Cohesive Powers of Linear Orders

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Abstract. Cohesive powers of computable structures can be viewed as effective ultraproducts over effectively indecomposable sets called cohesive sets. We investigate the isomorphism types of cohesive powers $\Pi_C \mathcal{L}$ for familiar computable linear orders $\mathcal{L}$. If $\mathcal{L}$ is isomorphic to the ordered set of natural numbers $\mathbb{N}$ and has a computable successor function, then $\Pi_C \mathcal{L}$ is isomorphic to $\mathbb{N}+\mathbb{Q}\times\mathbb{Z}$. Here, $+$ stands for the sum and $\times$ for the lexicographical product of two orders. We construct computable linear orders $\mathcal{L}_1$ and $\mathcal{L}_2$ isomorphic to $\mathbb{N}$, both with noncomputable successor functions, such that $\Pi_C \mathcal{L}_1$ is isomorphic to $\mathbb{N}+\mathbb{Q}\times\mathbb{Z}$, while $\Pi_C \mathcal{L}_2$ is not. While cohesive powers preserve all Π^0_2 and Σ^0_2 sentences, we provide new examples of Π^0_3 sentences Φ and computable structures $\mathcal{M}$ such that $\mathcal{M} \vDash \Phi$ while $\Pi_C \mathcal{M} \vDash \neg\Phi$.

1 Introduction and Preliminaries

Skolem was the first to construct a countable nonstandard model of true arithmetic. Various countable nonstandard models of (fragments of) arithmetic have been later studied by Feferman, Scott, Tennenbaum, Hirschfeld, Wheeler, Lerman, McLaughlin and others (see [6], [8], [7], [9]). The following definition, and other notions from computability theory can be found in [10].

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Definition 1. *(i) An infinite set $C \subseteq \omega$ is* cohesive *(*r-cohesive*) if for every c.e. (computable) set W, either $W \cap C$ or $\overline{W} \cap C$ is finite.*

(ii) A set M is maximal *(*r-maximal*) if M is c.e. and $\overline{M}$ is cohesive (*r-*cohesive).*

(iii) If M is maximal, then $\overline{M}$ is called co-maximal.

(iv) A set B is quasimaximal *if it is the intersection of finitely many maximal sets.*

In the definition above ω denotes the set of natural numbers. We will use $=^*$(and $\subseteq^*$) to refer to equality (inclusion) of sets up to finitely many elements. Let A be a fixed r-cohesive set. For computable functions f and g, Feferman, Scott, and Tennenbaum (see [6]) defined an equivalence relation $f \sim_A g$ if $A \subseteq^* \{n : f(n) = g(n)\}$. They then proved that the structure $\mathcal{R}/\sim_A$, with domain the set of recursive functions modulo $\sim_A$, is a model of only a fragment of arithmetic. They constructed a particular Π^0_3 sentence Φ such that for the standard model of arithmetic, $\mathcal{N}$, we have $\mathcal{N} \models \Phi$ but $\mathcal{R}/\sim_A \not\models \Phi$. The sentence Φ provided in [6] essentially uses Kleene's T predicate.

Cohesive powers of computable structures are effective versions of ultrapowers. They have been introduced in [2] in relation to the study of automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ of effective vector spaces. Cohesive powers of the field of rational numbers were used in [1] to characterize certain principal filters of $\mathcal{L}^*(V_\infty)$. Their isomorphism types and automorphisms were further studied in [4]. They were also used in [1] and [3] to find interesting orbits in $\mathcal{L}^*(V_\infty)$.

The goal of this paper is to show that the presentation of a computable structure matters for the isomorphism type of its cohesive power. We give computable presentations of the ordered set of natural numbers such that their cohesive powers are not elementary equivalent. Furthermore, we provide examples of computable structures $\mathcal{M}$ and Π^0_3 sentences Ψ, which do not use Kleene's T predicate, such that $\mathcal{M} \models \Psi$ while the cohesive power $\Pi_C\mathcal{M} \models \neg\Psi$. We will now present some additional definitions and known results.

Definition 2. *[2] Let $\mathcal{A}$ be a computable structure for a computable language L and with domain A. Let $C \subseteq \omega$ be a cohesive set. The* cohesive power of $\mathcal{A}$ over C*, denoted by $\Pi_C\mathcal{A}$, is a structure $\mathcal{B}$ for L defined as follows:*

(i) Let $D = \{\varphi \mid \varphi : \omega \to A$ is a partial computable function, and $C \subseteq^* dom(\varphi)\}$.

For $\varphi_1, \varphi_2 \in D$, define $\varphi_1 =_C \varphi_2$ iff $C \subseteq^* \{x : \varphi_1(x)\downarrow = \varphi_2(x)\downarrow\}$.

Let $B = (D/=_C)$ be the domain of $\mathcal{B} = \Pi_C\mathcal{A}$

(ii) If $f \in L$ is an n-ary function symbol, then $f^{\mathcal{B}}$ is an n-ary function on B such that for every $[\varphi_1], \ldots, [\varphi_n] \in B$, $f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi]$, where for every $x \in A$,

$$\varphi(x) \simeq f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)),$$

where $\simeq$ stands for equality of partial functions.

(iii) If $P \in L$ is an m-ary predicate symbol, then $P^{\mathcal{B}}$ is an m-ary relation on B such that for every $[\varphi_1], \ldots, [\varphi_m] \in B$,

$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \Leftrightarrow C \subseteq^* \{x \in A \mid P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

(iv) If $c \in L$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the (total) computable function on A with constant value $c^{\mathcal{A}}$.

The following is the fundamental theorem of cohesive powers due to Dimitrov (see [2]).

