DEFINABILITY IN THE LOCAL THEORY OF THE ω-TURING DEGREES

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Abstract

The present paper continues the studying of the definability in the local substructure $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees, started in [Sariev and Ganchev 2014]. We show that the class $\mathbf{I}$ of the intermediate degrees is definable in $\mathcal{G}_{T,\omega}$.

1 Introduction

The investigation of definability issues in degree structures is a main part of the research in Computability theory. Suppose given a degree structure possessing an ordering relation and a least element. If additionally the structure is augmented with a jump operation, natural questions about the first-order definability (in the language of the structure order) of degree classes, determined by the structure jump operation, arise. The same questions can be transferred to its local substructure (namely, the substructure consisting of the degrees bounded by the jump of the least element) as well. As an interesting special case one can ask about the definability of the classes from the jump hierarchy.

The jump hierarchy was firstly introduced for the local substructure of the Turing degrees in [Cooper 1972] and [Soare 1974]. In this hierarchy, the elements of the local substructure are partitioned in classes depending to their 'closeness' to the least element or to its first jump in the terms of the jump operation. Namely, a degree in the local substructure is low_n ($\mathbf{L}_n$) iff its n-th jump is as low as possible – the same as the n-th jump of the least element. Analogically, a degree is $high_n$ ($\mathbf{H}_n$) iff its n-th jump is as high as possible – the same as the n-th jump of the jump of the least element. We shall refer to the degrees in the local substructure, which are neither in $\mathbf{H}_n$, nor in $\mathbf{L}_n$ for any natural n, as *intermediate* ($\mathbf{I}$).

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This paper concerns the problem of the first order definability of the class $\mathbf{I}$ in the local substructure of the ω-Turing degrees $\mathcal{D}_{T,\omega}$. Unlike the well known structures of the Turing and the enumeration degrees, $\mathcal{D}_{T,\omega}$ is induced by a reducibility on the set $\mathcal{S}_\omega$ of the sequences of sets of natural numbers. The studying of the degree structures induced by a such kind of reducibilities has been initiated by Soskov. In the paper [Soskov 2007], he introduces the ω-enumeration reducibility $\leq_\omega$, generalizing the enumeration reducibility over sequences.

The structure an object of the current work, $\mathcal{D}_{T,\omega}$, is introduced in [Sariev and Ganchev 2014] as a "Turing" analogue of $\mathcal{D}_\omega$ in the following way. First, to each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ in $\mathcal{S}_\omega$, a jump-class $J_\mathcal{A}$ is assigned:

$$J_\mathcal{A} = \{\deg_T(X) \mid A_k \leq_T X^{(k)} \text{ uniformly in } k\}.$$

Then the ω-Turing reducibility $\leq_{T,\omega}$, which induces $\mathcal{D}_{T,\omega}$, is set in a such a way that for each sequences $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{S}_\omega$, $\mathcal{A} \leq_{T,\omega} \mathcal{B}$ if and only if $J_\mathcal{B} \subseteq J_\mathcal{A}$.

The jump $\mathcal{A}'$ of a sequence $\mathcal{A}$ is defined so that the class $J_{\mathcal{A}'}$ consists exactly of the jumps of the Turing degrees in $J_\mathcal{A}$, i.e. so that $J_{\mathcal{A}'} = J'_\mathcal{A}$. The jump operator on sequences is monotone and thus induces a jump operation $'$ in $\mathcal{D}_{T,\omega}$. Just like the jump operation in $\mathcal{D}_T$, the range of the jump operation in $\mathcal{D}_{T,\omega}$ is exactly the cone above the first jump of the least element $\mathbf{0}_{T,\omega}$. In other words a general jump inversion theorem is valid for $\mathcal{D}_{T,\omega}$. Moreover, an even stronger statement turns out to be true, namely for every ω-Turing degree $\mathbf{a}$ above $\mathbf{0}_{T,\omega}{}'$ there is a least degree with jump equal to $\mathbf{a}$. This property is true neither for $\mathcal{D}_T$ nor for $\mathcal{D}_e$.

The strong jump inversion theorem makes the structure $\mathcal{D}_{T,\omega}$ worth studying, since using it one may consider a natural copy of the structure $\mathcal{D}_T$ definable in $\mathcal{D}_{T,\omega}$ augmented by the jump operation. Moreover, the automorphism groups of the structures $\mathcal{D}_T$ and $\mathcal{D}_{T,\omega}{}'$ are isomorphic.

Just like the Turing and the enumeration degree structures, the jump operation in each one of $\mathcal{D}_\omega$ and $\mathcal{D}_{T,\omega}$ gives rise to the corresponding local substructure: $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$. What do we know about the definability of the jump classes in these two local structures? First, in both $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$, for each n the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first-order definable, as it is shown respectively in [Ganchev and Soskova 2012] and [Sariev and Ganchev 2014]. The first-order definability of the class $\mathbf{I}$ of the intermediate degrees in $\mathcal{G}_\omega$ is shown in [Ganchev and Sariev 2015]. Note that the latter does not hold in the local substructures neither of the Turing degrees, nor of the enumeration degrees.

