Correspondence problem on several classes of frames for intuitionistic propositional formulas.

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Introduction

Correspondence problems

Correspondence problems

Correspondence theory is a classical topic of modal logic, stemming from the natural duality between modal or intuitionistic Kripke frames and FO-models. Van Benthem states three problems of definability:

1. Is there an algorithm that can determine whether a modal(or intuitionistic) formula φ has an FO-definition A?
2. Is there an algorithm that can determine whether an FO-sentence A has a modal (or intuitionistic) definition φ?
3. Is there an algorithm that can determine whether an FO-sentence A and a modal (or intuitionistic) formula φ express the same property of frames?

Undecidability of the correspondence problems

All three problems of correspondence receive a negative answer in the general case, due to Chagrova. Her results show that correspondence is undecidable in both the modal and intuitionistic case.

Correspondence problems

Definability with respect to a class $\mathcal{K} \subseteq PO$

Despite the negative result in general, definability with respect to a class is not always undecidable, depending on the class $\mathcal{K}$ in consideration.

Definition (Intuitionistic definition/FO definition)

We say that the propositional formula φ defines the FO sentence A (and that A defines φ) with respect to a class $\mathcal{K}$ of partial orders if for every structure $\mathfrak{F} \in \mathcal{K}$, $\mathfrak{F} \vDash A$ precisely when $\mathfrak{F} \vDash \varphi$.

Scope of interest

Our aim is to analyse the correspondence problem modulo classes of frames, with respect to which the logic $LC = IPL + (p \to q) \vee (q \to p)$ is complete.
The class of frames validating LC contains exactly the frames $\mathfrak{F}$ such that the upper closure $a \uparrow$ of any element $a \in \mathfrak{F}$ is a chain.

The formulas $\varphi_{depth\leq n}$

Definition ($\varphi_{depth\leq n}$)

Let $1 \leq n < \omega$, the formula $\varphi_{depth\leq n}$ is defined recursively:

- $\varphi_{depth\leq 1} = p_1 \vee \neg p_1$
- $\varphi_{depth\leq n+1} = p_{n+1} \vee (p_{n+1} \to \varphi_{depth\leq n})$

Models of $\varphi_{depth\leq n}$

The models of $\varphi_{depth\leq n}$ are precisely the partial orders $\mathfrak{F}$ such that every chain contains at most n elements.

FO definability

FO definability

Properties of Int formulas

Let $\mathcal{K}$ be a class of frames for LC and φ be a propositional formula.

- If $var(\varphi) = \{p_1, \cdots, p_n\}$ and $\mathfrak{F} \vDash \varphi$, where $\mathfrak{F}$ is an infinite chain or a chain with at least 2^n elements, then $LC \vDash \varphi$.
- If the chain with n elements $\mathfrak{F}_n \vDash \varphi$ and $0 < k \leq n$ then $\mathfrak{F}_k \vDash \varphi$.
- For a model $\mathfrak{F} \vDash LC$, $\mathfrak{F} \vDash \varphi$ precisely when every chain in $\mathfrak{F}$ validate φ.

FO definability is decidable wrt every class $\mathcal{K}$ of frames for LC

Let $\mathcal{K}$ be any class of frames for LC. In conclusion of the above arguments, every propositional formula φ is definable with respect to $\mathcal{K}$, moreover we can effectively find a definition of φ:

- If $LC \vDash \varphi$, then $\exists x(x \doteq x)$ is an FO definition of φ wrt $\mathcal{K}$.
- If φ is not validated by the singleton frame, then $\exists x(\neg x \doteq x)$ is an FO definition of φ wrt $\mathcal{K}$.
- If the maximal chain that validates φ has n elements (where $n \leq 2^{|vars(\varphi)|}$) then $\forall x_1 \cdots \forall x_n \forall x_{n+1}(\bigwedge_{1\leq i\leq n}(x_i \leq x_{i+1}) \to \bigvee_{1\leq i<j\leq n}(x_i \doteq x_j))$ is an FO definition of φ wrt $\mathcal{K}$.

The monadic theory of at most countable frames for LC

Frames for LC

Frames for LC

From now on, we will denote by PL the class of all frames for LC and by $\mathcal{PL}^{countable}$ the class of all at most countable frames for LC. We will call such frames postlinear orders.
The class $\mathcal{PL}$ is axiomatized by the axiom for partial orders +
$\forall x \forall y \forall z (x \leq y \wedge x \leq z \rightarrow y \leq z \vee z \leq y)$.
We will be interested in proving that the MSO theory of the class $\mathcal{PL}^{countable}$ is decidable.

Connected frames

We say that a frame $\mathfrak{F} \in \mathcal{PL}$ is connected if $\mathfrak{F} \vDash \forall x \forall y \exists z (x \leq z \wedge y \leq z)$. Denote the class of all connected frames by $\mathcal{PL}_{connected}$ and of all countable connected frames by $\mathcal{PL}^{countable}_{connected}$.
We can readily see that any (countable) frame $\mathfrak{F} \in \mathcal{PL}$ is the disjoint union of (countable) connected frames.