Theorem 3. *Let C be a cohesive set and let $\mathcal{A}$ and $\mathcal{B}$ be as in the definition above.*

1. *If $\tau(y_1, \ldots, y_n)$ is a term in L and $[\varphi_1], \ldots, [\varphi_n] \in B$, then $[\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ is the equivalence class of a partial computable function such that*
$$\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = \tau^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)).$$
2. *If $\Phi(y_1, \ldots, y_n)$ is a formula in L that is a Boolean combination of Σ_1^0 and Π_1^0 formulas and $[\varphi_1], \ldots, [\varphi_n] \in B$, then*
$$\mathcal{B} \vDash \Phi([\varphi_1], \ldots, [\varphi_n]) \text{ iff } C \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$
3. *If Φ is a Π_2^0 (or Σ_2^0) sentence in L, then $\mathcal{B} \vDash \Phi$ iff $\mathcal{A} \vDash \Phi$.*
4. *If Φ is a Π_3^0 sentence in L, then $\mathcal{B} \vDash \Phi$ implies $\mathcal{A} \vDash \Phi$.*

Note that $\mathcal{A}$ is a substructure of $\mathcal{B} = \Pi_C \mathcal{A}$. For $c \in A$ let $[\varphi_c] \in B$ be the equivalence class of the total function φ_c such that $\varphi_c(x) = c$ for every $x \in \omega$. The map $d : A \to B$ such that $d(c) = [\varphi_c]$ is called *canonical embedding* of $\mathcal{A}$ into $\mathcal{B}$.

2 Cohesive Powers of Linear Orders

We will now investigate various algebraic and computability-theoretic properties of cohesive powers of linear orders. We first provide some definitions and notational conventions we will use. Let $C \subseteq \omega$ be a cohesive set. Let $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ be a fixed computable bijection, and let the (computable) functions π_1 and π_2 be such that $\pi_1(\langle m, n \rangle) = m$ and $\pi_2(\langle m, n \rangle) = n$.

Definition 4. *Let $\mathcal{L}_0 = \langle L_0, <_{\mathcal{L}_0} \rangle$ and $\mathcal{L}_1 = \langle L_1, <_{\mathcal{L}_1} \rangle$ be linear orders. Then*

(1) $\mathcal{L}_0 + \mathcal{L}_1 = \langle \{\langle 0, l \rangle : l \in L_0\} \cup \{\langle 1, l \rangle : l \in L_1\}, <_{\mathcal{L}_0 + \mathcal{L}_1} \rangle$, where
$$\langle i, l \rangle <_{\mathcal{L}_0 + \mathcal{L}_1} \langle j, m \rangle \text{ iff } (i < j) \vee (i = j \wedge l <_{\mathcal{L}_i} m).$$

(2) $\mathcal{L}_0 \times \mathcal{L}_1 = \langle L_0 \times L_1, <_{\mathcal{L}_0 \times \mathcal{L}_1} \rangle$, where
$\langle k, m \rangle <_{\mathcal{L}_0 \times \mathcal{L}_1} \langle l, n \rangle$ iff $(k <_{\mathcal{L}_0} l) \vee (k =_{\mathcal{L}_0} l \wedge m <_{\mathcal{L}_1} n)$.

Remark 5. (1) By $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ we denote the usual ordered sets of natural numbers, integers, and rational numbers. The order types of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are denoted by ω, ζ, and η.

(2) In Definition 4 we use $\mathcal{L}_0 \times \mathcal{L}_1$ to denote the lexicographical product of the linear orders $\mathcal{L}_0$ and $\mathcal{L}_1$. This product is also denoted by $\mathcal{L}_1 \cdot \mathcal{L}_0$. (For example, $\mathbb{Q} \times \mathbb{Z}$ is also denoted by $\mathbb{Z} \cdot \mathbb{Q}$, and its order type is denoted $\zeta \cdot \eta$.)

(3) We will use $\mathcal{L}^{rev}$ to denote the reverse linear order of $\mathcal{L}$. (In the literature it is also denoted by $\mathcal{L}^*$.)

(4) Let the quantifier $\forall^\infty n$ stand for "infinitely many n." Note that if $\{n \mid \varphi(n)\}$ is a c.e. set, then $(\forall^\infty n \in C)\,[\varphi(n)]$ will mean that $\varphi(n)$ is satisfied "for almost all $n \in C$."

Before we state the next theorem, we would like to remind that $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$ is the order type of a countable non-standard model of PA.

Theorem 6. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be computable linear orders and let C be a cohesive set. Then*

(1) $\Pi_C\,(\mathcal{L}_0 + \mathcal{L}_1) \cong \Pi_C \mathcal{L}_0 + \Pi_C \mathcal{L}_1$

(2) $\Pi_C\,(\mathcal{L}_0 \times \mathcal{L}_1) \cong \Pi_C \mathcal{L}_0 \times \Pi_C \mathcal{L}_1$

(3) $\Pi_C \mathcal{L}_0^{rev} \cong (\Pi_C \mathcal{L}_0)^{rev}$

(4) Let $\mathcal{A}$ be a computable presentation of the linear order $\mathbb{N}$ with a computable successor function. Then $\Pi_C \mathcal{A} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.

(5) If $\mathcal{L}$ is a computable dense linear order without endpoints, then $\mathcal{L} \cong \Pi_C \mathcal{L}$.

Proof. (1) Let $\mathcal{A} = \Pi_C\,(\mathcal{L}_0 + \mathcal{L}_1)$ and $\mathcal{B} = \Pi_C \mathcal{L}_0 + \Pi_C \mathcal{L}_1$. We will define an isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$. Suppose $[\varphi]_C \in \Pi_C\,(\mathcal{L}_0 + \mathcal{L}_1)$ for a partial computable function φ.