All of these definability results rely on the existence of a class of remarkable degrees having no analogue in either $\mathcal{R}$, $\mathcal{G}_T$ or $\mathcal{G}_e$. These degrees are denoted by $\mathbf{o}_n$, $n < \omega$, and are defined so that $\mathbf{o}_n$ is the least degree whose n-th jump is equal to the $(n+1)$-th jump of $\mathbf{0}_\omega$. In other words, $\mathbf{o}_n$ is the least high$_n$ degree. The degrees $\mathbf{o}_n$ are also connected to low$_n$ degrees. Indeed, a degree in $\mathcal{G}_\omega$ is low$_n$ *iff* it forms a minimal pair with $\mathbf{o}_n$. The same connections hold and in $\mathcal{G}_{T,\omega}$.

Each one of the degrees $\mathbf{o}_n$ turns out to be definable both in $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$ [Ganchev and Soskova 2012], [Sariev and Ganchev 2014] and hence so are the classes $\mathbf{H}_n$ and $\mathbf{L}_n$, for $n \in \omega$. Although the abovementioned similarities between $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$, the corresponding definitions of the degrees $\mathbf{o}_n$ are quite different. The definition in $\mathcal{G}_\omega$ of $\mathbf{o}_n$, given in [Ganchev and Soskova 2012], is based on the notion of Kalimullin pairs — a notion first introduced and studied by Kalimullin in the context of the enumeration degrees. Just like in the Turing degrees, Kalimullin pairs do not exist in the structure of the ω-Turing degrees. The definition here is based on the notion of *noncuppable* degrees. Namely, in [Sariev and Ganchev 2014] it is shown that for each $n < \omega$,

$$\mathbf{o}_{n+1} \text{ is the greatest degree below } \mathbf{o}_n \text{which is noncuppable to } \mathbf{o}_n.$$

Further, to obtain a first-order definition of the class $\mathbf{I}$ (no matter in $\mathcal{G}_\omega$ or $\mathcal{G}_{T,\omega}$) it is sufficient to first-order define the class $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$ (of course, in the corresponding local structure). Indeed, for each degree $\mathbf{x}$ in the corresponding local structure,

$$\mathbf{x} \in \mathbf{I} \iff (\forall n)[\mathbf{x} \notin \mathbf{H}_n \text{ and } \mathbf{x} \notin \mathbf{L}_n] \iff$$

$$\iff (\forall n)[\mathbf{x} \not\leq \mathbf{o}_n \text{ and } \mathbf{x} \wedge \mathbf{o}_n \neq \mathbf{0}] \iff$$
$$\iff (\forall \mathbf{o} \in \mathfrak{O})[\mathbf{x} \not\leq \mathbf{o} \text{ and } \mathbf{x} \wedge \mathbf{o} \neq \mathbf{0}].$$

In the case of $\mathcal{G}_\omega$, Ganchev and Sariev [Ganchev and Sariev 2015] exploit additional connections between the $\mathbf{o}_n$ degrees and the elements of the Kalimullin pairs in order to find a definition for the class $\mathfrak{O}$ and hence, a definition for $\mathbf{I}$.

The existing strong parallel between the ω-Turing and the ω-enumeration degree structures yields the hypothesis that the class of the intermediate degrees is first-order definable also in $\mathcal{G}_{T,\omega}$. In this paper we find an interesting cupping property of the $\mathbf{o}_n$ degrees – for each degree $\mathbf{b}$ above $\mathbf{o}_n$, the equation $\mathbf{o}_n \vee \mathbf{x} = \mathbf{b}$ always has a least solution. Further we use the above property as our main component in the first-order definition of the classes $\mathfrak{O}$ and $\mathbf{I}$.

2 Preliminaries

We shall denote the set of all natural numbers by ω. If not stated otherwise, a, b, c, ... shall stand for natural numbers, A, B, C, ... for sets of natural numbers,

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

We assume that the reader is familiar with the notion of Turing reducibility, $\leq_T$, and with the structure of the Turing degrees.

For every natural e and every function $f \in \omega^\omega$, we denote by $\{e\}^f$ the partial function computed by the oracle Turing machine with index e and using f as an oracle. Thus for arbitrary sets $A, B \subseteq \omega$, $A \leq_T B \iff \chi_A = \{e\}^{\chi_B}$ for some natural number e. Given a natural number x, we shall indicate by $\{e\}^f(x)\downarrow$ that the function $\{e\}^f$ is defined in x. In case that $\{e\}^f$ is not defined in x, we shall write $\{e\}^f(x)\uparrow$.

