.

3 Dynamic model of space. Point based definitions

In this section we will introduce a concrete point-based model of space modeling regions changing in time and various temporal relations between them. Taking some true facts from this model we will arrive at the abstract definitions of various kinds of dynamic mereological algebras.

3.1 Time structures

Classical physics describes changing objects by presenting their main features as functions of time. So it presupposes that the time is given by its sets of time points identifying them with real or rational numbers with their specific arithmetic structure. This structure of the set of time points is not obligatory for all situations where we have to describe change. Very often time structures have the form of an abstract relational system of the form $(T, \prec)$, where T is a non-empty set of time points and $\prec$ is a binary relation on T such that $m \prec n$ means that m is before n. This intuition motivates to call $\prec$ before after relation or time order. Temporal structures $(T, \prec)$ of such a kind are studied in temporal logic. For instance if T is the set of real numbers the time order coincides with one of the standard ordering relations $<$ or $\leq$ of strict or partial order of numbers. In the general time structures the relation $\prec$ may satisfy various abstract properties. In the following list we describe some of them with their specific names and notations which will be used in this paper.

- **(RS)** Right seriality $(\forall m)(\exists n)(m \prec n)$
- **(LS)** Left seriality $(\forall m)(\exists n)(n \prec m)$
- **(Up Dir)** Updirectedness $(\forall i, j)(\exists k)(i \prec k \textit{ and } j \prec k)$
- **(Down Dir)** Downdirectedness $(\forall i, j)(\exists k)(k \prec i \textit{ and } k \prec j)$
- **(Dens)** Density $i \prec j \rightarrow (\exists k)(i \prec k \textit{ and } k \prec j)$
- **(Ref)** Reflexivity $(\forall m)(m \prec m)$
- **(Irr)** Irreflexivity $(\forall m)(\textit{not } m \prec m)$
- **(Lin)** Linearity $(\forall m, n)(m \prec n \textit{ or } n \prec m)$
- **(Tri)** Trichotomy $(\forall m, n)(m = n \textit{ or } m \prec n \textit{ or } n \prec m)$
- **(Tr)** Transitivity $(\forall i, j, k)(i \prec j \textit{ and } j \prec k \rightarrow i \prec k)$

We call the set of formulas (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) time conditions. Before-after relation satisfying the condition (Irr) will be called strict. If $\prec$ satisfies (Ref) the reading of m $\prec$ n should be more precise: m is equal or before n. Note that the above listed conditions for time ordering are not independent. Taking some meaningful subsets of them we obtain various notions of time order. For instance the subsets {(Ref), (Tr), (Lin)}, {(Irr), (Tr), (Tri), (Dens)}, {(Irr), (LS), (RS), (Tr), (Tri), (Dens)} are typical for the classical time, while for instance, the subset {(Ref), (Tr), (UpDir), (DownDir)} is used to characterize relativistic time.

Remark 3.1. We name the elements of the set T of a given time structure T with the more neutral name time points. Sometimes we will use the name moment of time, but it can be used also the unusual Whiteheads name epoch. In this paper we are going to use an extended version of the time structure $(T, \prec, \approx)$. The binary relation $\approx$ will acknowledge that two moments of time are proximate to each other. Obviously $\approx$ is both reflexive and symmetric relation.

3.2 Standard point-based dynamic model of space. Standard dynamic mereological algebras

Now we want to present a specific dynamic model of space based on a given time structure $(T, \prec, \approx)$. The intuition based on this model is the following. Suppose that we want to describe a dynamic environment consisting of a regions changing in time. First we suppose that we are given a time structure $\underline{\mathrm{T}} = (T, \prec, \approx)$ and want to know what is the spatial configuration of regions at each moment of time $m \in T$. We assume that for each $m \in T$ the spatial configuration of the regions forms a mereological algebra $(\underline{B}_m) = (B_m, 0_m, 1_m, \leq_m, +_m, \cdot_m, *_m)$. In other words $(\underline{B}_m)$ is a "snapshot" of this configuration. We identify a given changing region a with the series $< a_m >_{m \in T}$ of snapshots and call such a series a dynamic region. In a sense this series can be considered also as a "trajectory" or "time history" of a. We denote by $B(\underline{T})$ the set of all dynamic regions. If $a =< a_m >_{m \in T}$ is a given dynamic region then a_m can be considered as a at the time point m. The "static" region a_m will be called also the m-th coordinate of a. For instance the expression $a_m \neq 0_m$ means that a exists at the time point m. Thus $(\underline{B}_m)$ contains all m-th coordinates of the changing regions. We assume that the set $B(\underline{T})$ is a Boolean algebra, i.e. a mereology with Boolean constants and operations defined as follows: $1 =< 1_m >_{m \in T}$, $0 =< 0_m >_{m \in T}$, Boolean ordering $a \leq b$ iff $(\forall m \in T)(a_m \leq_m b_m)$ and Boolean operations are defined coordinatewise: $a + b =_{def}< a_m(+_m)b_m >_{m \in T}$, $a \cdot b =_{def}< a_m(\cdot_m)b_m >_{m \in T}$, $a^* =_{def}< a^*_m >_{m \in T}$.

The above informal reasoning suggests the following formal definition.

Definition 3.2. Formal definition of a dynamic model of space with explicit moments of time and time ordering As it is described above, we need first to have the time structure $(T, \prec, \approx)$. Then we assume that for each $m \in T$ we have a Boolean algebra $(\underline{B}_m)$, called coordinate mereological algebras. Then we take all sequences $< a_m >_{m \in T}$ and assume that their set $B(\underline{T})$ forms a Boolean algebra with the operations and relations defined coordinatewise as above. We call $B(\underline{T})$ a **dynamic model of space** over the time structure $(T, \prec, \approx)$. The elements of $B(\underline{T})$ are called **dynamic regions** and can be considered as formal analogs of Whiteheadeans processes. Note that the Boolean algebra $B(\underline{T})$ is a subalgebra of the Cartesian product $\prod_{m \in T} B_m$

of the mereological algebras $(\underline{B}_m)$, $m \in T$. A model which coincides with the Cartesian product is called a **full model**. $B(\underline{T})$ is called a **rich model** if it contains all dynamic regions a such that for all $m \in T$ we have $a_m = 0_m$ *or* $a_m = 1_m$. Obviously full models are rich.

Dynamic model of space is a quite rich spatio-temporal structure in which one can give explicit definitions of various spatio-temporal relations between dynamic regions. First we will study the following relations, the first two taken from [3]:

- **Time contact** $aC^t b$ *iff* $(\exists m \in T)(a_m \neq 0_m$ *and* $b_m \neq 0_m)$.
  Intuitively time contact between a and b means that there exists a time point in which a and b exist simultaneously. It can be considered also as a kind of simultaneity relation or contemporaneity relation studied in Whiteheads works. This suggests to call a and b contemporaries if $aC^t b$. The time contact means having a common time point when both a and b exist simultaneously.

- **Precedence** $a\mathcal{B}b$ *iff* $(\exists m, n \in T)(m \prec n$ *and* $a_m \neq 0_m$ *and* $b_n \neq 0_n)$.
  Intuitively a is in a precedence relation with b (in words a precedes b) means that there is a time point in which a exists which is before a time point in which b exists, which motivates the name of B as a precedence relation. This relation is mentioned by Whitehead without explicit formal definition.

- **Proximity** $a\mathcal{P}b$ *iff* $(\exists m, n \in T)(m \approx n$ *and* $a_m \neq 0_m$ *and* $b_n \neq 0_n)$.
  Intuitively a is in a proximity relation with b means that there is a time point in which a exists which is near to a time point in which b exists, which motivates the name of $\mathcal{P}$ as a proximity relation.

Definition 3.3. Dynamic model of space $B(\underline{T})$ supplied with the relations C^t, $\mathcal{B}$ and $\mathcal{P}$ is called a standard dynamic mereological algebra, standard DMA for short. It is called rich if the dynamic model of space is rich and full if the dynamic model of space is full.

Our aim in this paper is to give an abstract point-free characterization of dynamic mereological algebras. First we will study some formal properties of the three basic relations, which in the abstract setting will be taken as axioms.

Lemma 3.4.

(i) C^t *is a contact relation.*

(ii) If the algebra is rich then C^t *satisfies the Efremovich axiom*
(C^tE) *If* $a\overline{C}^t b$*, then there exists* c *such that* $a\overline{C}^t c$ *and* $c^*\overline{C}^t b$

(iii) $\mathcal{B}$ *is a precontact relation.*

(iv) $\mathcal{P}$ *is a contact relation.*

(v) $aC^t b \rightarrow a\mathcal{P}b$

Proof. (i) Let us check that C^t satisfies the contact relation axioms.
By definition aC^tb *iff* $(\exists m \in T)(a_m \neq 0_m$ *and* $b_m \neq 0_m)$ from which we have (C1) and (C5) axioms.

Let aC^tb *and* $a \leq a'$ *and* $b \leq b'$ then $(\exists m \in T)(a_m \neq 0_m$ *and* $b_m \neq 0_m)$.
and $a_m \leq a'_m$ which means that $a'_m \neq 0_m$. Analogously we have $b'_m \neq 0_m$ and so $a'C^tb'$ which is (C2) axiom.

Let $aC^t(b+c)$ then $(\exists m \in T)(a_m \neq 0_m$ *and* $(b+c)_m \neq 0_m)$.
Then we have that $b_m \neq 0_m$ *or* $c_m \neq 0_m$, so aC^tb *or* aC^tc which is (C3) axiom.

The proof of (C4) axiom is analogous to the one of (C3).

