New results on subrecursive degrees of representations of irrational numbers

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Logic conference[1] on the occasion of the 85th anniversary of prof. Dimiter Vakarelov

Strelcha, 18-21 September 2023

[1]funded by FNI-SU project 80-10-210/22.5.2023

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A real number α is computable, if there exists a recursive Cauchy name C for α.

Intersection of intervals

Theorem (Skordev, Mostowski)

A real number α *is computable if and only if there exist primitive recursive functions* $A, B : \mathbb{N} \to \mathbb{Q}$*, such that* $A(n) < \alpha < B(n)$ *for all* n *and* $\lim_{n\to\infty} B(n) - A(n) = 0$.

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where $C(k+1)$ is computable in $n + x + 1$ steps.

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In particular, $\mathcal{F}$ can be any of the classes $\mathcal{E}^m$ for $m \geq 2$ or $\mathcal{M}^2$, $\mathcal{L}^2$.

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Theorem (Peshev, Skordev)

For any fixed N there exists an $\mathcal{E}^2$-computable operator Γ of $N+1$ arguments, such that whenever $C_0, C_1, \ldots, C_N$ are Cauchy names for the coefficients of the polynomial

$$P(z) \ = \ \alpha_0 z^N + \alpha_1 z^{N-1} + \ldots + \alpha_{N-1} z + \alpha_N,$$

$\Gamma(C_0, C_1, \ldots, C_N)$ is a Cauchy name for some root of P.

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Our main question: *is it possible to transform one representation into another without using unbounded search*?

Subrecursive reducibility

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Note that $R_2 \not\leq_S R_1$ means that there exists α, for which the transformation is not possible.

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$$D(q) = \begin{cases} 0, & \text{if } q < \alpha, \\ 1, & \text{if } q > \alpha. \end{cases}$$
- for $b \geq 2$, the *base-b expansion* of α is the function $E_b : \mathbb{N} \to \{0,1,\ldots,b-1\}$, such that
$$\alpha \;=\; \sum_{n=0}^{\infty} E_b(n) \cdot b^{-n}.$$

$E_b \leq_S D$

Assume we have computed $E_b(1), E_b(2), \ldots, E_b(n)$ and let

$$q_n \;=\; E_b(1) \cdot b^{-1} + E_b(2) \cdot b^{-2} + \ldots + E_b(n) \cdot b^{-n}.$$

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To compute $E_b(n+1)$: we search for the unique $\mathtt{D} \in \{0, 1, \ldots, b-1\}$, such that

$$D(q_n + \mathtt{D} \cdot b^{-n-1}) = 0 \quad \& \quad D(q_n + (\mathtt{D}+1) \cdot b^{-n-1}) = 1$$

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No unbounded search is used in this algorithm!

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Therefore, we have $E_b <_S D$.

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$C \leq_S E_b$ is obvious:
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Theorem (Skordev)

Suppose $b \geq 3$ and the base-b expansion of α does not contain the digit $b-1$. Then $E_b \leq_S C$ for any Cauchy sequence C for α.

Hurwitz characteristic

Let us form the Farey pair tree of intervals:

- the root is $(\frac{0}{1}, \frac{1}{1})$;
- the left descendant of $(\frac{a}{b}, \frac{c}{d})$ is $(\frac{a}{b}, \frac{a+c}{b+d})$;
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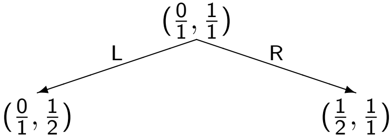

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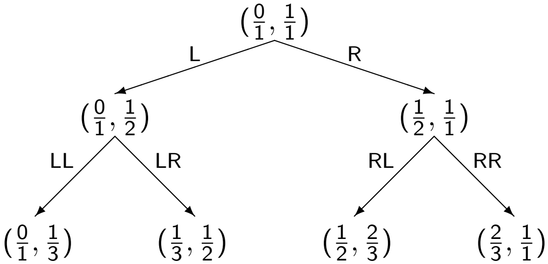

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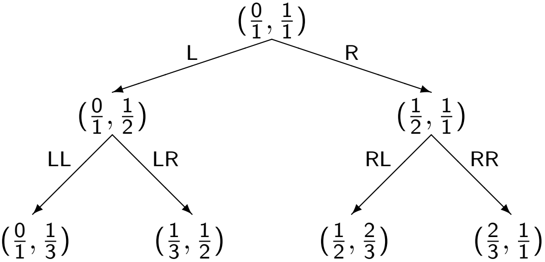

The Hurwitz characteristic H of α is the unique infinite path in the tree, which consists of all intervals containing α.