Recall, that A is computably enumerable (*c.e.*) in B *iff* there is an enumeration f of A (i.e. a surjection from ω onto A), such that $f = \{e\}^{\chi_B}$ for some natural number e.

By A' we shall denote the Turing jump of the set A, i.e.

$$A' = \{x \mid \{x\}^{\chi_A}(x)\downarrow\}.$$

Recall that A' is c.e. in A and every set c.e. in A is Turing reducible to A'. Further, the jump operator preserves uniformly the Turing reducibility, i.e. there is a computable function f, such that for arbitrary sets A and B, if A is Turing reducible to B via the oracle Turing machine with index e, then A' is Turing reducible to B' via the oracle Turing machine with index $f(e)$.

The relation $\leq_T$ is a preorder on the powerset $\mathcal{P}(\omega)$ of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Turing degrees. The Turing degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Turing reducibility between sets induces a partial order $\leq$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $(\mathbf{D}_T, \leq)$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\emptyset$. Also, the Turing degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_T$ is an upper semilattice with least element.

The jump operation gives rise to the local substructure $\mathcal{G}_T$, consisting of all degrees bellow $\mathbf{0}'_T$ – the jump of the least Turing degree. Infact, $\mathcal{G}_T$ is exactly the collection of all Δ^0_2 Turing degrees.

Finally we need the following definition, which we shall use in order to characterise ω-Turing reducibility. Given a sequence $\mathcal{A} \in S_\omega$ we define the *jump-sequence* $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ as the sequence $\{P_k(\mathcal{A})\}_{k<\omega}$ such that:

1. $P_0(\mathcal{A}) = A_0$;
2. $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$.

3 The ω-Turing degrees

The structure of the ω-Turing degrees $\mathcal{D}_{T,\omega}$ is introduced by Sariev and Ganchev [Sariev and Ganchev 2014] in the following way. For every sequence $\mathcal{A} \in \mathcal{S}_\omega$, we define its jump class $J_\mathcal{A}$ to be the set:

$$J_\mathcal{A} = \{\deg_T(X) \mid A_k \leq_T X^{(k)} \text{ uniformly in } k\}. \tag{1}$$

We set

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff J_\mathcal{B} \subseteq J_\mathcal{A}.$$

Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \;\&\; \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-Turing degrees. In particular the equivalence class $\deg_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Turing degree of $\mathcal{A}$. The relation $\leq$ defined by

$$\mathbf{a} \leq \mathbf{b} \iff (\exists \mathcal{A} \in \mathbf{a})(\exists \mathcal{B} \in \mathbf{b})[\mathcal{A} \leq_{T,\omega} \mathcal{B}]$$

is a partial order on the set of all ω-Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $(\mathbf{D}_{T,\omega}, \leq)$. The ω-Turing degree $\mathbf{0}_{T,\omega}$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_\omega(\mathcal{A})$ and $\mathbf{b} = \deg_\omega(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

An explicit characterisation of the ω-Turing reducibility is derived in [Sariev and Ganchev 2014]. According to it, $\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff A_n \leq_T P_n(\mathcal{B})$ uniformly in n. More formally, $\mathcal{A} \leq_{T,\omega} \mathcal{B}$ *iff* there is a computable function f, such that for every natural number

k, $\chi_{A_k} = \{f(k)\}^{P_k(\mathcal{B})}$. From here, one can show that each sequence is ω-Turing equivalent with its jump-sequence, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2}$$

Further, for the sake of convenience, for sequences $\mathcal{A}, \mathcal{B} \in S_\omega$ we shall write $\mathcal{A} \leq_T \mathcal{B}$ if and only if for each $k < \omega$, $A_k \leq_T B_k$ uniformly in k. So $\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_T \mathcal{P}(\mathcal{B})$. Note that there exist only countably many computable functions, so that there could be only countably many sequences ω-Turing reducible to a given sequence. In particular every ω-Turing degree cannot contain more than countably many sequences and hence there are continuum many ω-Turing degrees.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \ldots, \emptyset, \ldots)$. From the definition of $\leq_{T,\omega}$ and the uniformity of the jump operation, we have that for all sets A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{3}$$

The last equivalence means, that the mapping $\kappa : \mathcal{D}_T \to \mathcal{D}_{T,\omega}$, defined by, $\kappa(\mathbf{x}) = \deg_\omega(X \uparrow \omega)$, where X is an arbitrary set in $\mathbf{x}$, is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Further, the so defined embedding κ preserves the least element and the binary least upper bound operation. We shall denote the range of κ by $\mathbf{D}_1$.

