Combining contact and measure

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Point-free theory of space, I

The point-free theories to space aim to replace the abstract primitive notions and relations ‘point’, ‘line’r, ‘between’ etc. by more realistic ones. In other words, the aim is to reverse the atomistic idea about the space taking as primitive some sets of points, regions, and to define points as some sets of regions.

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Standard model. Let T be a topological space with interior operator *Int* and closure operator *Cl*. A set A is called *regular closed* if $Cl(Int(A)) = A$. The regular closed sets in T with the constants $\varnothing$ and T and $\subseteq$ *part-of* form a complete Boolean algebra, $RC(T)$. The binary relation *contact*, C_T, is defined by $C_T(A, B) \iff A \cap B \neq \varnothing$.

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The models of the universal fragment of the $Th(\langle RC(T), C_T \rangle)$ are natural algebraic structures called *contact algebras*.

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Remarks. 1. In $RC(T)$ the Boolean meet and complement are not the corresponding set theoretic operations:
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2. An isomorphic variant is to take the algebra of regular open sets $RO(T)$ with the following contact relation:
$C_T(A, B) \iff Cl(A) \cap Cl(B) \neq \varnothing$.

Point-free theory of space, II

Arntzenius' approach to point-free space is to take as regions the Borel sets modulo null sets with respect to some σ-additive measure. So, the most standard model of the regions will be $Bor(\mathbb{R}^m)/Null$ where $Null$ is the set of all sets in $\mathbb{R}^m$ with Lebesgue measure 0. This Boolean algebra has nice representation theory, but the Lebesgue measure is not finitely additive over $RC(\mathbb{R}^m)$, see Arntzenius'08, Lando'18, Lando and Scott'19.

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In this talk we bring together both approches in the structures called *contact algebras with qualitative measure*.

Measure on Boolean algebra

Let $\mathbf{B} = \langle B, 0_B, 1_B, \sqcup, \sqcap, * \rangle$ be a Boolean algebra. *A measure on* $\mathbf{B}$ is a function $\mu : B \to \mathbb{R} \cup \{+\infty\}$ such that

$$\mu(0_B) = 0,\ \mu(1_B) > 0$$
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A measure μ on $\mathbf{B}$ is *positive* if for all $a \in B$, $a > 0_B$ implies $\mu(a) > 0$.

A measure μ on $\mathbf{B}$ is *probability measure* if $\mu(1_B) = 1$.

Contact algebras with qualitative measure — CAQM-structures

CAQM-structure is a tuple $\mathcal{F}$ of the kind $\langle\langle \mathbf{B}, C\rangle, \mu\rangle$, where

— $\mathbf{B} = \langle B, 0_B, 1_B, \sqcup, \sqcap, *\rangle$ is a Boolean algebra

— $\langle \mathbf{B}, C\rangle$ is a contact algebra, i.e. for all $a, a_1, a_2, b \in B$

$$C(a, b) \Rightarrow a \neq 0_B \text{ and } b \neq 0_B$$
$$a \neq 0_B \Rightarrow C(a, a)$$
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$\mathcal{F}$ is called *contact algebras with qualitative probability measure (CAQPM)* if μ is probability measure.

$\mathcal{F}$ is called *connected* if for all $a \in B \setminus \{0_B, 1_B\}$ it holds $C(a, a^*)$.

Examples for connected CAQM-structures, I

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1. $\langle POL(\mathbb{R}^m), \mu \rangle$ is a connected CAQM-structure, where $POL(\mathbb{R}^m)$ is the generated subalgebra of $RC(\mathbb{R}^m)$ from the set of all basic polytops in $\mathbb{R}^m$ and μ is the Lebesgue measure.
(A basic polytop in $\mathbb{R}^m$ is a finite intersection with non empty interior of closed halfspaces (hyperplanes) or the empty set.)