Let $a \cdot b \neq 0$ then $(\exists m \in T)(a_m \neq 0_m$ *and* $b_m \neq 0_m)$ implies aC^tb which is (C6) axiom.

(ii) We will verify the Efremovich axiom for C^t. Suppose the algebra is rich and let $a\overline{C}^tb$ and define c coordinatewise as follows:

$$c_k = \begin{cases} 0_k, & \text{if } a_k \neq 0_k. \\ 1_k, & \text{if } a_k = 0_k. \end{cases} \tag{1}$$

Since the algebra is rich then c exists. The verification of the conclusion $a\overline{C}^tc$ and $c^*\overline{C}^tb$ is straightforward, but still we are going to verify it as an exercise.
Let $a\overline{C}^tb$ by definition this means $(\forall m \in T)(a_m = 0_m$ *or* $b_m = 0_m)$. Let $a_m = 0$ this implies $c_m = 1_m$, i.e. $a\overline{C}^tc$. On the other side $c^*_m = 0_m$, i.e. $b\overline{C}^tc^*$. Note that the condition of richness of the algebra for the Efremovich axiom is only sufficient.

(iii) Analogous to the proof of (i) without (C5) and (C6).

(iv) Analogous to the proof of (i).

(v) Let aC^tb then $(\exists m \in T)(a_m \neq 0_m$ *and* $b_m \neq 0_m)$, but $m \approx m$ so we have $a\mathcal{P}b$.

□

3.3 A correlation between abstract properties of time structures and time axioms in standard DMA

We do not presuppose in the formal definition of dynamic model of space that the time structure $(T, \prec, \approx)$ satisfies some abstract properties of the precedence relation. In this section we shall see that all abstract properties of the time structure mentioned in Section 3.1, are in an exact correlation with some properties of time contact C^t and precedence relation $\mathcal{B}$ given in the next table:

(RS) Right seriality $(\forall m)(\exists n)(m \prec n) \iff$

(rs) $a \neq 0 \rightarrow a\mathcal{B}1$,

(LS) Left seriality $(\forall m)(\exists n)(n \prec m) \iff$

(ls) $a \neq 0 \rightarrow 1\mathcal{B}a$,

(Up Dir) Updirectedness $(\forall i, j)(\exists k)(i \prec k \textit{ and } j \prec k) \iff$

(up dir) $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}p \vee b\mathcal{B}p^*$,

(Down Dir) Downdirectedness $(\forall i, j)(\exists k)(k \prec i \textit{ and } k \prec j) \iff$

(down dir) $a \neq 0 \wedge b \neq 0 \rightarrow p\mathcal{B}a \vee p^*\mathcal{B}b$,

(Dens) Density $i \prec j \rightarrow (\exists k)(i \prec k \wedge k \prec j) \iff$

(dens) $a\mathcal{B}b \rightarrow a\mathcal{B}p \textit{ or } p^*\mathcal{B}b$,

(Ref) Reflexivity $(\forall m)(m \prec m) \iff$

(ref) $aC^tb \rightarrow a\mathcal{B}b$,

(Irr) Irreflexivity $(\forall m)(\textit{not } m \prec m) \iff$

(irr) $a\mathcal{B}b \rightarrow (\exists c, d)(c\mathcal{B}d \textit{ and } aC^tc \textit{ and } bC^td \textit{ and } c\overline{C}^td)$,

(Lin) Linearity $(\forall m, n)(m \prec n \vee n \prec m) \iff$

(lin) $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}b \vee b\mathcal{B}a$,

(Tri) Trichotomy $(\forall m, n)(m = n \textit{ or } m \prec n \textit{ or } n \prec m) \iff$

(tri) $(aC^tc \textit{ and } bC^td \textit{ and } c\overline{C}^td) \rightarrow (a\mathcal{B}b \textit{ or } b\mathcal{B}a)$,

(Tr) Transitivity $i \prec j \textit{ and } j \prec k \rightarrow i \prec k \iff$

(tr) $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

Let **B** be a standard DMA. Then the correspondence in the above table ... $\iff$ ... means that the condition from the left side is universally true in the time structure $(T, \prec, \approx)$ iff the condition from the right side is universally true in the algebra **B**.

Lemma 3.5. ***Correspondence Lemma.*** *Let **B** be a rich standard DMA with time structure $(T, \prec, \approx)$. Then all the correspondences in the above table are true.*

Proof. We will demonstrate the proofs only for two examples. The other proofs can be obtained in a similar way.

(**Lin**) $\Rightarrow$ (**lin**). Let (**Lin**) holds and suppose $a \neq 0, b \neq 0$. So $\exists i : a_i \neq 0_i$ *and* $\exists j : b_j \neq 0_j$. If we apply (**Lin**) we get that either $i \prec j$ or $j \prec i$.
Case 1: $i \prec j$ but we also have $a_i \neq 0_i$ *and* $b_j \neq 0_j$ which implies $a\mathcal{B}b$.
Case 2: $j \prec i$ but we also have $a_i \neq 0_i$ *and* $b_j \neq 0_j$ which implies $b\mathcal{B}a$.

(**Lin**) $\Leftarrow$ (**lin**). Let have rich DMA and let (**lin**) hold. For obtaining a contradiction suppose that (**Lin**) is not true:
$\exists i, j : i \nprec j$ *and* $j \nprec i$.

$$a_k = \begin{cases} 1_k, & \text{if } k = i. \\ 0_k, & \text{if } k \neq i. \end{cases} \tag{2}$$

$$b_l = \begin{cases} 1_l, & \text{if } l = j. \\ 0_l, & \text{if } l \neq j. \end{cases} \tag{3}$$

$a_i = 1_i \neq 0_i \Rightarrow a \neq 0, b_j = 1_j \neq 0_j \Rightarrow b \neq 0$.
Suppose that $a\mathcal{B}b, \exists k, l : k \prec l : a_k \neq 0_k, a_k = 1_k \Rightarrow k = i, b_l \neq 0_l, b_l = 1_l \Rightarrow l = j$.
So we have $i \prec j$, which is a contradiction.
The case $b\mathcal{B}a$ is similar.

(**Up Dir**) $\Rightarrow$ (**up dir**). Let (**Up Dir**) holds and suppose $a \neq 0, b \neq 0$.
So $\exists i : a_i \neq 0_i$ *and* $\exists j : b_j \neq 0_j$.
If we apply (**Up Dir**) we get that $\exists k : i \prec k$ *and* $j \prec k$.
Let p be arbitrary dynamic region. There are two cases:
Case 1: $p_k \neq 0_k$ we have also $a_i \neq 0_i, i \prec k \Rightarrow a\mathcal{B}p$.
Case 2: $p_k = 0_k \Rightarrow p^*_k = 1_k \neq 0_k$ and we have $b_j \neq 0_j, j \prec l \Rightarrow b\mathcal{B}p^*$.

(**Up Dir**) $\Leftarrow$ (**up dir**). Let have rich DMA and let (**up dir**) hold. For obtaining a contradiction suppose that (**Up Dir**) is not true:
$\exists i, j : \forall k : i \nprec k$ *or* $j \nprec k$. We shall define a, b, p which make (**up dir**) not true.

$$a_k = \begin{cases} 1_k, & \text{if } k = i. \\ 0_k, & \text{if } k \neq i. \end{cases} \tag{4}$$

$$b_l = \begin{cases} 1_l, & \text{if } l = j. \\ 0_l, & \text{if } l \neq j. \end{cases} \tag{5}$$

$$p_s = \begin{cases} 1_s, & \text{if } j \prec s. \\ 0_s, & \text{if } j \nprec s. \end{cases} \tag{6}$$

$k = i, a_k = a_i = 1_i \neq 0_i \Rightarrow a \neq 0, b_j = 1_j \neq 0_j \Rightarrow b \neq 0$.
Case 1: Suppose $a\mathcal{B}p$.
$\exists k : a_k \neq 0_k, \exists s : p_s \neq 0_s, p_s = 1_s \Rightarrow j \prec s, k \prec s \Rightarrow a_k = 1_k \Rightarrow k = i$.
$k \prec s \Rightarrow i \prec s$, but we have $j \prec s, k = i \Rightarrow \exists k : i \prec k \, and \, j \prec k$, which is a contradiction.
Case 2: Suppose $b\mathcal{B}p^*$.
$\exists l, s : l \prec s, b_l \neq 0_l, l = j, p^*_s \neq 0_s, p_s \neq 0^*_s = 1_s \Rightarrow p_s = 0_s \Rightarrow j \not\prec s$, $j = l$, but we have that $j \prec s$ which is a contradiction. □

The above lemma is very important because it states that the abstract properties of the time structure of a given rich standard dynamic mereological algebra are determined by some abstract properties of the relations C^t, $\mathcal{B}$ and $\mathcal{P}$ containing only variables for regions. These properties are formulated in the language of DMA and they will be taken later on as axioms in the abstract definition of DMA. That is why we call the conditions (rs), (ls),(up dir), (down dir), (dens), (ref), (irr), (lin), (tri) and (tr) time axioms.

3.4 Time representatives

Consider the phrases: the epoch of Leonardo, the epoch of Renaissance, the geological age of the dinosaurs, the time of the First World War. All these phrases indicate a concrete unit of time named by something which happened or existed at that time and not in some other time. These examples suggest to introduce in the standard dynamic mereological algebras a special set of dynamic regions called time representatives, which are regions existing at a unique time point. The formal definition is the following:

Definition 3.6. A region c in a standard DMA is called a **time representative** if there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i, c_j = 0_j$. We say also that c is a representative of the time point i and indicate this by writing $c = c(i)$. In the case when $c_i = 1_i$, c is called **universal time representative**. We denote by TR (UTR) the set of (universal) time representatives in a given standard DMA.