4 The jump operator

Following the lines of Sariev and Ganchev [Sariev and Ganchev 2014], the ω-Turing jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots).$$

This operator is defined so that if $\mathcal{A}'$ is the jump of $\mathcal{A}$, then the jump class $J_{\mathcal{A}'}$ of $\mathcal{A}'$ contains exactly the jumps of the degrees in the jump class $J_\mathcal{A}$ of $\mathcal{A}$. Note also, that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_\omega \{P_{k+1}(\mathcal{A})\}$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_{T,\omega} \mathcal{A}'$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'$. This allows to define a jump operation on the ω-Turing degrees by setting

$$\mathbf{a}' = \deg_\omega(\mathcal{A}'),$$

where $\mathcal{A}$ is an arbitrary sequence in $\mathbf{a}$. Clearly $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathcal{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{4}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the ω-Turing degree $\mathbf{a}$.

Further, in [Sariev and Ganchev 2014] it is shown that for every natural number n, if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-Turing degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_k, \ldots), \tag{5}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

From here it follows that for every given $\mathbf{a} \in \mathbf{D}_{T,\omega}$ and $n < \omega$, the operation $\mathbf{I}^n_{\mathbf{a}}$ is monotone. Further its range is a downwards closed subset of the upper cone with least element $\mathbf{a}$. In fact even a stronger property holds: if $\mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$ are such that $\mathbf{a} \leq \mathbf{x}$, $\mathbf{a}^{(n)} \leq \mathbf{b}$ and $\mathbf{x} \leq \mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ then $\mathbf{x}$ is equal to $\mathbf{I}^n_{\mathbf{a}}(\mathbf{x}^{(n)})$. Detailed proofs of the abovementioned properties one can find in [Sariev and Ganchev 2014].

It what follows when $\mathbf{a} = \mathbf{0}_{T,\omega}$ we shall just write $\mathbf{I}^n$ instead of $\mathbf{I}^n_{\mathbf{0}_{T,\omega}}$.

5 The local theory, jump classes and the $\mathbf{o}_n$ degrees

The structure of the degrees lying beneath the first jump of the least element is usually referred to as the local structure of a degree structure. In the case of the ω-Turing degrees we shall denote this structure by $\mathcal{G}_{T,\omega}$. When considering a local structure, one is usually concerned with questions about the definability of some classes of degrees, which have a natural definition either in the context of the global structure (for example the classes of the high and the low degrees) or in the context of the basic objects from which the degrees are built (for example the class of the Turing degrees containing a c.e. set).

Recall that a degree in the local structure is said to be *high*$_n$ for some n *iff* its n-th jump is as high as possible. Similarly a degree in the local structure is said to be *low*$_n$ for some n *iff* its n-th jump is as low as possible. More formally, in the case of $\mathcal{G}_{T,\omega}$, a degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$ is high$_n$ *iff* $\mathbf{a}^{(n)} = (\mathbf{0}_{T,\omega}{}')^{(n)} = \mathbf{0}_{T,\omega}{}^{(n+1)}$, and is low$_n$ *iff* $\mathbf{a}^{(n)} = \mathbf{0}_{T,\omega}{}^{(n)}$.

As usual we shall denote by $\mathbf{H}_n$ the collection of all high$_n$ degrees, and by $\mathbf{L}_n$ the collection of all low$_n$ degrees. Also $\mathbf{H}$ shall denote the union of all the classes $\mathbf{H}_n$ and analogously $\mathbf{L}$ shall denote the union of all of the classes $\mathbf{L}_n$. Finally, $\mathbf{I}$ will stand for the collection of the degrees that are neither high$_n$ nor low$_n$ for any n. The degrees in $\mathbf{I}$ shall be referred to as intermediate degrees.

Using the corresponding results for the structure of the Turing degrees, it is easy to see that there exist intermediate degrees and for every natural number n, there are degrees in the local structure of the ω-Turing degrees, that are high$_{(n+1)}$ (respectively low$_{(n+1)}$) but are not high$_n$ (respectively low$_n$).

Sariev and Ganchev [Sariev and Ganchev 2014] give a characterisation of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ that does not involve directly the jump operation. Let us set $\mathbf{o}_n$ to be the least n-th jump invert of ${\mathbf{0}_{T,\omega}}^{(n+1)}$, i.e. $\mathbf{o}_n = \mathbf{I}^n({\mathbf{0}_{T,\omega}}^{(n+1)})$. Hence by (5), for each natural number n, the sequence $\mathcal{O}_n = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$ is an element of $\mathbf{o}_n$.