2. $\langle POL([0,1]^m), \mu \rangle$ is a connected CAQPM-structure, where $POL([0,1]^m)$ is the generated subalgebra of $RC([0,1]^m)$ from the set of all basic polytops in $[0,1]^m$ and μ is the Lebesgue measure.

Examples for connected CAQM-structures, II

3. Let R be a reflexive and symmetric binary relation on a nonempty set W. Define the binary relation C_R on W as follows:

$$\langle A, B\rangle \in C_R \iff (\exists x \in A)(\exists y \in B)(\langle x, y\rangle \in R)$$

The Boolean algebra of all subsets of W with C_R is a contact algebra. If μ is a positive measure on $\mathcal{P}(W)$ then we will call this CAQM-structure *relational* or *Kripke structure* $\mathcal{F} = \langle W, R, \mu\rangle$. The CAQM-structure $\mathcal{F}$ is connected iff the graph $\langle W, R\rangle$ is connected.

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Remark. If W is finite then probability measures on $\mathcal{P}(W)$ are determined by the functions $f : W \to (0, 1)$:
$\mu(X) := \sum_{x\in X} f(x)/\sum_{x\in W} f(x)$.

The language CLQM

1. Boolean terms for representing the regions:
   - Boolean constants: 0, 1
   - countable set of Boolean variables, Var, $p, q, \ldots$ etc.
   - if a and b are Boolean terms, then a^*, $(a \sqcap b)$, $(a \sqcup b)$ are Boolean terms.

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4. The abbreviations $a < b$, $a \neq b$, $a <_\mu b$, $a \neq_\mu b$ are standard, for example $a <_\mu b$ is $\neg(b \leq_\mu a)$.

Semantics of CLQM

Let $\mathcal{F} = \langle\langle \mathbf{B}, C\rangle, \mu\rangle$ be a CAQM-structure. An assignment in $\mathcal{F}$, as usual, is a function $v : Var \rightarrow B$. It can be extended to all Boolean terms in a standard way. The pair $\mathcal{M} = \langle \mathcal{F}, v\rangle$ is called model over $\mathcal{F}$.

Evaluation of the atomic formulas in a model $\mathcal{M}$:

$\mathcal{M} \models (a \leq b)$ iff $v(a) \leq v(b)$

$\mathcal{M} \models C(a, b)$ iff $\langle v(a), v(b)\rangle \in C_B$

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For arbitrary formulas φ, $\mathcal{M} \models \varphi$ is defined in a standard way. Satisfiability, validity in a CAQM-structure (denoted $\mathcal{F} \models \varphi$), and validity in a class of structures (denoted $\mathcal{C} \models \varphi$) have the usual meaning.

Some valid formulas

1. $\mathcal{C}_{all} \models a \neq 0 \leftrightarrow 0 <_\mu a$
2. $\mathcal{C}_{all} \models a < 1 \leftrightarrow a <_\mu 1$
3. $\mathcal{C}_{all} \models a \leq_\mu b \vee b \leq_\mu a$
4. $\mathcal{C}_{all} \models a \leq_\mu b \wedge b \leq_m d \rightarrow a \leq_\mu d$
5. $\mathcal{C}_{all} \models a \sqcap d = 0 \wedge b \sqcap d = 0 \wedge d <_\mu 1 \rightarrow (a \leq_\mu b \leftrightarrow a \sqcup d \leq_\mu b \sqcup d)$

$5'$. $\mathcal{C}_{all} \models a \sqcap d = 0 \wedge b \sqcap d = 0 \wedge d <_\mu 1 \rightarrow (a <_\mu b \leftrightarrow a \sqcup d <_\mu b \sqcup d)$

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6. Let $\mathcal{C}_{all,prob}$ be the class of all CAQPM-structures. Then in $\mathcal{C}_{all,prob}$ formulas 5, $5'$ and $5''$ without the conjunctive term $d <_\mu 1$ are valid.