As the next lemma states, time representatives exist.

Lemma 3.7. *Let $B(\underline{T})$ be a rich standard DMA with time structure $(T, \prec, \approx)$. Then for each time point $i \in T$ there exist an universal time representative $c(i)$ of i. If a is a region such that $a_i \neq 0_i$ and $a_i \neq 1_i$ then $c(i) \cdot a$ is a time representative which is not universal.*

Proof. Define $c(i)$ coordinatewise as follows:

$$c(i)_k = \begin{cases} 1_k, & \text{if } k = i. \\ 0_k, & \text{if } k \neq i. \end{cases} \tag{7}$$

Obviously $c(i)$ is a universal time representative of the time point i and $c(i) \cdot a$ is a time representative which is not universal. □

Time representatives and universal time representatives satisfy some formal properties listed in the next lemma, which in the abstract definition of DMA will be taken as axioms.

Lemma 3.8. ***Properties of time representatives.*** *Let $B(\underline{T})$ be a rich standard DMA. Then the following conditions for time representatives are true:*

(TR1) $c \in TR$ *iff* $c \neq 0$ *and* $(\forall a, b)(aC^t c$ *and* $bC^t c \rightarrow aC^t b)$

(TR2) $c \in UTR$ *iff* $c \in TR$ *and* $c\overline{C}^t c^*$

(TRC^t) If $aC^t b$, *then* $(\exists c \in UTR)(aC^t c$ *and* $bC^t c)$

(TR$\mathcal{B}$1) If $c \in TR, c\mathcal{B}b$ *and* $aC^t c$, *then* $a\mathcal{B}b$

(TR$\mathcal{B}$2) If $d \in TR, a\mathcal{B}d$ *and* $bC^t d$, *then* $a\mathcal{B}b$

(TR$\mathcal{B}$3) If $a\mathcal{B}b$, *then* $\exists c \in UTR$ *such that* $c\mathcal{B}b$ *and* $aC^t c$

(TR$\mathcal{B}$4) If $a\mathcal{B}b$, *then* $\exists d \in UTR$ *such that* $a\mathcal{B}d$ *and* $bC^t d$

(TR$\mathcal{P}$1) If $c \in TR, c\mathcal{P}b$ *and* $aC^t c$, *then* $a\mathcal{P}b$

(TR$\mathcal{P}$2) If $d \in TR, a\mathcal{P}d$ *and* $bC^t d$, *then* $a\mathcal{P}b$

(TR$\mathcal{P}$3) If $a\mathcal{P}b$, *then* $\exists c \in UTR$ *such that* $c\mathcal{P}b$ *and* $aC^t c$

(TR$\mathcal{P}$4) If $a\mathcal{P}b$, *then* $\exists d \in UTR$ *such that* $a\mathcal{P}d$ *and* $bC^t d$

Proof.
We will verify only (TR1) and (TR$\mathcal{P}$1) conditions, all others can be proven similiarly.
(**TR1**) ($\Rightarrow$) Let $c \in TR$ then there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i, c_j = 0_j$.
If $aC^t c$ *and* $bC^t c$ then $a_i \neq 0_i$ *and* $b_i \neq 0_i$ thus $aC^t b$.
(**TR1**) ($\Leftarrow$) Let $c \neq 0$ *and* $(\forall a, b)(aC^t c$ *and* $bC^t c \rightarrow aC^t b)(*)$.
There exist a time point i in which $c_i \neq 0_i$, lets assume that there is another time point j ($j \neq i$) in which $c_j \neq 0_j$. From $c_i \neq 0_i$ and $(*)$ we have that $a_i \neq 0_i$ and $b_i \neq 0_i$. From $c_j \neq 0_j$ and $(*)$ we have that $a_j \neq 0_j$ and $b_j \neq 0_j$. By given we have that if $c \neq 0$, $aC^t c$ and $bC^t c$ then $aC^t b$. So for $c_i \neq 0_i$, $c_j \neq 0_j$, $a_i \neq 0_i$ and $b_j \neq 0_j$, we have $c \neq 0$ and $aC^t c$ and $bC^t c$ then $aC^t b$, but this is a contradiction because $i \neq j$. The contradiction is due to the assuming that there is a second time point in which $c \neq 0$. Now having that $c_i \neq 0_i$ only for unique $i \in T$ and $c_j = 0_j$ for all other time points j ($j \neq i$), we can conclude that $c \in TR$.
(**TR$\mathcal{P}$1**) ($\Rightarrow$) Let $c \in TR, c\mathcal{P}b$ *and* $aC^t c$, then there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i, c_j = 0_j$.
From the statement above and $aC^t c$ we have that $a_i \neq 0_i$ and from $c\mathcal{P}b$ we have that $\exists k \in T$ such that $b_k \neq 0_k$ and $i \approx k$. From $a_i \neq 0_i$ and $b_k \neq 0_k$ and $i \approx k$ we have $a\mathcal{P}b$. □

The following lemma lists some additional properties of time representatives.

Lemma 3.9. *If $c(k)$ and $d(l)$ are universal time representatives correspondingly for k and l then:*

(i) $aC^t c(k)$ *iff* $a \cdot c(k) \neq 0$

(ii) $k = l$ *iff* $c(k)C^t d(l)$ *iff* $c(k) \cdot d(l) \neq 0$

(iii) $k \prec l$ *iff* $c(k)\mathcal{B}d(l)$

(iv) $k \approx l$ *iff* $c(k)\mathcal{P}d(l)$

Proof. The proof is straightforward following of the defintions, but still we are going to verify only the first statement as an exercise.
($\Rightarrow$) Let $aC^t c(k)$ then by definition there exists a time point $i \in T$ such that $a_i \neq 0_i$ *and* $c(k)_i \neq 0_i$. We know also that $c(k)$ is a universal time representative, thus we have that $i = k$ and more over $c(k) = 1_k$ from which we can conclude that $a_k \neq 0_k$ *and* $a_k \leq c_k$ and so $a \cdot c(k) \neq 0$.

($\Leftarrow$) Let $a \cdot c(k) \neq 0$, now applying (C6) axiom for the time contact C^t we have $aC^t c(k)$.

□

The above lemma suggests the following translation τ from the first order language of time structures into the language of DMA-s as follows. If i is a variable for a time point let $c(i)$ be a variable denoting universal time representative and let for different i and j, $c(i)$ and $c(j)$ be also different. Then replace all atomic formulas of the form $i = j$, $i \prec j$ and $i \approx j$ with the formulas $c(i)C^t c(j)$, $c(i)\mathcal{B}c(j)$ and $c(i)\mathcal{P}c(j)$ respectively. Example: $A = (\forall i)(\exists j)(i \prec j), \tau(A) = (\forall c(i))(\exists c(j))(c(i)\mathcal{B}c(j))$.

Lemma 3.10. ***Translation Lemma.*** *Let $B(\underline{T})$ be a rich standard DMA with time structure $(T, \prec, \approx)$. Then for any first-order formula A in the language of $(T, \prec, \approx)$ we have that: A is universally true in $(T, \prec, \approx)$ iff $\tau(A)$ is universally true in DMA. In particular, for all formulas A from the set {(RS), (LS),(Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri) and (Tr)} we have A is true in $(T, \prec, \approx)$ iff $\tau(A)$ is true $B(\underline{T})$.*

Proof. The proof follows by induction on the complexity of the formula A by the use of Lemma 3.9. □

Lemma 3.10 suggests to call the translations of the formulas of the list of time conditions (RS), (LS),(Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri) and (Tr) also time axioms. As a corollary from Lemma 3.10 we obtain:

Corollary 3.11. *Let $B(\underline{T})$ be a rich standard DMA and let A be any formula from the list of time conditions (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) and α be the corresponding formula from the list of time axioms (rs), (ls),* *(up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). Then* $\tau(A)$ *is true in* $B(\underline{T})$ *iff* α *is true in* $B(\underline{T})$*. (The correspondence between the two lists is obvious by the notations, for instance (down dir) corresponds to (Down Dir)).*

The following lemma lists some additional properties of universal time representatives, which will be taken as axioms in the abstract definition of DMA.

Lemma 3.12. *Let* $B(\underline{T})$ *be a rich standard DMA with time structure* $(T, \prec, \approx)$ *and let* $c(i)$ *and* $c(j)$ *be the universal time representatives for the time points* i *and* j *respectively. Then the following conditions are true:*

$(UTR\mathcal{B}11)$ $(a)(\forall p \in B)(p\mathcal{B}c(i)$ *or* $p^*\mathcal{B}c(j))$ *iff*

$(b)(\exists k \in T)(k \prec i$ *and* $k \prec j)$ *iff*

$(c)(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ *and* $c(k)\mathcal{B}c(j))$.

$(UTR\mathcal{B}12)$ $(a)(\forall p \in B)(p\mathcal{B}c(i)$ *or* $c(j)\mathcal{B}p^*)$ *iff*

$(b)(\exists k \in T)(k \prec i$ *and* $j \prec k)$ *iff*

$(c)(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ *and* $c(j)\mathcal{B}c(k))$.

$(UTR\mathcal{B}21)$ $(a)(\forall p \in B)(c(i)\mathcal{B}p$ *or* $p^*\mathcal{B}c(j))$ *iff*

$(b)(\exists k \in T)(i \prec k$ *and* $k \prec j)$ *iff*

$(c)(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ *and* $c(k)\mathcal{B}c(j))$.

$(UTR\mathcal{B}22)$ $(a)(\forall p \in B)(c(i)\mathcal{B}p$ *or* $c(j)\mathcal{B}p^*)$ *iff*

$(b)(\exists k \in T)(i \prec k$ *and* $j \prec k)$ *iff*

$(c)(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ *and* $c(j)\mathcal{B}c(k))$.