Note that $\mathbf{o}_n$ is the least element of the class $\mathbf{H}_n$. Thus for arbitrary $\mathbf{x} \in \mathcal{G}_{T,\omega}$,

$$\mathbf{x} \in \mathbf{H}_n \iff \mathbf{o}_n \leq \mathbf{x}. \tag{6}$$

In particular, since every high$_n$ degree is also high$_{(n+1)}$, $\mathbf{o}_{n+1} \leq \mathbf{o}_n$. On the other hand, since $\mathbf{H}_{n+1} \setminus \mathbf{H}_n \neq \emptyset$, the equality $\mathbf{o}_{n+1} = \mathbf{o}_n$ is impossible, so that

$$\mathbf{0}'_\omega = \mathbf{o}_0 > \mathbf{o}_1 > \mathbf{o}_2 > \cdots > \mathbf{o}_n > \dots.$$

In [Sariev and Ganchev 2014] it is shown that for arbitrary $\mathbf{x} \in \mathcal{G}_{T,\omega}$,

$$\mathbf{I}^n(\mathbf{x}^{(n)}) = \mathbf{x} \wedge \mathbf{o}_n. \tag{7}$$

Indeed, let us take an arbitrary $\mathbf{x} \in \mathcal{G}_{T,\omega}$. Clearly $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq \mathbf{x}$ and $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq \mathbf{o}_n$. On the other hand if $\mathbf{y}$ is such that $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{y} \leq \mathbf{o}_n$, then from the second inequality we have $\mathbf{y} = \mathbf{I}^n(\mathbf{z})$ for some $\mathbf{z}$. This together with the first inequality gives us $\mathbf{z} = (\mathbf{I}^n(\mathbf{z}))^{(n)} = \mathbf{y}^{(n)} \leq \mathbf{x}^{(n)}$. Thus $\mathbf{y} = \mathbf{I}^n(\mathbf{z}) \leq \mathbf{I}^n(\mathbf{x}^{(n)})$.

This gives us a characterisation of the low$_n$ degrees in terms of the partial order $\leq$ in $\mathcal{G}_{T,\omega}$ and the degrees $\mathbf{o}_n$, namely

$$\mathbf{x} \in \mathbf{L}_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_{T,\omega}. \tag{8}$$

Note that from the above characterisations, in order to show that the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first order definable in $\mathcal{G}_{T,\omega}$ it is sufficient to show this for the degree $\mathbf{o}_n$. For the definition of the classes $\mathbf{H} = \bigcup \mathbf{H}_n, \mathbf{L} = \bigcup \mathbf{L}_n$ and $\mathbf{I} = \mathcal{G}_{T,\omega} \setminus (\mathbf{H} \cup \mathbf{L})$ it is sufficient to show the definability of the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$.

Also in the paper [Sariev and Ganchev 2014] it is shown a characterisation of the degrees in $\mathbf{D}_1 \cap \mathcal{G}_{T,\omega}$ in terms of the ordering $\leq$ and the degree $\mathbf{o}_1$. Namely, for arbitrary $\mathbf{a} \in \mathcal{G}_{T,\omega}$, $\mathbf{a}$ is a degree in $\mathbf{D}_1$ *iff*

$$(\forall \mathbf{x} \in \mathcal{G}_{T,\omega})[\mathbf{x} \vee \mathbf{o}_1 = \mathbf{a} \vee \mathbf{o}_1 \to \mathbf{x} \geq \mathbf{a}]. \tag{9}$$

Moreover, a characterisation of the noncuppable degrees in $\mathcal{G}_{T,\omega}$ can be derived again in the terms of the structure order and the degree $\mathbf{o}_1$, [Sariev and Ganchev 2014]. Recall that a degree $\mathbf{a}$ is said to be noncuppable in a local structure *iff* the least upper bound of this degree and any degree strictly less than the top element of the structure is not equal to the top element of the structure. It is known by a result proved by Posner and Robinson [Posner and Robinson 1981] that the local structure of the Turing degrees features no nonzero noncuppable degrees. In particular every nonzero degree in $\mathbf{D}_1 \cap \mathcal{G}_{T,\omega}$ is cuppable in $\mathcal{G}_{T,\omega}$. On the other hand taking $\mathbf{a}$ to be $\mathbf{0}_{T,\omega}{}'$ in (9), we obtain that for all $\mathbf{x} \in \mathcal{G}_{T,\omega}$

$$\mathbf{x} \vee \mathbf{o}_1 = \mathbf{0}_{T,\omega}{}' \vee \mathbf{o}_1 \Longrightarrow \mathbf{x} \geq \mathbf{0}_{T,\omega}{}',$$

and hence $\mathbf{o}_1$ is noncuppable in $\mathcal{G}_{T,\omega}$.

Now let us fix a noncuppable $\mathbf{a} \in \mathcal{G}_{T,\omega}$ and let $\mathcal{A} \in \mathbf{a}$. The noncuppability of $\mathbf{a}$ yields that the first element A_0 of the sequence $\mathcal{A}$ is Turing equivalent to $\emptyset$, for otherwise $\kappa(\mathbf{d}_T(A_0))$ would be a nonzero cuppable degree beneath $\mathbf{a}$ and in particular $\mathbf{a}$ would be cuppable. This, together with the definition of the jump operator and the definition of the operator $I^1_{\emptyset_\omega}$, implies $\mathcal{A} \equiv_{T,\omega} I^1_{\emptyset_\omega}(\mathcal{A}')$.