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7. Let $\mathcal{C}_\infty$ be the class of all CAQM-structures such that $+\infty$ belongs to the range of μ and for any positive r there exists $a \in B$ such that $\mu(a) > r$. Then

$$\mathcal{C}_\infty \models (a =_\mu 1) \vee (a^* =_\mu 1)$$

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Our purpose is to present a formal system $L_{pol,[0,1]}$ which is complete with respect to $\langle POL([0,1]^m), \mu \rangle$. What about $\mathcal{C}_{all}$?

The formal system $L_{pol,[0,1]}$

Axioms:
1. The axioms for the connected contact algebras;

2.1. $0 <_\mu 1$
$(p \leq_\mu q) \vee (q \leq_\mu p)$
$(p \leq_\mu q) \wedge (q \leq_\mu r) \rightarrow (p \leq_\mu r)$
$(p \sqcap q = 0) \wedge (p \sqcap r = 0) \wedge (q \sqcap r = 0) \rightarrow ((p \leq_\mu q) \leftrightarrow (p \sqcup r \leq_\mu q \sqcup r))$

2.2. For any integer $n > 2$ finitely many formulas of the following kind $part(p_1, p_2, \ldots, p_n) \rightarrow \neg\varphi_\sigma$ described on a next slide.

Rules: modus ponens and uniform substitution

Some systems of simple linear inequalities, I

Let $x_1, x_2, \ldots, x_n$ be real variables. We consider finite systems (σ) of linear inequalities of the following kind:

$$\sum_{i \in I_l} x_i < \sum_{i \in I_r} x_i \qquad \text{or} \qquad \sum_{i \in I_l} x_i \leq \sum_{i \in I_r} x_i,$$

where $I_l \cup I_r \subseteq \{1, 2, \ldots, n\}$ and $I_r \neq \varnothing$, providing $\sum_{i \in \varnothing} x_i = 0$.
For any n there is finetely many such systems (σ) and it is decidable whether given system has a positive rational solution (all components to be positive).
With any inequality e of the above mentioned type we associate the following formula ϕ_e:

$$(\sqcup_{i \in I_l} p_i) <_\mu (\sqcup_{i \in I_r} p_i) \qquad \text{or} \qquad (\sqcup_{i \in I_l} p_i) \leq_\mu (\sqcup_{i \in I_r} p_i),$$

where $p_1, p_2, \ldots, p_n$ are different Boolean variables.
Let φ_σ be the conjunction of all ϕ_e for all inequalities e from σ.

Some systems of simple linear inequalities, II

Let $part(p_1, p_2, \ldots, p_n)$ be the formula saying that $v(p_1), \ldots, v(p_n)$ is a partition of the Boolean 1, i.e.
$part(p_1, p_2, \ldots, p_n) := \bigwedge_{1 \le i < j \le n} (p_i \sqcap p_j = 0) \wedge (\bigsqcup_{1 \le i \le n} p_i = 1)$

Proposition. The following are equivalent
(i) $part(p_1, p_2, \ldots, p_n) \to \neg\varphi_\sigma$ is unsatisfiable;
(ii) the system (σ) has no solution in $\mathbb{Q}_{>0}$.

Axioms 2.2.

For $n > 2$ let $\Phi_n := \bigwedge_{(\sigma)\in\Xi_n}(part(p_1, p_2, \ldots, p_n) \rightarrow \neg\varphi_\sigma)$, where Ξ_n is the set of all systems of inequalities over $x_1, x_2, \ldots, x_n$ which have no positive solution.

The axioms of this set are all Φ_n for $n > 2$.

Remark. The set of axioms from $L_{pol,[0,1]}$ is infinite and recognizible (recursive).

The main result

The logic $L_{pol,[0,1]}$ is correct and complete with respect the class of all connected CAQPM-structures. All of the structures $\langle POL([0,1]^m), \mu\rangle$ validate the same formulas, i.e. it holds the following

Theorem. Let φ be a formula. The following are equivalent:
(i) φ is a theorem of $L_{pol,[0,1]}$;
(ii) $\mathcal{C}^{prob}_{all,conn} \models \varphi$;
(iii) for a given $m \geq 1$, $\langle POL([0,1]^m), \mu\rangle \models \varphi$;
(iv) $\langle POL([0,1]), \mu\rangle \models \varphi$;
(v) For every finite relational connected CAQPM-structure (Kripke frame) $\mathcal{F}$ it holds $\mathcal{F} \models \varphi$.