Proof. Since all four cases of the lemma are similar we will prove only one case, for instance $(UTR\mathcal{B}11)$. The proof of the other cases can be obtained in a similar way (note the numbering of the statements which reflects the internal structure of the formulas and can help how to obtain the proofs for the other cases by analogy).
$(UTR\mathcal{B}11)$ $(a) \Leftarrow (b)$ Suppose that $(\exists k \in T)(k \prec i$ *and* $k \prec j)$ and let p be an arbitrary element of B.
Case 1: $p_k \neq 0_k$, then by $k \prec i$ we obtain $p\mathcal{B}c(i)$.
Case 2: $p_k = 0_k$, then $p_k^* = 1_k \neq 0_k$ *and* $k \prec j$ we obtain $p^*\mathcal{B}c(j)$.
$(a) \Rightarrow (b)$. Suppose (a) holds and for the sake of contradiction suppose that (b) is not true. Define p coordinatewise as follows:

$$p_k = \begin{cases} 1_k, & \text{if } k \prec j. \\ 0_k, & \text{if } k \not\prec j. \end{cases} \tag{8}$$

It can easily be verified that this p makes the assumption (a) not true - a contradiction.

$(b) \Rightarrow (c)$. Suppose (b) is fulfilled and let k_0 be such that $k_0 \prec i \textit{ and } k_0 \prec j$ and let $c(k_0)$ be the universal time representative for k_0. Then $c(k) = c(k_0)$ verifies $(c)(c(k)\mathcal{B}c(i) \textit{ and } c(k)\mathcal{B}c(j))$.

$(b) \Leftarrow (c)$. Let the condition (c) be satisfied and take $c(k)$ to be one element of UTR satisfying (c). Then by Lemma 3.9 (iii) is fullfilled by k. □

3.5 Definability of some notions of Whiteheadean type by means of time representatives

In this section we will show that many new spatio-temporal relations have direct definitions by the use of time representatives.

3.5.1 Society of contemporaries

Time representatives make possible to extend time contact C^t, precedence relation $\mathcal{B}$ and proximity relation $\mathcal{P}$ for arbitrary non-empty sets of dynamic regions and to define the notions of a society of contemporaries, also precedence and proximity between two societies of contemporaries. Note that the notion of society is one of the important notions of Whiteheads epochal theory of time. First we give the relevant definitions in a given dynamic model of space $B(\underline{T})$:

Definition 3.13. Let A and B be nonempty sets of dynamic regions in B($\underline{\text{T}}$). Then we define:

- A is a **society of contemporaries**, $C^t(A)$ in symbols $\Leftrightarrow_{def}$ there exists $i \in T$ such that for every $a \in A : a_i \neq 0_i$.
- A is in a **precedence relation** with B in symbols $A\mathcal{B}B \Leftrightarrow_{def}$ there exist $i, j \in T$ such that $i \prec j$ and for every $a \in A \textit{ and } b \in B, a_i \neq 0_i \textit{ and } b_j \neq 0_j$.
- A is in a **proximity relation** with B in symbols $A\mathcal{P}B \Leftrightarrow_{def}$ there exist $i, j \in T$ such that $i \approx j$ and for every $a \in A \textit{ and } b \in B, a_i \neq 0_i \textit{ and } b_j \neq 0_j$.

Lemma 3.14. ***Definability Lemma.*** *Let $B(\underline{T})$ be a rich standard DMA and A, B be non-empty sets of dynamic regions. Then:*

(i) $C^t(A) \Leftrightarrow_{def} (\exists c \in TR)(\forall a \in A)(aC^tc)$

(ii) $A\mathcal{B}B \Leftrightarrow_{def} (\exists c, d \in TR)(c\mathcal{B}b \textit{ and } (\forall a \in A)(aC^tc) \textit{ and } (\forall b \in B)(bC^td))$

(iii) $A\mathcal{P}B \Leftrightarrow_{def} (\exists c, d \in TR)(c\mathcal{P}b \textit{ and } (\forall a \in A)(aC^tc) \textit{ and } (\forall b \in B)(bC^td))$

Proof. Straightforward by the corresponding definitions, but still we are going to verify only the first statement as an exercise.

($\Rightarrow$) Let $C^t(A)$ by definition there exists $i \in T$ such that for every $a \in A : a_i \neq 0_i$ and we also know by Lemma 3.7 that for every time point $i \in T$ there exists $c(i)$ a time representative. $a_i \neq 0_i$ *and* $c(i) \neq 0_i$ so we have $aC^t c(i)$ for every $a \in A$.

($\Leftarrow$) Let $(\exists c \in TR)(\forall a \in A)(aC^t c)$, so there is a $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i$, $c_j = 0_j$. By the definition of C^t and $aC^t c$ we know that there $(\exists k \in T)(a_k \neq 0_k \textit{ and } c_k \neq 0_k)$. So we have that $k = i$ which implies $a_i \neq 0_i$ for all $a \in A$ and $C^t(A)$. □

3.5.2 Present, Past and Future

Whitehead very often is talking in [5, 6] about Present epoch, Present cosmic epoch, Contemporary World, Actual World considering all these phrases as synonyms. In the common language Present epoch is just the state of all things which exist now. To represent the present epoch we introduce a special time representative named NOW and the point of time representing by NOW is denoted by **now**. Considering **now** requires to extend the signature of time structure - $(T, \prec, \approx, \mathbf{now})$ assuming $\mathbf{now} \in T$. By means of NOW we can define many interesting notions which in the standard DMA-s indeed have their expected meaning. For this definitions in the abstract DMA-s we have to postulate that $NOW \in TR$.

- a exists now $\Leftrightarrow_{def}$ $aC^t NOW$
- a exists sometimes in the future $\Leftrightarrow_{def}$ $NOW\mathcal{B}a$
- a exists sometimes in the near future $\Leftrightarrow_{def}$ $NOW\mathcal{B}a \textit{ and } NOW\mathcal{P}a$
- a exists sometimes in the remote future $\Leftrightarrow_{def}$ $NOW\mathcal{B}a \textit{ and } NOW\overline{\mathcal{P}}a$
- a exists always in the future $\Leftrightarrow_{def}$ $(\forall b \in TR)(NOW\mathcal{B}b \rightarrow aC^t b)$
- a exists always $\Leftrightarrow_{def}$ $(\forall b \in TR)(aC^t b)$
- a exists sometimes in the past $\Leftrightarrow_{def}$ $a\mathcal{B}NOW$
- a exists sometimes in the near past $\Leftrightarrow_{def}$ $a\mathcal{B}NOW \textit{ and } a\mathcal{P}NOW$
- a exists sometimes in the remote past $\Leftrightarrow_{def}$ $a\mathcal{B}NOW \textit{ and } a\overline{\mathcal{P}}NOW$
- a exists always in the past $\Leftrightarrow_{def}$ $(\forall b \in TR)(b\mathcal{B}NOW \rightarrow aC^t b)$

4 Dynamic mereological algebras

4.1 Abstract definition

Taking into account Lemma 3.4 and Lemma 3.8 we present the following abstract definition of dynamic mereological algebra:

Definition 4.1. By a dynamic mereological algebra (DMA for short) we mean any system $\underline{B} = (B, 0, 1, \cdot, +, *, C^t, \mathcal{B}, \mathcal{P}, TR, UTR, NOW)$ where $(B, 0, 1, \cdot, +, *)$ is a non-degenerate Boolean algebra and the following condition is satisfied:

(i) C^t is a contact relation on B, called **time contact** satisfying the axiom: (C^tE) If $a\overline{C}^tb$ then $(\exists c)(a\overline{C}^tb$ *and* $c^*\overline{C}^tb)$

(ii) $\mathcal{B}$ is a precontact relation in B called **precedence relation**

(iii) $\mathcal{P}$ is a contact relation in B called **proximity relation**

(iv) TR - **time representatives** and UTR - **universal time representatives** are subsets of B satisfying the following axioms:

$(TR1)$ $c \in TR$ *iff* $c \neq 0$ *and* $(\forall a, b)(aC^tc$ *and* $bC^tc \rightarrow aC^tb)$.

$(TR2)$ $c \in UTR$ *iff* $c \in TR$ *and* $c\overline{C}^tc^*$

(TRC^t) *If* aC^tb *then* $(\exists c \in UTR)(aC^tc$ *and* $bC^tc)$

$(TR\mathcal{B}1)$ *If* $c \in TR,$ $c\mathcal{B}b$ *and* aC^tc *then* $a\mathcal{B}b$.

$(TR\mathcal{B}2)$ *If* $d \in TR,$ $a\mathcal{B}d$ *and* bC^td *then* $a\mathcal{B}b$.

$(TR\mathcal{B}3)$ *If* $a\mathcal{B}b,$ *then* $\exists c \in UTR$ *such that* $c\mathcal{B}b$ *and* aC^tc

$(TR\mathcal{B}4)$ *If* $a\mathcal{B}b,$ *then* $\exists d \in UTR$ *such that* $a\mathcal{B}d$ *and* bC^td

$(TR\mathcal{P}1)$ *If* $c \in TR,$ $c\mathcal{P}b$ *and* aC^tc *then* $a\mathcal{P}b$.

$(TR\mathcal{P}2)$ *If* $d \in TR,$ $a\mathcal{P}d$ *and* bC^td *then* $a\mathcal{P}b$.