Hence for every noncuppable degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$,

$$\mathbf{a} = \mathbf{I}^1(\mathbf{a}')$$

and therefore $\mathbf{a} \leq \mathbf{o}_1$.

Thus a degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$ is noncuppable *iff* $\mathbf{a} \leq \mathbf{o}_1$. In particular $\mathbf{o}_1$ is the greatest noncuppable degree in $\mathcal{G}_{T,\omega}$, and hence $\mathbf{o}_1$ is first order definable in $\mathcal{G}_{T,\omega}$. From here, (6), (8) and (9) we obtain that the classes $\mathbf{H}_1$, $\mathbf{L}_1$ and $\mathbf{D}_1 \cap \mathcal{G}_{T,\omega}$ are first order definable in $\mathcal{G}_{T,\omega}$.

With a reasoning analogous to the one we did for $\mathbf{o}_1$, we can prove that $\mathbf{o}_{n+1}$ is the greatest degree beneath $\mathbf{o}_n$, that is noncuppable to $\mathbf{o}_n$. Thus for each natural n, the degree $\mathbf{o}_n$ is first order definable in $\mathcal{G}_{T,\omega}$ and hence the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are also first order definable in $\mathcal{G}_{T,\omega}$.

In the above mentioned work of Sariev and Ganchev the notion of *almost zero* (*a.z.*) degrees is introduced. Namely, the degree $\mathbf{x}$ is *a.z.* *iff* there is a representative $\mathcal{X} \in \mathbf{x}$ such that

$$(\forall k)[P_k(\mathcal{X}) \equiv_T \emptyset^{(k)}]. \tag{10}$$

It is clear that the class of the $a.z.$ degrees is downward closed. Note also that there are continuum many $a.z.$ degrees and hence not all $a.z.$ degrees are in $\mathcal{G}_{T,\omega}$. The $a.z.$ degrees in $\mathcal{G}_{T,\omega}$ are exactly the degrees bounded by every degree $\mathbf{o}_n$, i.e.

$$\mathbf{x} \in \mathcal{G}_{T,\omega} \text{ is } a.z. \iff (\forall n < \omega)[\mathbf{x} \leq \mathbf{o}_n]. \tag{11}$$

We finish this section with some observations concerning the minimal[1] ω-Turing degrees. As has been shown in [Sariev and Ganchev 2014] there are exactly countably many minimal ω-Turing degrees and all of them are bounded by $\mathbf{0}_{T,\omega}{}'$. This follows from a characterisation of the minimal degrees in $\mathcal{D}_{T,\omega}$ by Sariev and Ganchev, [Sariev and Ganchev 2014]. Namely, an ω-Turing degree is minimal, if and only if it contains a sequence of the form $(\underbrace{\emptyset, \emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots)$, where the Turing degree of A is a minimal cover of $\mathbf{0}_T^{(n)}$ and $A' \equiv_T \emptyset^{(n+1)}$. Note, that no $a.z.$ degree is minimal. Since each $a.z.$ degree bounds only $a.z.$ degrees, then no $a.z.$ degree bounds a minimal degree. In converse, one can easily show that each of the degrees $\mathbf{o}_n$ bounds (countably many) minimal degrees.

6 Definability in $\mathcal{G}_{T,\omega}$

In this last section we shall show how to first order define the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$. This definition is based on observations for the local theory of the Turing degrees. Finally, from the fact that the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$ is first order definable in $\mathcal{G}_{T,\omega}$, by (6), we conclude the definability of the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$.

Further, if $\mathcal{D} = (\mathbf{D}, \leq, \vee)$ is an upper semi-lattice, and $\mathbf{a}, \mathbf{l}, \mathbf{r} \in \mathbf{D}$ are such that $\mathbf{l} \leq \mathbf{r}$, then by $\mathrm{Cup}(\mathbf{a}, \mathbf{l}, \mathbf{r})$ we shall denote the set of all solutions $\mathbf{x}$ of the equation $\mathbf{a} \vee \mathbf{x} = \mathbf{r}$ such that $\mathbf{l} \leq \mathbf{x} \leq \mathbf{r}$,

$$\mathrm{Cup}_{\mathcal{D}}(\mathbf{a}, \mathbf{l}, \mathbf{r}) = \{\mathbf{x} \mid \mathbf{l} \leq \mathbf{x} \leq \mathbf{r} \text{ and } \mathbf{a} \vee \mathbf{x} = \mathbf{r}\}.$$