S2S

- S2S is the monadic second-order theory of the complete infinite binary tree in the language $\{l, r\}$ where l and r are interpreted as the unary functions giving respectively the first and second successor of their argument.
- We express elements of the complete infinite binary tree as words over $\{0, 1\}$ and abbreviate $l(x)$ and $r(x)$ as respectively $x0$ and $x1$.
- Both the prefix and the lexicographic order are definable in S2S.
- As shown by Rabin [1], $S2S$ is decidable and is expressive enough to interpret interesting theories. In particular, Rabin shows that the monadic second-order theory of the class of all at most countable linear orders is decidable.

Rabin's proof of decidability of the MSO theory of at most countable linear orders

Embedding of $\mathbb{Q}$ inside the complete infinite binary tree

Rabin shows that the definable set if all words containing a unique occurrence of 101 at the end under the lexicographic order is a dense linear order without endpoints. By Cantor's theorem, this ordered set is isomorphic to $\mathbb{Q}$. Therefore, the MSO theory of $\mathbb{Q}$ is decidable.

Decidability of the MSO teory of at most countable linear orders

Every at most countable linear order embeds into $\mathbb{Q}$. Take an MSO sentence A about linear orders, relativize all individual quantifiers to elements of a set variable X and all set quantifiers to subsets of X and obtain $A\star$. Now A is a theorem of at most countable linear orders precisely when $(\forall X \subseteq \mathbb{Q})A\star$ is a theorem of S2S.

Strategy for $\mathcal{K}$

We will use the same methodology: find a countable connected postlinear order which embeds every other and find a definable copy of it inside the binary tree.

The feather structure

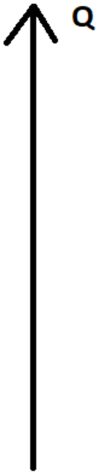
The feather structure

We will define the countable connected postlinear order $\mathfrak{F}$ by recursion:

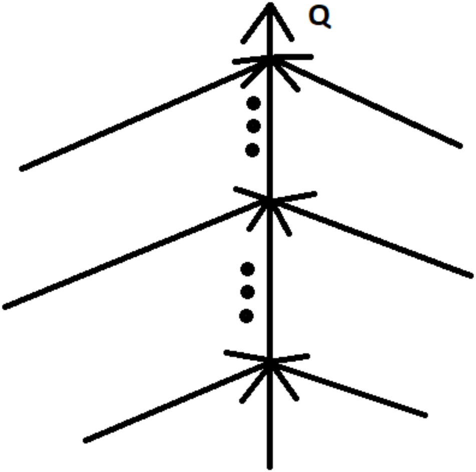
- Define $\mathfrak{F}_0$ to be the empty linear order.
- Define $\mathfrak{F}_1$ to be $\mathbb{Q}$ with its usual ordering.
- For every $n < \omega$, for each element of $\mathfrak{F}_{n+1} \setminus \mathfrak{F}_n$ take ω fresh isomorphic copies of $\mathbb{Q}$ and attach them below the element.

We obtain $\mathfrak{F} = \bigcup_{n<\omega} \mathfrak{F}_n$.

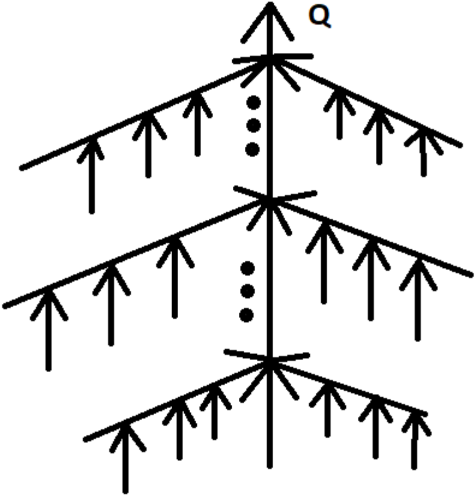
Feather construction step 1



Feather construction step 2



Feather construction step 3

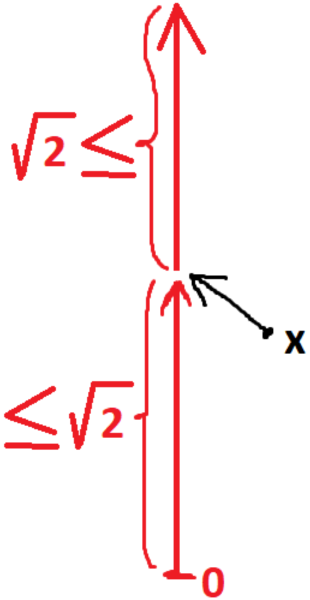


Every at most countable postlinear order embeds into $\mathfrak{F}$

General idea

- Take an at most countable postlinear order $\mathfrak{A}$ and enumerate its elements $a_0, a_1, \cdots$ and construct an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{F}$ by steps.
- At step 0 match a_0 with any element b in $\mathfrak{F}$ and embed $a_0 \uparrow$ inside $b \uparrow$.
- At step n+1, if a_{n+1} is not already matched, some part of the chain $a_{\{}n+1\} \uparrow$ is already embedded and we attach the lower half of the chain in a suitable branch.