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(iv) $\langle POL([0,1]), \mu\rangle \models \varphi$;
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For (v)$\Rightarrow$(iv) the main idea is step by step to untie the given finite connected graph, to modify the measure function in an evident way, to change the assignment in an appropriated way and at the end to realize the obtained CAQPM-structure by 1-dimensional polytops.

The idea of the proof, cont. 1

The implication (v)$\Rightarrow$(i) follows from the following

Proposition. There is an algoritm which for every formula φ gives either a finite relational connected CAQPM-structure (Kripke frame) $\mathcal{F}$ such that $\mathcal{F} \models \varphi$ or a proof of $\neg\varphi$ in $L_{pol,[0,1]}$.

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Sketch of the proof. Let $p_1, \ldots, p_n$ be different Boolean variables. Let us call, as usually, *a monom over* $p_1, \ldots, p_n$ Boolean term from the following kind $\sqcap_{1 \leq i \leq n} \lambda_i p_i$, where λ_i is either the empty word or $\neg$.
Let us call *good conjunction* a formula of the following kind

$$\bigwedge_{1 \leq i \leq 2^n} \lambda_i (m_i > 0) \wedge \bigwedge_{1 \leq i < j \leq 2^n} \delta_{ij} C(m_i, m_j) \wedge \chi,$$

where $m_1, \ldots, m_{2^n}$ are all monoms over $p_1, \ldots, p_n$, in χ occurs only μ-atomic formulas and all λ_i, δ_{ij} are either the empty word or $\neg$.

The idea of the proof, cont. 2

It should be clear that there exists an algorithm which for any φ gives a finite disjunction Ψ from good conjunctions such that $\varphi \leftrightarrow \Phi$ is provable in $L_{pol,[0,1]}$.

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Now the proof relies on analizing at two levels the contact part of every good conjunction containing at least one positive monom. As result we choose some of these good conjunctions. With any choosen conjunction we associate a finite connected graph and a proof of the negation of any non-choosen is presented.

The idea of the proof, cont. 2

It should be clear that there exists an algorithm which for any φ gives a finite disjunction Ψ from good conjunctions such that $\varphi \leftrightarrow \Phi$ is provable in $L_{pol,[0,1]}$.

Now the proof relies on analizing at two levels the contact part of every good conjunction containing at least one positive monom. As result we choose some of these good conjunctions. With any choosen conjunction we associate a finite connected graph and a proof of the negation of any non-choosen is presented.

At the end, the χ-part of any choosen good conjunction is analized. As result we have either probability measure over the associated graph or a proof of the negation of this conjunction.

Close results

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4. The logic of the polytops in $\mathbb{R}^2$ is also known.

Complexity

In 2019 Balbiani and T. proved that:

1. Satisfiability in $\mathcal{C}_{all}^{prob}$ is NP-complete problem.
2. Satisfiability in $\mathcal{C}_{all,conn}^{prob}$ is PSPACE-complete problem.

Finite axiomatizability

The presented here axiomatization of $\mathcal{C}_{all}^{prob}$ contains infinitely many axioms, Φ_n for $n > 2$. I strongly believe that this logic is not finitely axiomatizable: for any system (σ_n) of inequalities over $x_1, \ldots, x_n$ which has a positive solution I can effectively show a system (σ_{n+1}) over $x_1, \ldots, x_n, x_{n+1}$ without positive solution. If we suppose that some formula θ with n Boolean variables axiomatizes $\mathcal{C}_{all}^{prob}$ then (???) Φ_{n+1} **is not derivable from** θ.

Thank you for your attention!