Let us now consider a degree $\mathbf{b}$ in the local substructure $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees, which is above $\mathbf{o}_1$. Since $(\emptyset, \emptyset^{(2)}, \emptyset^{(3)}, \dots) \in \mathbf{o}_1$, then $\mathbf{b}$ contains a sequence of the form $(B, \emptyset^{(2)}, \emptyset^{(3)}, \dots)$. Then it is easy to notice that the degree containing the sequence $B \uparrow \omega$ is the least degree, which cups $\mathbf{o}_1$ to $\mathbf{b}$. Using an analogous reasoning, one can show that each of the degrees $\mathbf{o}_n$ satisfies the formula:

$$\Phi(\mathbf{a}) \rightleftharpoons (\forall \mathbf{b} \geq \mathbf{a})[\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{b}) \text{ has a least element}].$$

[1] A degree $\mathbf{m}$ is said to be minimal in a degree structure $\mathcal{D} = (\mathbf{D}, \mathbf{0}, \leq)$, if the only degree strictly less than $\mathbf{m}$ is the least element $\mathbf{0}$ of $\mathcal{D}$. Also, $\mathbf{m}$ is a minimal cover of $\mathbf{a}$ iff $\mathbf{a} < \mathbf{m}$ and the interval $\mathbf{D}(\mathbf{a}, \mathbf{m})$ is empty.

Indeed, fix a natural number n. Let $\mathbf{b} \in \mathcal{G}_{T,\omega}$ be a degree above $\mathbf{o}_n$. Recall that $\mathbf{o}_n$ contains the sequence $\mathcal{O}_n = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$. Hence there are sets $B_0, B_1, \dots, B_{n-1}$ such that the sequence $(B_0, B_1, \dots, B_{n-1}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$ belongs to $\mathbf{b}$. Note that the degree $\widetilde{\mathbf{b}} = \deg_\omega(B_0, B_1, \dots, B_{n-1}, \emptyset, \emptyset, \dots)$ is in the set $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{o}_n, \mathbf{0}_{T,\omega}, \mathbf{b})$. Now, let $\mathbf{x}$ be an arbitrary degree, which cups $\mathbf{o}_n$ to $\mathbf{b}$ and let $\mathcal{X} \in \mathbf{x}$. Hence, for all $i < n$, we have that $B_i \leq_T P_i(\mathcal{X} \oplus \mathcal{O}_n) \equiv_T P_i(\mathcal{X})$. Thus, $\widetilde{\mathbf{b}} \leq \mathbf{x}$ and therefore, $\widetilde{\mathbf{b}}$ is the least element of $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{o}_n, \mathbf{0}_{T,\omega}, \mathbf{b})$.

Further, recall that no $a.z.$ degree bounds a minimal degree. On the other hand for each n, $\mathbf{o}_n$ bounds a minimal degree. Thus the property

$$\Psi(\mathbf{a}) \rightleftharpoons (\exists \mathbf{m})[\mathbf{m} \text{ is a minimal degree } \& \ \mathbf{m} < \mathbf{a}]$$

separates $\mathfrak{O}$ from the $a.z.$ degrees.

In fact, the formula $\Phi \,\&\, \Psi$ defines exactly the degrees $\mathbf{o}_n$. In order to prove it, first we need some additional observations concerning the local theory of the Turing degrees. By [Posner and Robinson 1981], for all Turing degrees $\mathbf{a}, \mathbf{b} \in \mathbf{D}_T(\mathbf{0}_T, \mathbf{0}'_T)$, there is a (low) Turing degree $\mathbf{c} \in \mathbf{D}_T(\mathbf{0}_T, \mathbf{0}'_T)$, such that $\mathbf{a} \vee \mathbf{c} = \mathbf{0}'_T$ and $\mathbf{b} \not\leq \mathbf{c}$. Hence, if $\mathbf{a}$ is a Turing degree strictly between the least element $\mathbf{0}_T$ and its first jump $\mathbf{0}'_T$, then the set

$$\mathrm{Cup}_{\mathcal{D}_T}(\mathbf{a}, \mathbf{0}_T, \mathbf{0}'_T) = \{\mathbf{x} \mid \mathbf{0}_T \leq \mathbf{x} \leq \mathbf{0}'_T \text{ and } \mathbf{a} \vee \mathbf{x} = \mathbf{0}'_T\}$$

is not empty, but does not have a least element. The result in [Posner and Robinson 1981] can be relativized straightforward[2]. Thus for each Turing degree $\mathbf{d}$, if $\mathbf{a}, \mathbf{b} \in \mathbf{D}_T(\mathbf{d}, \mathbf{d}')$ then there is a (low over $\mathbf{d}$) degree $\mathbf{c}$ in $\mathbf{D}_T(\mathbf{d}, \mathbf{d}')$, which cups $\mathbf{a}$ to $\mathbf{d}'$ and avoids $\mathbf{b}$. Using this relativization, we conclude that if $\mathbf{a}$ is in the interval $\mathbf{D}_T(\mathbf{d}, \mathbf{d}')$, then the set $\mathrm{Cup}_{\mathcal{D}_T}(\mathbf{a}, \mathbf{d}, \mathbf{d}')$ is not empty, but does not have a least element.