If $(\forall^\infty n \in C)\,[\varphi(n) \in \{0\} \times L_1]$, then let $\Phi([\varphi]_C) =_{def} \langle 0, [\pi_2 \circ \varphi]_C \rangle$.

If $(\forall^\infty n \in C)\,[\varphi(n) \in \{1\} \times L_2]$, then let $\Phi([\varphi]_C) =_{def} \langle 1, [\pi_2 \circ \varphi]_C \rangle$.

Since C is cohesive, exactly one of the two cases above applies, so it follows that so it follows that Φ is well defined. It is then easy to check that Φ is an isomorphism.

(2) Let $\mathcal{A} = \Pi_C\,(\mathcal{L}_0 \times \mathcal{L}_1)$ and $\mathcal{B} = \Pi_C \mathcal{L}_0 \times \Pi_C \mathcal{L}_1$. We will define an isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$. Suppose $[\varphi]_C \in \Pi_C\,(\mathcal{L}_0 \times \mathcal{L}_1)$, and let $\Phi([\varphi]_C) =_{def} \langle [\pi_1 \circ \varphi]_C, [\pi_2 \circ \varphi]_C \rangle$. We will prove that

$$[\varphi]_C <_{\mathcal{A}} [\psi]_C \Leftrightarrow \langle [\pi_1 \circ \varphi]_C, [\pi_2 \circ \varphi]_C \rangle <_{\mathcal{B}} \langle [\pi_1 \circ \psi]_C, [\pi_2 \circ \psi]_C \rangle .$$

By definition, $[\varphi]_C <_{\mathcal{A}} [\psi]_C$ iff $C \subseteq^* \{n : \varphi(n) < \psi(n)\}$. By cohesiveness of C, we will have either

$(\forall^\infty n \in C)\,[(\pi_1 \circ \varphi)(n) < (\pi_1 \circ \psi)(n)]$, or

$(\forall^\infty n \in C)\,[(\pi_1 \circ \varphi)(n) = (\pi_1 \circ \psi)(n) \wedge (\pi_2 \circ \varphi)(n) < (\pi_2 \circ \psi)(n)]$.

In the first case, $[\pi_1 \circ \varphi]_C <_{\Pi_C \mathcal{L}_0} [\pi_1 \circ \psi]_C$. In the second case, $[\pi_1 \circ \varphi]_C =_{\Pi_C \mathcal{L}_0} [\pi_1 \circ \psi]_C$ and $[\pi_2 \circ \varphi]_C <_{\Pi_C \mathcal{L}_1} [\pi_2 \circ \psi]_C$. Therefore,

$$\langle [\pi_1 \circ \varphi]_C, [\pi_2 \circ \varphi]_C \rangle <_{\mathcal{B}} \langle [\pi_1 \circ \psi]_C, [\pi_2 \circ \psi]_C \rangle .$$

(3) Let $\mathcal{A} = \Pi_C \mathcal{L}_0^{rev}$ and $\mathcal{B} = (\Pi_C \mathcal{L}_0)^{rev}$. We will define an isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$. If $[\varphi]_C \in \Pi_C \mathcal{L}_0^{rev}$, then let $\Phi([\varphi]_C) = [\varphi]_C$. We will prove that $[\varphi]_C <_{\mathcal{A}} [\psi]_C$ iff $[\varphi]_C <_{\mathcal{B}} [\psi]_C$. By definition we have

$$[\varphi]_C <_{\mathcal{B}} [\psi]_C \Leftrightarrow [\psi]_C <_{\Pi_C \mathcal{L}_0} [\varphi]_C \Leftrightarrow$$

$$(\forall^\infty n \in C)\,(\psi(n) <_{\mathcal{L}_0} \varphi(n)) \Leftrightarrow$$

$$(\forall^\infty n \in C)\,(\varphi(n) <_{\mathcal{L}_0^{rev}} \psi(n)) \Leftrightarrow [\varphi]_C <_{\mathcal{A}} [\psi]_C .$$

(4) The proof of this fact is omitted because it is a simplified version of the proof of Theorem 8.

(5) The theory of dense linear orders without endpoints is Π_2^0 axiomatizable and countably categorical. By Theorem 3 (part 4), $\Pi_C \mathcal{L}$ is also a dense linear order without endpoints. Since $\Pi_C \mathcal{L}$ is countable, we have $\mathbb{Q} \cong \mathcal{L} \cong \Pi_C \mathcal{L}$.

Item (5) in the previous Theorem provides an example of an infinite structure $\mathcal{L}$ such that $\mathcal{L} \cong \Pi_C \mathcal{L}$. The linear order $\mathbb{Q}$ is an ultrahomogeneous structure; it is the Fraïssé limit of the class of finite linear orders. The relationship between Fraïssé limits and cohesive powers is considered in ([5]). We now provide two more examples of structures isomorphic to their cohesive powers.

Example 7. (1) $\Pi_C\,(\mathbb{Q} \times \mathbb{Z}) \cong \mathbb{Q} \times \mathbb{Z}$
(2) $\Pi_C\,(\mathbb{N} + \mathbb{Q} \times \mathbb{Z}) \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$

Proof. (1) $\Pi_C \mathbb{Q} \times \Pi_C \mathbb{Z} \cong \mathbb{Q} \times \Pi_C(\mathbb{N}^{rev} + \mathbb{N}) \cong \mathbb{Q} \times (\Pi_C \mathbb{N}^{rev} + \Pi_C \mathbb{N}) \cong$
$\cong \mathbb{Q} \times [(\mathbb{N} + \mathbb{Q} \times \mathbb{Z})^{rev} + (\mathbb{N} + \mathbb{Q} \times \mathbb{Z})] \cong \mathbb{Q} \times [\mathbb{Q} \times \mathbb{Z} + \mathbb{N}^{rev} + \mathbb{N} + \mathbb{Q} \times \mathbb{Z}] \cong$
$\cong \mathbb{Q} \times [\mathbb{Q} \times \mathbb{Z} + \mathbb{Z} + \mathbb{Q} \times \mathbb{Z}] \cong \mathbb{Q} \times [\mathbb{Q} \times \mathbb{Z}] \cong \mathbb{Q} \times \mathbb{Z}$
(2) $\Pi_C\,(\mathbb{N} + \mathbb{Q} \times \mathbb{Z}) \cong \Pi_C \mathbb{N} + \Pi_C\,(\mathbb{Q} \times \mathbb{Z}) \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z} + \mathbb{Q} \times \mathbb{Z} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$

Theorem 6, part (4), demonstrates that having a computable successor function is a sufficient condition for the cohesive power of a computable linear order of type ω to be isomorphic to $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$. The next theorem shows that this condition is not necessary.