$(TR\mathcal{P}3)$ *If* $a\mathcal{P}b,$ *then* $\exists c \in UTR$ *such that* $c\mathcal{P}b$ *and* aC^tc

$(TR\mathcal{P}4)$ *If* $a\mathcal{P}b,$ *then* $\exists d \in UTR$ *such that* $a\mathcal{P}d$ *and* bC^td
In the next axioms $c(i)$ *and* $c(j)$ are arbitrary elements of UTR.

$(UTR\mathcal{B}11)$ $(\forall p \in B)(p\mathcal{B}c(i)$ *or* $p^*\mathcal{B}c(j))$ *iff*

$$(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i) \text{ and } c(k)\mathcal{B}c(j)).$$

$(UTR\mathcal{B}12)$ $(\forall p \in B)(p\mathcal{B}c(i)$ *or* $c(j)\mathcal{B}p^*)$ *iff*

$(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ *and* $c(j)\mathcal{B}c(k))$.

$(UTR\mathcal{B}21)$ $(\forall p \in B)(c(i)\mathcal{B}p$ *or* $p^*\mathcal{B}p^*)$ *iff*

$(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ *and* $c(k)\mathcal{B}c(j))$.

$(UTR\mathcal{B}22)$ $(\forall p \in B)(c(i)\mathcal{B}p$ *or* $c(j)\mathcal{B}p^*)$ *iff*

$(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ *and* $c(j)\mathcal{B}c(k))$.

$(UTRNOW)$ $NOW \in UTR$.

We consider also DMA-s satisfying some of the **time axioms** (rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri) and (tr).

The elements of B are called dynamic regions and according to Whiteheads process philosophy they can be considered as formal analogs of processes. For the intuitions of TR, UTR and NOW see Section 3.4. All definitions by means of NOW and time representatives from Section 3.5 can be repeated in the context of the present abstract definition of DMA and by the representation theory of DMA-s they will obtain their expected meaning. Since DMA-s are algebraic systems, we adopt for them the standard definitions of subalgebra, homomorphism, isomorphism, isomorphic embedding, etc. Note that axioms $(TR1)$ and $(TR2)$ show that the sets TR and UTR are definable with first-order formulas of the relation C^t. We however include these sets in the signature of DMA, because we want they to be preserved by the embeddings in the representation theory of DMA-s.

Lemma 4.2. *Let $\underline{B}$ be a DMA. Then the following holds for every $a, b \in TR$ and $c, d \in B$: If aC^tc, bC^td and aC^tb, then cC^td.*

Proof. Trivial by axiom $(TR1)$. □

The following lemma presents some important properties of universal time representatives.

Lemma 4.3. ***Properties of UTR.*** *Let $\underline{B}$ be a DMA. Then:*

(i) *If* $c \in UTR$, *then for every* $a \in B : aC^tc \iff a \cdot c \neq 0$

(ii) aC^tb *iff* $(\exists c \in UTR)((a \cdot c)C^t(b \cdot c))$

(iii) $a\mathcal{B}b$ *iff* $(\exists c \in UTR)((a \cdot c)\mathcal{B}b)$

(iv) $a\mathcal{B}b$ *iff* $(\exists d \in UTR)(a\mathcal{B}(b \cdot d))$

(v) $a\mathcal{B}b$ *iff* $(\exists c, d \in UTR)((a \cdot c)\mathcal{B}(b \cdot d))$

(vi) $a\mathcal{P}b$ *iff* $(\exists c \in UTR)((a \cdot c)\mathcal{P}b)$

(vii) $a\mathcal{P}b$ *iff* $(\exists d \in UTR)(a\mathcal{P}(b \cdot d))$

(viii) $a\mathcal{P}b$ *iff* $(\exists c, d \in UTR)((a \cdot c)\mathcal{P}(b \cdot d))$

(ix) $a \neq 0$ *iff* $(\exists c \in UTR)(a \cdot c \neq 0)$

(x) *If* $a, b, c \in UTR$ *then* $a \cdot c \neq 0, b \cdot d \neq 0$ *and* $c \cdot b \neq 0$ *imply* $c \cdot d \neq 0$

Proof. (i) Let $c \in UTR$. Then by axiom $(TR2)$ we have $c\overline{C}^t c^*$. First we will establish the first equivalence.

$(\Rightarrow)$ Suppose aC^tc and for the sake of contradiction that $a \cdot c = 0$. This implies $a \leq c^*$ and by aC^tc this implies c^*C^tc, and since C^t is a contact relation we have cC^tc^* - a contradiction with $c\overline{C}^t c^*$.

$(\Leftarrow)$ Suppose $a \cdot c \neq 0$, since C^t is a contact relation, then aC^tc.

(ii) $(\Rightarrow)$ Suppose aC^tb. By axiom (TRC^t) there exists $c \in UTR$ such that aC^tc *and* bC^tc. By (i) we obtain $(a \cdot c) \cdot c = a \cdot c \neq 0$ *and* $(b \cdot c) \cdot c = b \cdot c \neq 0$. Again by (i) we obtain $(a \cdot c)C^tc$ *and* $(b \cdot c)C^tc$, which by axiom $(TR1)$ implies $(a \cdot c)C^t(b \cdot c)$.

$(\Leftarrow)$ Obvious.

(iii) $(\Rightarrow)$ Let $a\mathcal{B}b$. Then axiom $(TR\mathcal{B}3)$ implies $\exists c \in UTR$ such that $c\mathcal{B}b$, aC^tc. As in (ii) we deduce from aC^tc that $(a \cdot c)C^tc$. Then $c\mathcal{B}b$ and $(a \cdot c)C^tc$ imply by axiom $(TR\mathcal{B}1)$ $(a \cdot c)\mathcal{B}b$.

$(\Leftarrow)$ Obvious.

(iv) The proof is analogous to the proof of (iii).

(v) Follows from (iii) *and* (iv).

(vi) Similarly to (iii).

(vii) Similarly to (iv).

$(viii)$ Similarly to (v).

(ix) Having in mind that $a \neq 0$ is equivalent to aC^ta, the proof of this condition follows from (ii).

(x) Follows from Lemma 4.2 and (i). □

4.2 Expressing time axioms by universal time representatives

We introduced in Section 3.4 a translation τ which translates formulas from the language of time structures into formulas of the language of DMA containing only variables for universal time representatives. Corollary 3.11 states that each formula α from the list of time axioms (see Section 3.3) is equivalent in rich standard DMA-s to the formula $\tau(A)$, where A is the corresponding formula from the list of time conditions (see Section 3.1). We will prove in this section the same equivalence on the base of the abstract DMA using its axioms.

Proposition 4.4. *Let* $B(\underline{T})$ *be DMA and let* A *be any formula from the list of time conditions (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) and* α *be the corresponding formula from the list of time axioms (rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). Then* $\tau(A)$ *is true in* $B(\underline{T})$ *iff* α *is true in* $B(\underline{T})$.

Proof. The proof goes by considering separate cases. We will use many times the axioms for UTR and Lemma 4.3 sometimes without a direct reference. Let us recall the translation τ: for time variable $\tau(m) = c(m)$, where $c(m)$ is a variable running in UTR, $\tau(m = n) = c(m) \cdot c(n) \neq 0$ *and* $\tau(m \prec n) = c(m)\mathcal{B}c(n)$. For simplicity we replace different variables $c(i)$ by different letters from the Latin alphabet: a, b, c etc. We shall use in this proof many times the axioms for TR and Lemma 3.9, sometimes without explicit reference to them.

- Proof of $(rs) \Leftrightarrow \tau(RS)$. Recall that (rs) is $a \neq 0 \rightarrow a\mathcal{B}1, \tau(RS)$ is $(\forall a \in UTR)(\exists b \in UTR)(a\mathcal{B}b)$

  $(rs) \Rightarrow \tau(RS)$. Suppose (rs) and let $a \in UTR$. Then by axiom $(TR2)$ $a \neq 0$ and by (rs) we obtain $a\mathcal{B}1$. By Lemma 4.3 there exists $b \in UTR$ such that $a\mathcal{B}(1 \cdot b)$, which implies $a\mathcal{B}b$, thus $\tau(RS)$ is proved.

  $(rs) \Leftarrow \tau(RS)$. Suppose $\tau(RS)$ and for the proof of (rs) suppose $a \neq 0$. Then by Lemma 4.3 there exists $c \in UTR$ such that $a \cdot c \neq 0$, which implies aC^tc. By $\tau(RS)$ there exists $b \in UTR$ such that $c\mathcal{B}b$. Conditions $c\mathcal{B}b, aC^tc$ imply by axiom $(TR\mathcal{B}1)$ that $a\mathcal{B}b$.

- Proof of $(ls) \Leftrightarrow \tau(LS)$. Analogous to the above proof.

- Proof of $(up\ dir) \Leftrightarrow \tau(Up\ Dir)$. Recall that $(up\ dir)$ is $a \neq 0 \ \wedge \ b \neq 0 \rightarrow a\mathcal{B}p \ \vee \ b\mathcal{B}p^*$ and $\tau(Up\ Dir) = (\forall a, b \in UTR)(\exists c \in UTR)(a\mathcal{B}c \textit{ and } b\mathcal{B}c)$.

  $(up\ dir) \Rightarrow \tau(Up\ Dir)$. Suppose $(up\ dir)$ and let $a, b \in UTR$. Then $a \neq 0 \textit{ and } b \neq 0$ and by $(up\ dir)$ we obtain $(\forall p \in B)(a\mathcal{B}p \textit{ or } b\mathcal{B}p^*)$. Applying the axiom $(UTR\mathcal{B}22)$ we obtain that $(\exists c \in UTR)(a\mathcal{B}c \textit{ and } b\mathcal{B}c$, which proves $\tau(Up\ Dir)$.