Now we are ready to prove that $\Phi \,\&\, \Psi$ only defines the degrees in $\mathfrak{O}$. For the purpose, let $\mathbf{a}$ be a degree in $\mathcal{G}_{T,\omega}$, which is not in $\mathfrak{O}$. Suppose also $\Phi(\mathbf{a})$ and $\Psi(\mathbf{a})$ hold in $\mathcal{G}_{T,\omega}$. Since $\mathbf{a}$ bounds a minimal degree, then $\mathbf{a}$ is not $a.z.$ degree. Hence, there exists a greatest natural number n, such that $\mathbf{a} < \mathbf{o}_n$. Since $(\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$ is an element of $\mathbf{o}_n$, then there are sets $A_n, A_{n+1}, A_{n+2}, \dots$ such that the sequence $\mathcal{A} = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, A_n, A_{n+1}, A_{n+2}, \dots)$ has a degree $\mathbf{a}$. By the choice of n, $\mathbf{a}$ is not below $\mathbf{o}_{n+1}$, so $\emptyset^{(n)} <_T P_n(\mathcal{A})$. On the other hand, $\mathbf{a} \leq \mathbf{0}_{T,\omega}{}'$ and $\mathbf{a} \neq \mathbf{o}_n$, so $P_n(\mathcal{A}) <_T \emptyset^{(n+1)}$.

Since $\mathbf{a} < \mathbf{o}_n$ and $\Phi(\mathbf{a})$, then there is a least degree $\mathbf{x}$ such that $\mathbf{a} \vee \mathbf{x} = \mathbf{o}_n$. Note that $\mathbf{x} \leq \mathbf{o}_n$, so it contains a sequence $\mathcal{X}$ of the form

[2] Posner and Robinson themselves use the relativization in the same paper.

$(\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, X_n, X_{n+1}, X_{n+2}, \ldots)$, for some sets $X_n, X_{n+1}, X_{n+2}, \ldots$. From $\mathbf{a} \vee \mathbf{x} = \mathbf{o}_n$ we have that $P_n(\mathcal{A}) \oplus P_n(\mathcal{X}) \equiv_T \emptyset^{(n+1)}$ and since $P_n(\mathcal{A}) <_T \emptyset^{(n+1)}$, then $\emptyset^{(n)} <_T P_n(\mathcal{X})$.

Further, note that if the set of natural numbers Y has a Turing degree in the set $\mathrm{Cup}_{\mathcal{D}_T}(\deg_T(P_n(\mathcal{A})), \mathbf{0}_T^{(n)}, \mathbf{0}_T^{(n+1)})$, then $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$ contains the ω-Turing degree of the sequence $(\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, Y, \emptyset, \emptyset, \ldots)$. By the relativization of the result of Posner and Robinson for $\mathbf{d} = \mathbf{0}_T^{(n)}$, $\mathrm{Cup}_{\mathcal{D}_T}(\deg_T(P_n(\mathcal{A})), \mathbf{0}_T^{(n)}, \mathbf{0}_T^{(n+1)})$ has an element not equal to $\mathbf{0}_T^{(n+1)}$. Therefore, $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$ has an element strictly below $\mathbf{o}_n$, and so $P_n(\mathcal{X}) <_T \emptyset^{(n+1)}$.

Finally, since $\emptyset^{(n)} <_T P_n(\mathcal{X}) <_T \emptyset^{(n+1)}$, again by the relativization of the Posner and Robinson result, there is a set Y, such that $\emptyset^{(n)} <_T Y <_T \emptyset^{(n+1)}$, $P_n(\mathcal{A}) \oplus Y \equiv_T \emptyset^{(n+1)}$ and $P_n(\mathcal{X}) \not\leq_T Y$. Then the ω-Turing degree $\mathbf{y}$ of the sequence $(\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, Y, \emptyset, \emptyset, \ldots)$ is an element of $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$, which is not above $\mathbf{x}$. A contradiction.

Combining all together, we have that for each degree $\mathbf{a}$ below $\mathbf{0}_{T,\omega}{}'$, $\mathbf{a} \in \mathfrak{O} \iff \mathcal{G}_{T,\omega} \models \Phi(\mathbf{a}) \,\&\, \Psi(\mathbf{a})$. In order to conclude our main result, we only have to notice that $\Phi \,\&\, \Psi$ is equivalent to a first order formula in the language of the partial orders.

Theorem 1. *The classes* $\mathbf{H}, \mathbf{L}$ *and* $\mathbf{I}$ *are first order definable in* $\mathcal{G}_{T,\omega}$

A direct consequence of the latter and (11) is the following corollary.

Corollary 2. *The class of the a.z. degrees is first order definable in* $\mathcal{G}_{T,\omega}$.

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