Theorem 8. *There is a computable linear order $\mathcal{L}$ of order type ω with a non-computable successor function such that for every cohesive set C we have $\Pi_C \mathcal{L} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.*

Proof. Fix a non-computable c.e. set A, and let f be a total computable injection on the set of natural numbers with range A. Let $\mathcal{L} = (\omega, <_{\mathcal{L}})$ be the linear order obtained by ordering the even numbers according to their natural order, and by setting $2a <_{\mathcal{L}} 2k + 1 <_{\mathcal{L}} 2a + 2$ if and only if $f(k) = a$. Specifically, we set

$$\begin{array}{rcl} 2c <_{\mathcal{L}} 2d & \leftrightarrow & 2c < 2d \\ 2c <_{\mathcal{L}} 2k+1 & \leftrightarrow & c \le f(k) \\ 2k+1 <_{\mathcal{L}} 2c & \leftrightarrow & f(k) < c \\ 2k+1 <_{\mathcal{L}} 2\ell+1 & \leftrightarrow & f(k) < f(\ell). \end{array}$$

Then $\mathcal{L}$ is a computable linear order of type ω. Let $S^{\mathcal{L}}$ denote the successor function of $\mathcal{L}$. Then $A \leq_{\mathrm{T}} S^{\mathcal{L}}$ (indeed, $A \equiv_{\mathrm{T}} S^{\mathcal{L}}$) because $a \in A$ if and only if $S^{\mathcal{L}}(2a) \neq 2a+2$. Thus $S^{\mathcal{L}}$ is not computable.

Let C be cohesive, and let $\mathcal{P} = \Pi_C \mathcal{L}$. We show that $\mathcal{P} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$. To do this, we begin by establishing the following properties of $\mathcal{P}$.

(a) $\mathcal{P}$ has an initial segment of type ω.
(b) Every element of $\mathcal{P}$ has a $<_{\mathcal{P}}$-immediate successor.
(c) Every element of $\mathcal{P}$ that is not the least element has an $<_{\mathcal{P}}$-immediate predecessor.

For (a), note that the range of the canonical embedding of $\mathcal{L}$ into $\mathcal{P}$ is an initial segment of $\mathcal{P}$ of type ω.

For (b), consider a $[\psi] \in \mathcal{P}$. We define a partial computable φ such that, for almost every $n \in C$, $\varphi(n)$ is the $<_{\mathcal{L}}$-immediate successor of $\psi(n)$. It then follows that $[\varphi]$ is the $<_{\mathcal{P}}$-immediate successor of $[\psi]$. To define φ, observe that, by the cohesiveness of C, exactly one of the following three cases occurs.

1. $(\forall^\infty n \in C)(\psi(n) \text{ is odd})$
2. $(\forall^\infty n \in C)(\exists a \in A)(\psi(n) = 2a)$
3. $(\forall^\infty n \in C)(\exists a \notin A)(\psi(n) = 2a)$

Note that we cannot effectively decide which case occurs, but in each case we can define a particular φ_i such that $[\varphi_i]$ is the $<_{\mathcal{P}}$-immediate successor of $[\psi]$.

If case (1) occurs, define

$$\varphi_1(n) = \begin{cases} 2a+2 & \text{if } \psi(n)\downarrow,\ \psi(n) = 2k+1, \text{ and } f(k) = a; \\ \uparrow & \text{otherwise.} \end{cases}$$

If case (2) occurs, define

$$\varphi_2(n) = \begin{cases} 2k+1 & \text{if } \psi(n)\downarrow,\ \psi(n) = 2a,\ a \in A, \text{ and } f(k) = a; \\ \uparrow & \text{otherwise.} \end{cases}$$

If case (3) occurs, define

$$\varphi_3(n) = \begin{cases} 2a+2 & \text{if } \psi(n)\downarrow \text{ and } \psi(n) = 2a; \\ \uparrow & \text{otherwise.} \end{cases}$$

In each case (i) $(i = 1, 2, 3)$ we have that for almost every $n \in C$, $\varphi_i(n)$ is the $<_{\mathcal{L}}$-immediate successor of $\psi(n)$.

The proof of (c) is analogous to the proof of (b).

For $[\psi], [\varphi] \in \mathcal{P}$, write $[\psi] \ll_{\mathcal{P}} [\varphi]$ if $[\psi] <_{\mathcal{P}} [\varphi]$ and the interval $([\psi], [\varphi])_{\mathcal{P}}$ in $\mathcal{P}$ is infinite. Using the cohesiveness of C, we check that $[\psi] \ll_{\mathcal{P}} [\varphi]$ if and only if $[\psi] <_{\mathcal{P}} [\varphi]$ and $\limsup_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$, where $|(a,b)_{\mathcal{L}}|$ denotes the cardinality of the interval $(a,b)_{\mathcal{L}}$ in $\mathcal{L}$. Notice that for even numbers $2a$ and $2b$, $2a <_{\mathcal{L}} 2b$ if and only if $2a < 2b$. Therefore, if $2a < 2b$, then $|(2a, 2b)_{\mathcal{L}}| \geq b - a - 1$.