  $(up\ dir) \Leftarrow \tau(Up\ Dir)$. Suppose $\tau(Up\ Dir)$ and to prove $(up\ dir)$ suppose $a \neq 0 \textit{ and } b \neq 0$. Then there are $a', b' \in UTR$ such that $aC^ta' \textit{ and } (bC^tb'$. By $\tau(Up\ Dir)$ we obtain $(\exists c \in UTR)(a'\mathcal{B}c \textit{ and } b'\mathcal{B}c)$. Conditions $aC^ta' \textit{ and } a'\mathcal{B}c$ imply $a\mathcal{B}c$. Analogously $(bC^tb' \textit{ and } b'\mathcal{B}c$ imply $b\mathcal{B}c$ and hence $\tau(Up\ Dir)$ is proved.

- Proof of $(down\ dir) \Leftrightarrow \tau(Down\ Dir)$. Analogous to the proof of $(up\ dir) \Leftrightarrow \tau(Up\ Dir)$ by the use of axiom $(UTR\mathcal{B}11)$.

- Proof of $(dense) \Leftrightarrow \tau(Dense)$. Analogous to the proof of $(up\ dir) \Leftrightarrow \tau(Up\ Dir)$ by the use of axiom $(UTR\mathcal{B}21)$.

- Proof of $(ref) \Leftrightarrow \tau(Ref)$. Recall that (ref) is $aC^tb \rightarrow a\mathcal{B}b \textit{ and } \tau(Ref)$ is $(\forall a \in UTR)(a\mathcal{B}a)$.

  $(ref) \Rightarrow \tau(Ref)$. Suppose (ref) and let $a \in UTR$. Then $a \neq 0$ and by (ref) we obtain $a\mathcal{B}a$, hence $\tau(Ref)$ holds.

$(ref) \Leftarrow \tau(Ref)$. Suppose $\tau(Ref)$ and let $a \neq 0$. Then there exists $a' \in UTR$ such that aC^ta'. By $\tau(Ref)$ we have $a'\mathcal{B}a'$. This condition together with aC^ta' and axioms $(TR\mathcal{B}1)$ *and* $(TR\mathcal{B}2)$ imply $a\mathcal{B}a$, so (ref) holds.

- Proof of $(irr) \Leftrightarrow \tau(Irr)$. Recall that (irr) is $a\mathcal{B}b \rightarrow (\exists c, d)$ $(c\mathcal{B}d$ *and* aC^tc *and* bC^td *and* $c\overline{C}^td)$ *and* $\tau(Irr)$ is $(\forall a \in UTR)(a\overline{\mathcal{B}}a)$. Note that another equivalent form of (Irr)(which we also denote by (Irr)) is: $(\forall i, j \in T)(i \prec j \rightarrow i \neq j)$. This condition translated by τ is $(\forall a, b \in UTR)(a\mathcal{B}b \rightarrow a\overline{C}^tb)$. We will use below this condition.

  $(irr) \Rightarrow \tau(irr)$. Suppose (irr) and assume that $a\mathcal{B}b$ for some $a, b \in UTR$. Then by (irr) we obtain $(\exists c, d \in B)(c\mathcal{B}d$ *and* aC^tc *and* bC^td *and* $c\overline{C}^td)$. By Lemma 4.2 conditions aC^tc *and* bC^td *and* $c\overline{C}^td$ imply by Lemma 4.2 that $a\overline{C}^tb$, hence $\tau(Irr)$ holds.

  $(irr) \Leftarrow \tau(irr)$. Let $\tau(Irr)$ and suppose $a\mathcal{B}b$. Then by Lemma 4.3 (v) $(\exists c, d \in UTR)(a \cdot c)\mathcal{B}(b \cdot d)$. From here we obtain: $a \cdot c \neq 0$ and consequently $aC^tc, b \cdot c \neq 0$ and consequently bC^td *and* $c\mathcal{B}d$. The last condition implies by $\tau(Irr)$ that $c\overline{C}^td$. Thus we have just proved that $a\mathcal{B}b$ implies $(\exists c, d \in UTR)$ $(c\mathcal{B}d$ *and* aC^tc *and* bC^td *and* $c\overline{C}^td)$ - just what has to be proved.

- Proof of $(lin) \Leftrightarrow \tau(Lin)$. Analogous to the proof of $(ref) \Leftrightarrow \tau(Ref)$.

- Proof of $(tri) \Leftrightarrow \tau(Tri)$. Analogous to the proof of $(irr) \Leftrightarrow \tau(Irr)$, by an application of Lemma 4.2.

- Proof of $(tr) \Leftrightarrow \tau(Tr)$. Analogous to the proof of $(up\ dir) \Leftrightarrow \tau(Up\ Dir)$, by an application of axiom $(UTR\mathcal{B}21)$.

□

4.3 Ultrafilters and clusters in DMA

Let $\underline{B} = (B, 0, 1, \cdot, +, *, C^t, \mathcal{B}, \mathcal{P}, TR, UTR, NOW)$ be a DMA. We denote by $Ult(B)$ the set of ultrafilters in B. We denote by $R^t, \prec$ *and* $\approx$ the canonical relations between ultrafilters defined as in Section 2.2, corresponding respectively to $C^t, \mathcal{B}$ *and* $\mathcal{P}$. Since C^t is a contact relation satisfying the Efremovich axiom, R^t is an equivalence relation (Section 2.4). Clusters with respect to the contact C^t will be called t-clusters.

Remark 4.5. Let us note that ultrafilters and t-clusters will be taken as building material in the canonical constructions in the representation theory of DMA-s: ultrafilters as time atoms, t-clusters (build by ultrafilters) as time points.

Definition 4.6. TR- and UTR-clusters. Let $\underline{B}$ be a DMA and Γ be a t-cluster in B. Γ is called a TR-cluster if there exists a time representative $c \in TR$ such that $c \in \Gamma$. If $c \in UTR$ then Γ is called UTR-cluster. If Γ is a TR-cluster (UTR-cluster) and $c \in TR$ $(c \in UTR)$ is one of its time representatives, we will denote this by $\Gamma(c)$. We denote by

$Ult(\Gamma)$ the set of ultrafilters included in Γ. The set of all UTR-clusters (TR-clusters) in $\underline{B}$ is denoted by $UTR - clusters(B)$ $(TR - clusters(B))$. Note that $Ult(\Gamma)$ is always non-empty set.

Lemma 4.7.

(i) Every UTR-cluster is a TR-cluster

(ii) Let $\Gamma(c)$ and $\Delta(c)$ be TR-clusters, then $\Gamma(c) = \Delta(c)$

(iii) Let $\Gamma(c)$ be a TR-cluster, then $a \in \Gamma(c)$ iff aC^tc

(iv) Let $\Gamma(c)$ and $\Delta(d)$ be TR-clusters, then $\Gamma(c) = \Delta(d)$ iff cC^td

(v) Let $\Gamma(c)$ be an UTR-cluster, then for every ultrafilter $U \subseteq \Gamma$ we have $c \in U$

(vi) Let $\Gamma(c)$ be an UTR-cluster, then $a \in \Gamma(c)$ iff $a \cdot c \neq 0$

Proof. (i) follows from the fact that $UTR \subseteq TR$

(ii) Let $\Gamma(c)$ *and* $\Delta(c)$ be TR-clusters and proceed to show that $\Gamma(c) \subseteq \Delta(c)$. Suppose for the sake of contradiction that $\Gamma(c) \nsubseteq \Delta(c)$. Then there exists $a \in \Gamma(c)$ (and consequently aC^tc) and $a \notin \Delta(c)$. By the definition of cluster there exists $b \in \Delta(c)$ (and consequently bC^tc) such that $a\overline{C}b$. Conditions $c \in TR, aC^tc$ *and* bC^tc imply by axioms $(TR1)$ that aC^tb, which is a contradiction.

(iii) Let $\Gamma(c)$ be a TR-cluster (this implies that $c \in TR$). Then $a \in \Gamma(c)$ obviously implies aC^tc. Let now aC^tc, then by Lemma 2.27 there exists a t-cluster Δ containing c *and* a. Since $c \in TR$, this implies that $\Delta = \Delta(c)$ is a TR-cluster. This implies by (ii) that $\Gamma(c) = \Delta(c)$, hence $a \in \Gamma(c)$.

(iv) Let $\Gamma(c)$ *and* $\Delta(d)$ be TR-clusters. If $\Gamma(c) = \Delta(d)$, then obviously cC^td. For the converse implication suppose cC^td. Then by (iii) $d \in \Gamma(c)$, so we may write $\Gamma(c) = \Gamma(d)$. Then by (ii) we obtain $\Gamma(c) = \Delta(d)$.

(v) Let $\Gamma(c)$ be an UTR-cluster and U be an ultrafilter such that $U \subseteq \Gamma$. By the axiom $(TR2)c \in UTR$ implies $c\overline{C}c^*$. Then conditions $c \in \Gamma$ *and* $\overline{C}c^*$ imply $c^* \notin \Gamma$ and since $U \subseteq \Gamma$, then $c^* \notin U$, so $c \in U$.

(vi) follows from (iii) and Lemma 4.3. □

The following lemma shows further important properties of UTR-clusters.

Lemma 4.8.

(i) aC^tb iff there exists an UTR-cluster Γ such that $a, b \in \Gamma$ iff there exist an UTR-cluster Γ and ultrafilters $\Delta, \Theta \in Ult(\Gamma)$ such that $a \in \Delta$ and $b \in \Theta$.

(ii) $a \neq 0$ iff there exists an UTR-cluster Γ containing a iff there exists an UTR-cluster Γ and an ultrafilter $\Delta \in Ult(\Gamma)$ such that $a \in \Delta$.