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Both algorithms do not use unbounded search!

Continued fraction

The continued fraction of α is the unique sequence $c : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that

$$\alpha \;=\; 0 + \cfrac{1}{c(0) + \cfrac{1}{c(1) + \cfrac{1}{\ddots}}}$$

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The following equality relates the continued fraction to the Hurwitz characteristic:

$$H = \underbrace{\mathrm{LL}\ldots\mathrm{L}}_{c(0)-1}\underbrace{\mathrm{RR}\ldots\mathrm{R}}_{c(1)}\underbrace{\mathrm{LL}\ldots\mathrm{L}}_{c(2)}\underbrace{\mathrm{RR}\ldots\mathrm{R}}_{c(3)}\ldots$$

$H \leq_S [\,]$

Therefore, we can compute H from the continued fraction $c = [\,]$: given n, compute the unique $x \leq n+1$, such that

$$c(0)+c(1)+\ldots+c(x-1) \;<\; n+2 \;\leq\; c(0)+c(1)+\ldots+c(x).$$

Then $H(n) \;=\; \begin{cases} \mathrm{L}, & \text{if } x \text{ is even,} \\ \mathrm{R}, & \text{if } x \text{ is odd.} \end{cases}$

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The two inequalities may be checked using the graph of the bounded sum of the continued fraction!

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For any representation R (considered as a function) we define a new representation $\mathcal{G}(R)$ by:

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Main question: Is $\mathcal{G}(R)$ subrecursively equivalent to a known representation, or it gives rise to a new subrecursive degree?

Two technical tools

(Tool 1) : There exists a function $t : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $\mathcal{G}(t) <_S t$.

Informally, t is a complex function, but its graph is simple.

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For a function s, let s^Σ be the bounded sum of s, $s^\Sigma(x) = \sum_{y=0}^{x} s(y)$.

(Tool 2) : There exists a function $s : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $\mathcal{G}(s^\Sigma) <_S \mathcal{G}(s)$.

Informally, the graph of s is complex, but the graph of its bounded sum is simple.

Applications

Let us take α to be the irrational number with continued fraction t, where t is the function given by Tool 1. We obtain

$$\mathcal{G}([\,]) <_S [\,].$$

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We also have shown: $H \leq_S \mathcal{G}([\,]^{\Sigma})$ (in fact, $H \equiv_S \mathcal{G}([\,]^{\Sigma})$). Combining these results we obtain:

Theorem

$$D \equiv_S H <_S \mathcal{G}([\,]) <_S [\,].$$

Best approximations and Baire sequences

Let $\alpha \in (0, 1)$ be irrational with Hurwitz characteristic H and $(l_1, r_1), (l_2, r_2), \ldots, (l_n, r_n), \ldots$ be the corresponding sequence of intervals in the Farey pair tree.

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Write $H = R^{B(0)} L R^{B(1)} L \ldots R^{B(n)} L \ldots$. The function B is called the standard Baire sequence of α.

Graphs of L and R

It is known that

$$D <_S L \equiv_S A <_S [\,], \quad D <_S R \equiv_S B <_S [\,], \quad \{L, R\} \equiv_S [\,],$$

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Graphs of Baire sequences

By using Tool 1 and Tool 2 in the same way, as for continued fractions, we obtain:

$$\mathcal{G}(A^{\Sigma}) <_S \mathcal{G}(A) <_S A, \quad \mathcal{G}(B^{\Sigma}) <_S \mathcal{G}(B) <_S B.$$

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Therefore, $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are subrecursively incomparable.

General results on representations

Definition (informal)

A representation R of an irrational number α is a set of functions, which is computably equivalent to the Dedekind cut of α.

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Thanks for your attention!