To finish the proof, we show the following.

(d) If $[\psi], [\varphi] \in \mathcal{P}$ satisfy $[\psi] \ll_{\mathcal{P}} [\varphi]$, then there is a $[\theta] \in \mathcal{P}$ such that $[\psi] \ll_{\mathcal{P}} [\theta] \ll_{\mathcal{P}} [\varphi]$.
(e) If $[\psi] \in \mathcal{P}$, then there is a $[\varphi] \in \mathcal{P}$ with $[\psi] \ll_{\mathcal{P}} [\varphi]$.

For (d), suppose that $[\psi], [\varphi] \in \mathcal{P}$ satisfy $[\psi] \ll_{\mathcal{P}} [\varphi]$. By (again) considering the cases (1)–(3) above, either $\psi(n)$ is odd for almost every $n \in C$, or $\psi(n)$ is even for almost every $n \in C$. In the case where $\psi(n)$ is odd for almost every $n \in C$, $\widehat{\psi}(n)$ is even for almost every $n \in C$, where $[\widehat{\psi}]$ is the $<_{\mathcal{P}}$-immediate successor of $[\psi]$. Thus we may assume that $\psi(n)$ and $\varphi(n)$ are even for almost every $n \in C$ by replacing $[\psi]$ and $[\varphi]$ by their $<_{\mathcal{P}}$-immediate successors if necessary. The condition $\limsup_{n\in C} |(\psi(n), \varphi(n))_{\mathcal{L}}| = \infty$ is now equivalent to $\limsup_{n\in C}(\varphi(n) - \psi(n)) = \infty$.

Define a partial computable θ by

$$\theta(n) = \begin{cases} \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor & \text{if } \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor \text{ is even;} \\ \\ \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor + 1 & \text{if } \left\lfloor \frac{\psi(n)+\varphi(n)}{2} \right\rfloor \text{ is odd.} \end{cases}$$

By the definition of θ, we have that $\limsup_{n\in C}(\theta(n) - \psi(n)) = \infty$ and that $\limsup_{n\in C}(\varphi(n) - \theta(n)) = \infty$. Since $\psi(n)$, $\varphi(n)$, and $\theta(n)$ are even for almost all $n \in C$, we have that:

$$\limsup_{n\in C} |(\psi(n), \theta(n))_{\mathcal{L}}| = \infty \text{ and } \limsup_{n\in C} |(\theta(n), \varphi(n))_{\mathcal{L}}| = \infty.$$

Thus, $[\psi] \ll_{\mathcal{P}} [\theta] \ll_{\mathcal{P}} [\varphi]$, as desired.

For (e), consider $[\psi] \in \mathcal{P}$. As argued above, we may assume that $\psi(n)$ is even for almost every $n \in C$ by replacing $[\psi]$ by its $<_{\mathcal{P}}$-immediate successor, if necessary. If $\limsup_{n\in C} \psi(n)$ is finite, then by the cohesiveness of C, the function ψ must be eventually constant on C. In this case, $[\psi] \ll_{\mathcal{P}} [2\mathrm{id}]$. If $\limsup_{n\in C} \psi(n) = \infty$, then $[\psi] \ll_{\mathcal{P}} [2\psi]$.

This completes the proof since the properties (a)–(e) ensure that $\mathcal{P} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.

3 Non-Isomorphic Cohesive Powers of Isomorphic Structures

Theorem 9. *For every co-maximal set $C \subseteq \omega$ there exist two isomorphic computable structures $\mathcal{A}$ and $\mathcal{B}$ such the cohesive powers $\prod_C \mathcal{A}$ and $\prod_C \mathcal{B}$ are not isomorphic.*

Proof. Note that it suffices to prove the theorem for an arbitrary co-maximal set consisting of even numbers only. Indeed, if C is an arbitrary co-maximal set, then $C_1 = \{2s \mid s \in C\}$ is also a co-maximal set, and for any computable structure $\mathcal{M}$, we have $\prod_C \mathcal{M} \cong \prod_{C_1} \mathcal{M}$. Then, if $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic computable structures such that $\prod_{C_1} \mathcal{M}_0 \not\cong \prod_{C_1} \mathcal{M}_1$, then $\prod_C \mathcal{M}_0 \not\cong \prod_C \mathcal{M}_1$.

Let $S = \{2s \mid s \in \omega\}$. Let $A \subseteq S$ be such that $A_1 = S - A$ is infinite and c.e. For every such A we will define a computable structure $\mathcal{M}_A$ with a single ternary relation.

Let $F = \{4s+1 \mid s \in \omega\}$ and $B = \{4s+3 \mid s \in \omega\}$. Fix a computable bijection f from the set $\{\langle i,j\rangle \in S \mid i < j\}$ onto F. Let also b be a computable bijection from the set $\{\langle j,i\rangle \in S \mid i < j \wedge (i \in A_1 \vee j \in A_1)\}$ onto B. For the function f, we write f_{ij} instead of $f(i,j)$ and similarly for the function b. Define a ternary relation P as follows:

$$P = \{(x, f_{xy}, y) \mid x, y \in S \wedge x < y\} \cup \{(y, b_{yx}, x) \mid x, y \in S \wedge x < y \wedge (x \in A_1 \vee y \in A_1)\}.$$

Finally, let $\mathcal{M}_A = \langle \omega; P\rangle$. Informally, we can view the triples x, w, y with the property $P(x,w,y)$ as labelled arrows (e.g., $x \xrightarrow{w} y$). We start with a structure consisting of the set $S \cup F$ with arrows $i \xrightarrow{f_{ij}} j$, that connect i with j for all $i, j \in S$ such that $i < j$. These arrows can be viewed as a way of redefining the natural ordering $<$ on S. Elements of S can be thought of as "stem elements" and elements of F can be thought of as "forward witnesses." Next, we start enumerating the c.e. set $A_1 = S - A$. At every stage a new element k is enumerated into A_1, we add new arrows together with appropriate elements from B, the "backward witnesses," which intend to exclude k from the initial ordering on S. More precisely, we add arrows $k \xrightarrow{b_{ki}} i$ for all i with $i < k$, and arrows $j \xrightarrow{b_{jk}} k$, for each j with $j > k$. Eventually, exactly the elements of A_1 will be excluded from the ordering, and the final ordering will be an ordering on the set A.