(iii) $a \in TR$ iff there exists a unique UTR-cluster Γ containing a iff there exists unique UTR-cluster Γ and an ultrafilter $\Delta \in Ult(\Gamma)$ such that $a \in \Delta$.

(iv) $a \in UTR$ *iff there exists a unique* UTR*-cluster* Γ *such that for all* $\Delta \in Ult(\Gamma), a \in \Delta$.

(v) There exists a unique UTR*-cluster (denoted by* ***now****) containing* NOW.

Proof.

(i) Proof of the first equivalence.

$(\Rightarrow)$ Let aC^tb. By Lemma 4.3 (ii) there exists $c \in UTR$ such that $(a \cdot c)C^t(b \cdot c)$. By Lemma 2.27 there is a t-cluster Γ containing a, b *and* c and since $c \in UTR$ then $\Gamma = \Gamma(c)$ is an UTR-cluster.

$(\Leftarrow)$ Obvious.

Proof of the second equivalence.

$(\Rightarrow)$ Suppose that there exists an UTR-cluster Γ such that $a, b \in \Gamma$. Since Γ is union of ultrafilters, there are ultrafilters $U, V \subseteq \Gamma$ such that $a \in U$ *and* $b \in V$.

$(\Leftarrow)$ Obvious.

(ii) $a \neq 0$ is equivalent to aC^ta then apply (i).

(iii) Proof of the first equivalence.

$(\Rightarrow)$ Let $c \in TR$ then $c \neq 0$ and by (ii) there exists an UTR-cluster Γ containing c. The uniqueness of Γ following from Lemma 4.7. (ii).

$(\Leftarrow)$ We will reason by contraposition. Suppose $c \notin TR$ then by $(TR1)$ either $c = 0$ or there are a, b such that aC^tc, bC^tc *and* $a\overline{C}^tb$. In the case $c = 0$ there are no clusters containing c. For the second case by using (i) there are UTR-clusters Γ, Δ such that $a, c \in \Gamma$ *and* $b, c \in \Delta$. Then $a\overline{C}^tb$ implies by Lemma 2.28 $\Gamma \neq \Delta$.

Proof of the second equivalence is similar to the proof of (i).

(iv) Obvious.

(v) Since $NOW \in UTR$ apply (iv).

□

Definition 4.9. Time order between t-clusters. Let Γ, Δ be t-clusters. Define $\Gamma \prec \Delta$ *iff* $(\forall U \in Ult(\Gamma))(\forall V \in Ult(\Delta))(U \prec V)$

Note that $\prec$ in the right part of the definition is the canonical relation between ultrafilters corresponding to the relation $\mathcal{B}$.

Lemma 4.10. ***Properties of time order between t-clusters.*** *Let* $\Gamma = \Gamma(c)$ *and* $\Delta = \Delta(d)$ *be UTR-clusters. Then the following conditions are equivalent:*

(i) $\Gamma \prec \Delta$

(ii) There exists $U \in Ult(\Gamma)$ *and* $V \in Ult(\Delta)$ *such that* $U \prec V$

(iii) $c\mathcal{B}d$

(iv) For all $a \in \Gamma$ *and* $b \in \Delta : a\mathcal{B}b$

Proof. $(i) \Rightarrow (ii)$ Suppose (i) holds. Since $Ult(\Gamma)$ *and* $Ult(\Delta)$ are non-empty sets, let $U \in Ult(\Gamma)$ *and* $V \in Ult(\Delta)$. By (i) we obtain $U \prec V$.

$(ii) \Rightarrow (iii)$ Suppose (ii) holds. Since $\Gamma(c)$ *and* $\Delta(d)$ are UTR-clusters, then by Lemma 4.7 (v) we get $c \in U$ *and* $d \in V$. Then by (ii) and by the definition of the canonical relation $\prec$ between ultrafilters we obtain $c\mathcal{B}d$.

$(iii) \Rightarrow (iv)$ Suppose (iii) holds. $a \in \Gamma$ *and* $b \in \Delta$. Then aC^tc, bC^td. From here and (iii) $(c\mathcal{B}d)$ we obtain by axioms $(TR\mathcal{B}1)$ *and* $(TR\mathcal{B}2)$ that $a\mathcal{B}b$.

$(iv) \Rightarrow (i)$ Suppose (iv) holds. For the sake of contradiction let (i) does not hold. Then for some $U \in Ult(\Gamma)$ and some $V \in Ult(\Delta)$ we have $U \not\prec V$. By the definition of $\prec$ there are $a \in U \subseteq \Gamma$ *and* $b \in V \subseteq \Delta$ such that $a\overline{\mathcal{B}}b$. But $a \in \Gamma$ *and* $b \in \Delta$ implies by (iv) that $a\mathcal{B}b$, which is a contradiction. □

Definition 4.11. Time proximity between t-clusters. Let Γ, Δ be t-clusters. Define $\Gamma \approx \Delta$ *iff* $(\forall U \in Ult(\Gamma))(\forall V \in Ult(\Delta))(U \approx V)$

Lemma 4.12. ***Properties of time proximity between t-clusters.*** *Let* $\Gamma = \Gamma(c)$ *and* $\Delta = \Delta(d)$ *be UTR-clusters. Then the following conditions are equivalent:*

(i) $\Gamma \approx \Delta$

(ii) There exists $U \in Ult(\Gamma)$ *and* $V \in Ult(\Delta)$ *such that* $U \approx V$

(iii) $c\mathcal{P}d$

(iv) For all $a \in \Gamma$ *and* $b \in \Delta : a\mathcal{P}b$

Proof. Similarly to the proof of Lemma 4.10. □

Lemma 4.13. $a\mathcal{B}b$ *iff there exists UTR-cluster* Γ *and* Δ *such that* $\Gamma \prec \Delta, a \in \Gamma$ *and* $b \in \Delta$.

Proof. $(\Rightarrow)$ Suppose $a\mathcal{B}b$. Then by Lemma 4.3 (v) there are $c, d \in UTR$ such that $(a \cdot c)\mathcal{B}(b \cdot d)$. Then by Lemma 2.19 there are ultrafilters U, V such that $U \prec V, a \cdot c \in U$ *and* $b \cdot d \in V$. Since U *and* V are also t-clans, by Lemma 2.24 there are maximal t-clans Γ *and* Δ such that $U \in \Gamma$ *and* $V \in \Delta$. But by Lemma 2.26 Γ *and* Δ are t-clusters. Also we have: $a, c \in \Gamma$, so $\Gamma = \Gamma(c)$ is an UTR-cluster containing a and $\Delta = \Delta(d)$ is an UTR-cluster containing d. From $(a \cdot c)\mathcal{B}(b \cdot d)$ we obtain $c\mathcal{B}d$ which by Lemma 4.10 implies $\Gamma \prec \Delta$.

$(\Leftarrow)$ Suppose that $a \in \Gamma, b \in \Delta$ *and* $\Gamma \prec \Delta$. Then by Lemma 4.10 it follows that $a\mathcal{B}b$. □

Lemma 4.14. *$a\mathcal{P}b$ iff there exists UTR-cluster Γ and Δ such that $\Gamma \approx \Delta, a \in \Gamma$ and $b \in \Delta$.*

Proof. Similarly to the proof of Lemma 4.13. $\square$

5 Representation theory for DMA's

The aim of this section is to show that each abstract DMA is isomorphic to a certain canonical dynamic mereological algebra. This means that if we have at hand a abstract DMA $\underline{B}$, we first have to extract from $\underline{B}$ a time structure $(T, \prec, \approx, now)$, second for each $m \in T$ to define in some way a contact algebra B_m (coordinate Boolean algebra of the time point m) and third, following the method described in Section 3, to define a standard DMA B^{can} (called canonical DMA) determined by the time structure $(T, \prec, \approx, now)$ and by the set of coordinate Boolean algebras $(\underline{B}_t; C_t)$, $t \in T$. The last step is to define an embedding h from $\underline{B}$ into the constructed canonical dynamic mereological algebra B^{can}. All these steps will be realized in the subsequent subsections.

5.1 Extracting the time structure

Definition 5.1. Formal definition of canonical time structure. Let $\underline{\mathrm{B}}$ be a DMA. The canonical time structure $T(B)^{can} = (T, \prec, \approx, now)$ of $\underline{B}$ is defined as follows: $T =_{def} UTR{-}clusters(B)$, the relation $\prec$ is the time order between UTR-clusters (see Definition 4.9), the relation $\approx$ is the time proximity between UTR-clusters (see Definition 4.11) and now is the unique UTR-cluster containing NOW (Lemma 4.8).

The UTR-clusters as elements of the canonical time structure are considered as the abstract time points of $\underline{B}$, now as the time point of present epoch and NOW is just the universal time representative of now. We restate in an abstract level the Lemma 3.5 formulated in Section 3.3 for standard DMA-s.

Lemma 5.2. ***Correspondence Lemma.*** *Let $\underline{B}$ be a DMA and $T(B)^{can} = (T, \prec, \approx, now)$ be the canonical time structure of $\underline{B}$. Let A be any formula from the list of time conditions (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) (Section 3.1) and α be the corresponding formula from the list of time axioms (rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr) (Section 3.3). Then α is true in $\underline{B}$ iff $\tau(A)$ is true in $T(B)^{can} = (T, \prec, \approx, now)$.*

Proof. By Proposition 4.4 α is true in $\underline{B}$ iff $\tau(A)$ is true in $\underline{B}$. It is easy to see, using Lemma 4.7 and Lemma 4.10 that $\tau(A)$ is true in $\underline{B}$ iff A is true in $T(B) = (T, \prec, \approx, now)$. Let us demonstrate this by example:
$A = (Tri) = (\forall \Gamma, \Delta \in T)(\Gamma = \Delta \textit{ or } \Gamma \prec \Delta \textit{ or } \Delta \prec \Gamma)$, $\tau(A) = (\forall a, b \in UTR)(aC^t b \textit{ or } a\mathcal{B}b \textit{ or } b\mathcal{B}a)$.