Pathological situation



Embedding into $\mathfrak{F}$

Partial completion of $\mathfrak{A}$

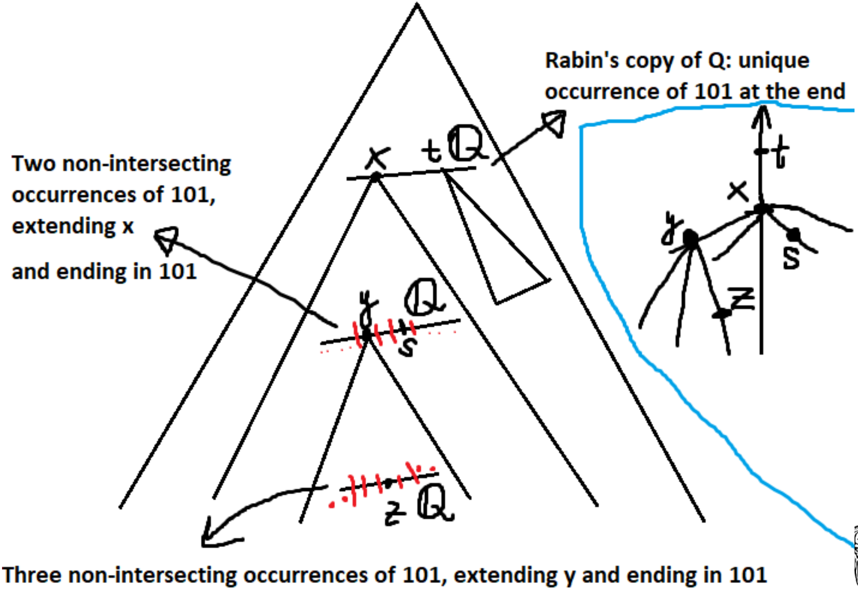
In order to avoid the pathological case, we first embed $\mathfrak{A}$ inside a structure, containing some of the missing limit points. Define $\mathfrak{B}$ as follows:

- $|\mathfrak{B}| = |\mathfrak{A}| \cup \{a\uparrow \cap b\uparrow \mid a, b \in |\mathfrak{A}|$ and $a\uparrow \cap b\uparrow$ does not contain a least element$\}$
- $\leq^{\mathfrak{A}} \subseteq \leq^{\mathfrak{B}}$, every new element $a\uparrow \cap b\uparrow$ is smaller than every element in $a\uparrow \cap b\uparrow$ and bigger than every lower bound for $a\uparrow \cap b\uparrow$, and two new elements are ordered by inclusion.

Now $\mathfrak{A}$ embeds into $\mathfrak{B}$ and for every $a, b \in |\mathfrak{B}|$, $a\uparrow \cap b\uparrow$ contains a least element. Since the new elements are at most as many as $|\mathfrak{A}| \times |\mathfrak{A}|$, $\mathfrak{B}$ remains countable.

Embedding of $\mathfrak{B}$ into $\mathfrak{F}$

At step n+1 of the embedding, $a_{n+1}\uparrow \cap a_i\uparrow$ has a least element for every $0 \leq i \leq n$. Take the least of those least elements and call it m. Now take an unused branch of m and isomorphically embed $a_{n+1}\uparrow \setminus m\uparrow$.


Rabin's copy of Q: unique
occurrence of 101 at the end
Two non-intersecting
occurrences of 101,
extending x
and ending in 101
Three non-intersecting occurrences of 101, extending y and ending in 101

Intuitionistic definability and correspondence

Correspondence problem

Decidability of the correspondence problem

Since the theory of PL is decidable and we can effectively compute a definition of a propositional formula if such exists, the Correspondence problem's decidability follows directly:

1. Given a propositional formula φ and FO sentence A, if φ is undefinable, A and φ are not correspondent.
2. Otherwise, take an FO definition B of φ and check if $PL \models A \leftrightarrow B$.

Intuitionistic definability

Definability by propositional formulas is decidable

The following algorithm determines whether a given FO sentence A has a definition, and if so produces it:

1. If $PL \models A$, $\top$ defines A.
2. Otherwise, if $PL \models \neg A$, $\bot$ defines A.
3. Otherwise, if A has a model in PL with an infinite chain, A is undefinable (otherwise its definition would be a theorem of LC)
4. Otherwise, we know that all chains in any model of A are bounded in size by some natural number n (if not, by a compactness argument there is a model of the theory of that model which has an infinite chain and then A would have a model with an infinite chain). By standard results from Ehrenfeucht-Fraisse games, we can determine that this number n is at most $2^{qr(A)}$.
5. Check if there is an $n \leq 2^{qr(A)}$ such that all postlinear orders with at most n elements validate A and no postlinear order with more than n elements validates A. If such exists A is definable by $\varphi_{depth \leq n}$ and is otherwise undefinable.

Bibliography

[1] Michael O. Rabin. “Decidability of Second-Order Theories and Automata on Infinite Trees”. In: **Transactions of the American Mathematical Society** 141 (1969), pp. 1–35.