In the resulting structure, every element $x \in A_1$ is connected with every element $y \in S$ such that $x \neq y$ with exactly two arrows: $x \xrightarrow{w} y$ and $y \xrightarrow{w_1} x$. If $x, y \in A$ are such that $x \neq y$ then they are connected with arrows of the type $x \xrightarrow{w} y$ exactly when $x < y$. In other words, the formula

$$\Phi(x,y) =_{def} \exists w P(x,w,y) \wedge \neg \exists w_1 P(y, w_1, x)$$

will be satisfied by exactly those $x, y \in A$ such that $x < y$. The formula Φ will not be satisfied by any pair (x,y) for which at least one of x or y has been excluded.

The following properties of the structure $\mathcal{M}_A$ follow immediately from the definition above.

(1) For every w there is at most one pair x, y such that $P(x,w,y)$.

(2) If $x \in S - A$, then for any $y \in S,\ y \neq x$, there is a unique w_1 such that $P(x, w_1, y)$ and a unique w_2 such that $P(y, w_2, x)$.

(3) If $x, y \in A$, then $x < y \Leftrightarrow \exists w P(x,w,y)$.

(4) $\mathcal{M}_A$ is computable.

To prove (4) note that the relation P is computable because

$$P(x,z,y) \Leftrightarrow x, y \in S \wedge \left[(x < y \wedge z = f_{xy}) \vee (x > y \wedge z \in B \wedge b^{-1}(z) = \langle x, y\rangle)\right].$$

(5) Let D, $E \subseteq S$ be infinite and such that $S-D$ and $S-E$ are infinite and c.e. Then $\mathcal{M}_D \cong \mathcal{M}_E$.

Since D and E are infinite, the orders $(D,<)$ and $(E,<)$, where $<$ is the natural order, are isomorphic to $\mathbb{N}$. The isomorphism between these orders, extended by any bijection between $S-D$ and $S-E$, has a unique natural extension to a map from the domain of $\mathcal{M}_D$ to the domain of $\mathcal{M}_E$. That is, the arrows in $\mathcal{M}_D$ (the elements of F and B) can be uniquely mapped to corresponding arrows in $\mathcal{M}_E$.

To continue with the proof, we let

$$\Theta(x) =_{def} (\exists t)\left[\Phi(x,t) \vee \Phi(t,x)\right].$$

The formula $\Theta(x)$ defines the set A in $\mathcal{M}_A$.

For any structure $\mathcal{M}=(M,P)$ in the language with one ternary predicate symbol we will use the following notation:

$L_{\mathcal{M}} =_{def} \{x \in M | \mathcal{M} \models \Theta(x)\}$,and

$<_{L_{\mathcal{M}}} =_{def} \{(x,y) \in M \times M | \mathcal{M} \models \Phi(x,y)\}$.

Fix $A \subseteq S$ such that $S-A$ is infinite and c.e.

It follows from the discussion above that the formula $\Phi(x,y)$ defines in $\mathcal{M}_A$ the restriction of the natural order $<$ to A. Clearly, $\left(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}}\right)$ has order type ω.

Let $\mathcal{M}_A^\sharp = \prod_C \mathcal{M}_A$. For partial computable functions g and h such that $[g], [h] \in \mathrm{dom}(\mathcal{M}_A^\sharp)$ we have:

(i) $\mathcal{M}_A^\sharp \models \Phi([g],[h]) \Leftrightarrow C \subseteq^* \{i | (g(i) \in A) \wedge (h(i) \in A) \wedge (g(i) < h(i))\}$

(ii) $L_{\mathcal{M}_A^\sharp} = \{[g] \in \mathcal{M}_A^\sharp |\ g(C) \subseteq^* A\}$ and $\left(L_{\mathcal{M}_A^\sharp}, <_{L_{\mathcal{M}_A^\sharp}}\right)$ is a linear order.

Note that (i) follows from Theorem 3, part (2), since $\Phi(x,y)$ is a Boolean combination of Σ_1^0 and Π_1^0 formulas.

For the proof of (ii) notice that for any $[g] \in \mathcal{M}_A^\sharp$ we have either $C \subseteq^* \{i | g(i) \in A\}$ or $C \subseteq^* \{i | g(i) \in \omega - A\}$ because C is cohesive and $\omega - A$ is c.e. Since

$$[g] \in L_{\mathcal{M}_A^\sharp} \Leftrightarrow (\exists x)\left[\Phi([g],x) \vee \Phi(x,[g])\right],$$

the equivalence in part (i) implies that $L_{\mathcal{M}_A^\sharp} = \{[g] \in \mathcal{M}_A^\sharp |\ g(C) \subseteq^* A\}$. It is easy to show that the relation $<_{L_{\mathcal{M}_A^\sharp}}$ is a linear order on $L_{\mathcal{M}_A^\sharp}$.

For any $a \in A$ let $h_a(i) = a$ for all $i \in \omega$. We will call the element $[h_a]$ in $\mathcal{M}_A^\sharp$ a constant in $\mathcal{M}_A^\sharp$.