($\Rightarrow$). Suppose $\tau(A)$ is true in $\underline{B}$ and let $\Gamma = \Gamma(a)$ *and* $\Delta = \Delta(b)$ be UTR-clusters. Then by the assumption for $\tau(A)$ we have: $aC^t b \textit{ or } a\mathcal{B}b \textit{ or } b\mathcal{B}a$. Then by Lemma 4.7 and Lemma 4.10 we obtain $\Gamma = \Delta \textit{ or } \Gamma \prec \Delta \textit{ or } \Delta \prec \Gamma$.

($\Leftarrow$). Suppose A holds in $T(B) = (T, \prec, \approx, now)$ and let $a, b \in UTR$. By axiom $(TR2) a \neq 0$. Then there exists an ultrafilter U containing a. Since U is a t-clan it is contained in a maximal t-clan $\Gamma = \Gamma(a)$ which is an UTR-cluster. Analogously b is contained in an UTR-cluster $\Delta = \Delta(b)$. By the assumption on A we have: $\Gamma(a) = \Delta(b) \textit{ or } \Gamma(a) \prec \Delta(b) \textit{ or } \Delta(b) \prec \Gamma(a)$. Then by Lemma 4.7 and Lemma 4.10 we get $(aC^t b \textit{ or } a\mathcal{B}b \textit{ or } b\mathcal{B}a)$. $\square$

5.2 Extracting the coordinate Boolean algebras

Definition 5.3. Coordinate Boolean algebras, canonical dynamic model of space and the embedding. Let $\underline{B}$ be a DMA and Γ be a time point in $\underline{B}$, i.e. Γ is a UTR-cluster and let $Ult(\Gamma)$ be the set of ultrafilters included in Γ.
First we construct the factor Boolean algebra $(\underline{B}_{Ult(\Gamma)})$ by the set $Ult(\Gamma)$ following the construction described in Section 2.3. The algebra $(\underline{B}_{Ult(\Gamma)})$ is called the **canonical coordinate Boolean algebra corresponding to the time point** Γ. Recall that the set $Ult(\Gamma)$ determines a congruence relation in B and the elements of $(\underline{B}_{Ult(\Gamma)})$ are just equivalence classes $|a|_{Ult(\Gamma)}$ determined by this congruence relation. For simplicity of notation we will write $\underline{B}_\Gamma$ instead of $(\underline{B}_{Ult(\Gamma)})$, similiarly for $|a|_\Gamma$.
Next we will define the full standard DMA denoted by $\underline{\mathbf{B}}^{can}$ with $C^t, \mathcal{B}\ and\ \mathcal{P}, TR, UTR, NOW$ in it as in Section 3 by means of the canonical time structure $T(B)^{can} = (T, \prec, \approx, now)$ and by the canonical coordinate algebras $(\underline{B}_\Gamma)$, $\Gamma \in T$.
$\underline{\mathbf{B}}^{can}$ is called **full canonical standard DMA** corresponding to $\underline{B}$.
The **canonical embedding** h is defined coordinatewise as follows for each $a \in \underline{B}\ and\ \Gamma \in T$:

$$h(a)_\Gamma = |a|_\Gamma$$

We consider the subalgebra $h(B)$ of $\underline{\mathbf{B}}^{can}$, denoted by $\underline{\mathbf{B}}$ as the **canonical standard DMA** corresponding to $\underline{B}$.

Remark 5.4. The definition of the coordinate Boolean algebra $\underline{B}_\Gamma$ as a factor Boolean algebra with respect to the set $Ult(\Gamma)$ is based on the following intuition taken from standard DMA-s. If we look at dynamic regions as trajectories of changing regions, then for different a and b we may have that $|a|_\Gamma = |b|_\Gamma$, which is an equivalence relation determined by Γ. The abstract definition of this equivalence in DMA-s is just the congruence, which determines the coordinate Boolean algebra $\underline{B}_\Gamma$.

The next lemma is important because it shows that the time axioms are preserved by the construction of the full canonical standard DMA.

Lemma 5.5. *Let $\underline{B}$ be a DMA and $\underline{B}^{can}$ be the full canonical standard dynamic contact algebra associated with $\underline{B}$. Then for each time axiom α from the list of time axioms (rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr) the following equivalence is true: α holds in $\underline{B}$ iff α holds in $\underline{B}^{can}$.*

Proof. By Proposition 4.4 α is true in B iff the corresponding condition A from the list time conditions (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) is true in the canonical time structure $T(B)^{can} = (T, \prec, \approx, now)$ of $\underline{B}$ iff (by Lemma 3.5 A is true in the full standard DMA $\underline{B}^{can}$). $\square$

The next lemma shows that the function h is an embedding from $\underline{B}$ into $\underline{B}^{can}$.

Lemma 5.6. ***Embedding Lemma.*** *Let $\underline{B}$ be a DMA and h be the mapping defined in Definition 5.3. Then:*

(i) *h preserves Boolean operations.*

(ii) *aC^tb in $\underline{B}$ iff there exists $\Gamma \in T$ such that $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Gamma \neq |0|_\Gamma$ iff $h(a)C^th(b)$ in $\underline{B}^{can}$*

(iii) *$a\mathcal{B}b$ in $\underline{B}$ iff there exist $\Gamma, \Delta \in T$ such that $\Gamma \prec \Delta$ and $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Delta \neq |0|_\Delta$ iff $h(a)\mathcal{B}h(b)$ in $\underline{B}^{can}$*

(iv) *$a\mathcal{P}b$ in $\underline{B}$ if there exist $\Gamma, \Delta \in T$ such that $\Gamma \approx \Delta$ and $|a|_\Gamma \neq |0|_\Gamma$ and $|b|_\Delta \neq |0|_\Delta$ iff $h(a)\mathcal{P}h(b)$*

(v) *$a \leq b$ in $\underline{B}$ iff for all $\Gamma \in T |a|_\Gamma \leq_\Gamma |b|_\Gamma$ iff $h(a) \leq h(b)$ in $\underline{B}^{can}$*

(vi) *$a = b$ iff $h(a) = h(b)$, i.e. h is an embedding*

(vii) *$a \in TR$ in $\underline{B}$ iff $h(a) \in TR$ in $\underline{B}^{can}$*

(viii) *$a \in UTR$ in $\underline{B}$ iff $h(a) \in UTR$ in $\underline{B}^{can}$*

(ix) *$h(NOW) = NOW$, where the second $NOW = NOW(now)$ is just the UTR representative of **now** in $\underline{B}^{can}$*

Proof. (i) The statement is obvious, because the elements of the coordinate algebras are equivalence classes determined by a congruence relations in $\underline{B}$ and that Boolean operations in $\underline{B}^{can}$ are defined coordinatewise.

(ii) aC^tb in $\underline{B}$ iff (By Lemma 4.8) there exists an UTR-cluster Γ and ultrafilters $\Delta, \Theta \in Ult(\Gamma)$ such that $a \in \Delta$ *and* $b \in \Theta$ iff there exist $\Gamma \in T$ such that $|a|_\Gamma \neq |0|_\Gamma$ *and* $|b|_\Gamma \neq |0|_\Gamma$ iff $h(a)C^th(b)$ in $\underline{\mathbf{B}}^{can}$.

(iii) $a\mathcal{B}b$ in $\underline{\mathrm{B}}$ iff there exists UTR-clusters Γ *and* Δ such that $\Gamma \prec \Delta$ and there exist ultrafilter $\Theta \in Ult(\Gamma)$ *and* $\Lambda \in Ult(\Delta)$ such that $a \in \Theta$ *and* $b \in \Lambda$ iff there exist $\Gamma, \Delta \in T$ such that $\Gamma \prec \Delta$, $|a|_\Gamma \neq |0|_\Gamma$ *and* $|b|_\Delta \neq |0|_\Delta$ iff $h(a)\mathcal{B}h(b)$ in $\underline{\mathbf{B}}^{can}$.

(iv) The proof is similar to that of (iii).

(v) For the proof of this equivalence use the fact that $a \leq b$ iff $a \cdot b^* = 0$ iff $(a \cdot b^*)\overline{C}^t(a \cdot b^*)$ (because C^t is a contact relation) and apply (ii).

(vi) Follows from (v)

(vii) $a \in TR$ iff (By Lemma 4.8) there exists a unique UTR-cluster Γ and an ultrafilter $\Delta \subseteq \Gamma$ such that $a \in \Delta$ iff there exists a unique element $\Gamma \in T$ such that $|a|_\Gamma \neq |0|_\Gamma$ iff $h(a) \in TR$ in $\underline{\mathbf{B}}^{can}$.

$(viii)$ The proof is similar to that of (vii) by an application of Lemma 4.8.

(ix) The proof follows from $(viii)$ and the fact that NOW is universal time representative of the UTR-cluster denoted by **now**. □

5.3 The Representation Theorem

Theorem 5.7. ***Representation Theorem for DMA-s.*** *Let $\underline{B}$ be a DMA. Then there exists a full standard DMA $\mathbb{B}$ and an isomorphic embedding h of $\underline{B}$ into $\mathbb{B}$. Moreover, $\underline{B}$ satisfies some of the time axioms iff the same axioms are satisfied in $\mathbb{B}$.*

Proof. The proof is a direct corollary of Lemma 5.5 and Lemma 5.6 by taking $\mathbb{B} = \underline{B}^{can}$. $\square$

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