(6) The set of constants $\{[h_a] | a \in A\}$ in the structure $\mathcal{M}_A^\sharp$ forms an initial segment of $\left(L_{\mathcal{M}_A^\sharp}, <_{L_{\mathcal{M}_A^\sharp}}\right)$ of order type ω.

Clearly, if $a_0, a_1 \in A$, then $\Phi([h_{a_0}],[h_{a_1}])$ if and only if $a_0 < a_1$. Therefore, $\{[h_a] | a \in A\}$ is an ordered set of type ω. It remains to check that $\{[h_a] | a \in A\}$ is an initial segment. Suppose $[h] \in \mathcal{M}_A^\sharp$ and $a \in A$ are such that $\mathcal{M}_A^\sharp \models$

$\Phi([h], [h_a])$. Then

$$C \subseteq^* \{i | \mathcal{M}_A \models \Phi(h(i), a)\} = \{i | h(i) \in A \wedge h(i) < a\} = \bigcup_{k \in A \wedge k < a} \{i | h(i) = k\}.$$

The last expression is a union of a finite family of mutually disjoint c.e. sets. Since C is cohesive, there exists a $k \in A$ such that $C \subseteq^* \{i | h(i) = k\}$, which means that $[h] = [h_k]$ is a constant.

We now define the following Σ_3^0 sentence

$$\Psi =_{def} (\exists x) [\Theta(x) \wedge (\forall y) [\Theta(y) \Rightarrow \Phi(y, x)]] .$$

The intended interpretation of Ψ is that when $\Phi(x, t)$ defines a linear order $(L_{\mathcal{M}}, <_{L_{\mathcal{M}}})$, then the order has a greatest element. Note that $\mathcal{M}_A \models \neg \Psi$. This is because $\left(L_{\mathcal{M}_A}, <_{L_{\mathcal{M}_A}}\right)$ has order type ω and hence has no greatest element.

Before we continue with the proof we recall Proposition 2.1 from [8].

Proposition 10. *(Lerman [8]) Let R be a co-r-maximal set, and let f be a computable function such that $f(R) \cap R$ is infinite. Then the restriction $f \upharpoonright R$ differs from the identity function only finitely.*

We now fix a co-maximal (hence co-r-maximal) set $C \subseteq S$ and an infinite co-infinite computable set $D \subseteq S$. By property (5) above, we have $\mathcal{M}_C \cong \mathcal{M}_D$. Let $\mathcal{M}_C^\sharp = \prod_C \mathcal{M}_C$ and $\mathcal{M}_D^\sharp = \prod_C \mathcal{M}_D$.

It is not hard to show that, since C is co-maximal, for every partial computable function φ for which $C \subseteq^* dom(\varphi)$, there is a computable function f_φ such that $[\varphi] = [f_\varphi]$ (see [4]).

To finish the proof we will establish the following facts:

(7) $\mathcal{M}_C^\sharp \models \Psi$

(8) $\mathcal{M}_D^\sharp \models \neg \Psi$

To prove (7) recall that $L_{\mathcal{M}_C^\sharp} = \{[f] \in \mathcal{M}_C^\sharp | \ f(C) \subseteq^* C\}$. By Proposition 10 if $[f] \in \mathcal{M}_C^\sharp$ is such that $f(C) \subseteq^* C$ and $f(C)$ is infinite, then $[f] = [\mathrm{id}]$. If $f(C)$ is finite, then f is eventually a constant on C, because C is cohesive. Therefore, $L_{\mathcal{M}_C^\sharp} = \{[f_c] \mid c \in C\} \cup \{[\mathrm{id}]\}$. It is easy to see that if $c \in C$, then $\Phi([f_c], [id])$. Thus, $\left(L_{\mathcal{M}_C^\sharp}, <_{L_{\mathcal{M}_C^\sharp}}\right)$ has order type $\omega + 1$ with the greatest element $[id]$. Therefore, $\mathcal{M}_C^\sharp \models \Psi$.

To prove (8), let $D = \{d_0 < d_1 < \cdots\}$. The function g defined as $g(d_i) = d_{i+1}$ is computable. Suppose that $\mathcal{M}_D^\sharp \models \Psi$ and let $[f]$ be the greatest element in $\left(L_{\mathcal{M}_D^\sharp}, <_{L_{\mathcal{M}_D^\sharp}}\right)$. Since $[f] <_{L_{\mathcal{M}_D^\sharp}} [g \circ f]$, it follows that $\mathcal{M}_D^* \models \neg \Psi$.

In conclusion, we defined computable isomorphic structures $\mathcal{M}_C$ and $\mathcal{M}_D$ such that $\prod_C \mathcal{M}_C$ and $\prod_C \mathcal{M}_D$ are not even elementary equivalent. The structure $\mathcal{M}_C$ also provides a sharp bound for the fundamental theorem of cohesive powers. Namely, for the Σ_3^0 sentence Ψ, $\mathcal{M}_C \models \neg \Psi$ but $\prod_C \mathcal{M}_C \models \Psi$.

4 Orders of type ω with cohesive powers not isomorphic to $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$

We prove that if C is co-maximal, then there is a computable linear order $\mathcal{L}$ of type ω (necessarily with a non-computable successor function) such that $\Pi_C \mathcal{L} \ncong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.

Lemma 11. *Let $C \subseteq \omega$ be co-c.e., infinite, and co-infinite. Then there is a computable linear order $\mathcal{L} = (\omega, <_{\mathcal{L}})$ of type ω such that for every partial computable function φ,*

$$\forall^\infty n \in C(\varphi(n)\downarrow \Rightarrow \varphi(n) \text{ is not the } \mathcal{L}\text{-immediate successor of } n). \qquad (*)$$

Proof. Fix an infinite computable set $R \subseteq \overline{C}$. We define $<_{\mathcal{L}}$ in stages. By the end of stage s, $<_{\mathcal{L}}$ will have been defined on $X_s \times X_s$ for some finite $X_s \supseteq \{0, 1, \ldots, s\}$. At stage 0, set $X_0 = \{0\}$ and define $0 \not<_{\mathcal{L}} 0$. At stage $s > 0$, start with $X_s = X_{s-1}$ and update X_s and $<_{\mathcal{L}}$ according to the following procedure.

1. If $<_{\mathcal{L}}$ has not yet been defined on s (i.e., if $s \notin X_s$), then update X_s to $X_s \cup \{s\}$ and extend $<_{\mathcal{L}}$ to make s the $<_{\mathcal{L}}$-greatest element of X_s.
2. Consider each $\langle e, n\rangle < s$ in order. If
   (a) $\varphi_{e,s}(n)\downarrow \in X_s$,
   (b) $\varphi_e(n)$ is currently the $<_{\mathcal{L}}$-immediate successor of n in X_s,
   (c) $n \notin R$, and
   (d) n is not $<_{\mathcal{L}}$-below any of $0, 1, \ldots, e$,
   then let m be the least element of $R - X_s$. Update X_s to $X_s \cup \{m\}$, and extend $<_{\mathcal{L}}$ so that $n <_{\mathcal{L}} m <_{\mathcal{L}} \varphi_e(n)$.

This completes the construction.

We claim that for every k, there are only finitely many elements $<_{\mathcal{L}}$-below k. It follows that $\mathcal{L}$ is of type ω. Say that φ_e *acts for n and adds m* when $<_{\mathcal{L}}$ is defined on an $m \in R$ to make $n <_{\mathcal{L}} m <_{\mathcal{L}} \varphi_e(n)$ as in (2). Let s_0 be a stage with $k \in X_{s_0}$. Suppose at stage $s > s_0$ we add an m to X_s and define $m <_{\mathcal{L}} k$. This can only be due to a φ_e acting for an $n \notin R$ and adding m at stage s. At stage s, we must have $n <_{\mathcal{L}} k$ because $n <_{\mathcal{L}} m <_{\mathcal{L}} k$. Therefore, we must also have $e < k$, for otherwise k would be among $0, 1, \ldots e$, and condition (2d) would prevent the action of φ_e. Furthermore, m is chosen so that $m \in R$ and thus only elements of R are added $<_{\mathcal{L}}$-below k after stage s_0. Hence an m can only be added $<_{\mathcal{L}}$-below k after stage s_0 when a φ_e with $e < k$ acts for an $n <_{\mathcal{L}} k$ with $n \notin R$. Each φ_e acts at most once for every n, and no new $n \notin R$ appears $<_{\mathcal{L}}$-below k after stage s_0. Thus, after stage s_0, only finitely many m are ever added $<_{\mathcal{L}}$-below k.

We claim that for every e, (*) holds. Given e, let ℓ be the $<_{\mathcal{L}}$-greatest element of $\{0, 1, \ldots, e\}$. Suppose that $n >_{\mathcal{L}} \ell$ and $n \in C$. If $\varphi_e(n)\downarrow$, let s be large enough so that $\langle e, n\rangle < s$, $\varphi_{e,s}(n)\downarrow$, $n \in X_s$, and $\varphi_e(n) \in X_s$. Then either $\varphi_e(n)$ is already not the $\mathcal{L}$-immediate successor of n at stage $s+1$, or at stage $s+1$ the conditions of (2) are satisfied for $\langle e, n\rangle$, and an m is added such that $n <_{\mathcal{L}} m <_{\mathcal{L}} \varphi_e(n)$.

Theorem 12. *Let C be a co-maximal set. Then there is a computable linear order $\mathcal{L}$ of type ω such that* [id] *does not have a successor in $\Pi_C\mathcal{L}$. Therefore, $\Pi_C\mathcal{L} \ncong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.*

Proof. Let $\mathcal{L}$ be a computable linear order as in Lemma 11 for C. Suppose that φ is a partial computable function such that $[\mathrm{id}] <_{\Pi_C\mathcal{L}} [\varphi]$. We show that $[\varphi]$ is not the $<_{\Pi_C\mathcal{L}}$-immediate successor of [id]. The inequality $[\mathrm{id}] <_{\Pi_C\mathcal{L}} [\varphi]$ means that $(\forall^\infty n \in C)\,(n <_\mathcal{L} \varphi(n))$. However, by Lemma 11,

$$(\forall^\infty n \in C)\,(\varphi(n) \text{ is not the } \mathcal{L}\text{-immediate successor of } n).$$

Define a partial computable ψ so that, for every n,

$$\psi(n) = \begin{cases} \text{the least } m \text{ such that } n <_\mathcal{L} m <_\mathcal{L} \varphi(n) & \text{if there is such an } m; \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $(\forall^\infty n \in C)\,(n <_\mathcal{L} \psi(n) <_\mathcal{L} \varphi(n))$. Thus, $[\mathrm{id}] <_{\Pi_C\mathcal{L}} [\psi] <_{\Pi_C\mathcal{L}} [\varphi]$. So, $[\varphi]$ is not the $<_{\Pi_C\mathcal{L}}$-immediate successor of [id].

It follows that $\Pi_C\mathcal{L} \ncong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$ because every element of $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$ has an immediate successor, but $[\mathrm{id}] \in \Pi_C\mathcal{L}$ does not have an immediate successor.

Note that the sentence Ψ that states that every element has an immediate successor is Π^0_3. Then for the computable linear order $\mathcal{L}$ of type ω constructed above, $\mathcal{L} \vDash \Psi$ but $\Pi_C\mathcal{L} \vDash \neg\Psi$.

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