UNIFORM REGULAR ENUMERATIONS

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Abstract. In the paper we introduce and study the uniform regular enumerations for arbitrary recursive ordinals. As an application of the technique we obtain a uniform generalization of a theorem of Ash and a characterization of a class of uniform operators on transfinite sequences of sets of natural numbers.

1. Introduction

Let ζ be a recursive ordinal and let $\{B_\alpha\}_{\alpha\leq\zeta}$ be an arbitrary sequence of sets of natural numbers. The regular with respect to the sequence $\{B_\alpha\}_{\alpha\leq\zeta}$ enumerations were introduced in [10] and used to characterize the sets A satisfying the following condition:

$$(*)\qquad (\forall X)[(\forall\alpha\leq\zeta)(B_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)\Rightarrow A \text{ is r.e. in } X^{(\alpha)}],$$

where α is a recursive ordinal.

In the present paper we are concerned with the characterization of all sequences $\{A_\alpha\}_{\alpha\leq\zeta}$ of sets satisfying a uniform generalization of (*):

$$(**)\qquad \begin{aligned}&(\forall X)[(\forall\alpha\leq\zeta)(B_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)\Rightarrow\\ &\qquad\qquad(\forall\alpha\leq\zeta)(A_\alpha \text{ is r.e. in } X^{(\alpha)} \text{ uniformly in } \alpha)\ ].\end{aligned}$$

It turned out that the technique developed in [10] cannot be applied directly to obtain this characterization which led to the uniform regular enumerations presented here.

Using the technique of the uniform regular enumerations we also obtain a characterization of a class of uniform operators mapping transfinite sequences of sets of natural numbers to sequences of sets of natural numbers, which generalize the operators studied in [9], and show that a two sequences $\{B_\alpha\}_{\alpha\leq\zeta}$ and $\{A_\alpha\}_{\alpha\leq\zeta}$ satisfy (**) if and only if there exists a uniform operator Γ such that

$$\Gamma(\{B_\alpha\}_{\alpha\leq\zeta})=\{A_\alpha\}_{\alpha\leq\zeta}.$$

2. Preliminaries

2.1. **Ordinal notations.** In what follows we shall consider only recursive ordinals α which are below a fixed recursive ordinal η. We shall suppose that a notation $e\in\mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha<\eta$ are elements a of $\mathcal{O}$ such that $a<_o e$. For the definitions of the set $\mathcal{O}$ and the relation "$<_o$" the reader may consult [6] or [7]. We shall identify every ordinal with its notation and denote the ordinals by the letters α,β,γ and δ. In particular we shall write $\alpha<\beta$ instead

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of $\alpha <_o \beta$. If α is a limit ordinal then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we shall denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. **The enumeration jump.** Given two sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leq_e B$) if $A = \Phi_z(B)$ for some enumeration operator Φ_z. In other words, using the notation D_v for the finite set having canonical code v and $W_0, \dots, W_z, \dots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v(\langle v, x\rangle \in W_z \ \&\ D_v \subseteq B)).$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult Cooper [4].

Given a set A denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. Clearly A is recursively enumerable in B iff $A \leq_e B^+$ and A is recursive in B iff $A^+ \leq_e B^+$.

Since $B \leq_e B^+$, if $A \leq_e B$, then A is r.e. in B. It is easy to see that there exists a recursive function t such that if $A \subseteq \mathbb{N}$ then for every z,

$$\Phi_z(A) = W^A_{t(z)},$$

where as usual W^A_z denotes the domain of the z-th Turing machine using as oracle the characteristic function of A, in other words the z-th r.e. in A set.

In the reverse direction there exist recursive functions e_1 and e_2 such that for all total functions f,

$$W^f_z = \Phi_{e_1(z)}(G_f),$$

where $G_f = \{\langle x, y\rangle : f(x) \simeq y\}$, and for all $A \subseteq \mathbb{N}$,

$$W^A_z = \Phi_{e_2(z)}(A^+).$$

Notice that the graph of every total function is a total set.

The enumeration jump operator is defined in Cooper [3] and further studied by McEvoy [5]. Here we shall use the following definition of the e-jump which is m-equivalent to the original one, see [5]:

2.1. Definition. Given a set A, let $K^0_A = \{\langle x, z\rangle : x \in \Phi_z(A)\}$. Define the *e-jump* A'_e of A to be the set $(K^0_A)^+$.

The following properties of the enumeration jump are proved in [5]:

Let A and B be sets of natural numbers. Set $B^{(0)}_e = B$ and $B^{(n+1)}_e = (B^{(n)}_e)'_e$.

(J1) If $A \leq_e B$, then $A'_e \leq_e B'_e$.

(J2) A is Σ^0_{n+1} relative to B iff $A \leq_e (B^+)^{(n)}_e$.

Let α be a recursive ordinal. To define the α-th enumeration jump of a set A we are going to use a construction very similar to that used in the definition of the α-th Turing jump. The idea is to modify the definition of the sets H^A_α, see [6] or [7], by taking enumeration jump instead of Turing jump:

2.2. Definition.

(i) $E^A_0 = A$.

(ii) $E^A_{\beta+1} = (E^A_\beta)'_e$.

(iii) If $\alpha = \lim \alpha(p)$, then $E^A_\alpha = \{\langle p, x\rangle : x \in E^A_{\alpha(p)}\}$.

From now on $A_e^{(\alpha)}$ will stand for E^A_α.

Of course the definition of the set $A_e^{(\alpha)}$ depends on the fixed notation of the ordinal α. On the other hand, it is easy to see by a minor modification of the proof of the corresponding theorem of Spector for the sets H^A_α, see [6] or [7], that if α_1 and α_2 are two notations of the same recursive ordinal, then $A_e^{(\alpha_1)} \equiv_e A_e^{(\alpha_2)}$.

The following properties of the transfinite iteration of the enumeration jump follow easily from the definition:

(E1) If $\beta \leq \alpha$ are recursive ordinals, then $A_e^{(\beta)} \leq_e A_e^{(\alpha)}$ uniformly in β and α.

(E2) If $A \leq_e B$, then for every recursive ordinal α, $A_e^{(\alpha)} \leq_e B_e^{(\alpha)}$.

(E3) If $\alpha > 0$, then $A_e^{(\alpha)}$ is a total set.

Finally, we have that for total sets the α-th enumeration jump and the α-th Turing jump are equivalent. Namely the following is true:

2.3. Proposition. *Let A be a total set of natural numbers. Then for every recursive ordinal α, $E^A_\alpha \equiv_e (H^A_\alpha)^+$ uniformly in α.*

Since we are going to consider only e-jumps here, from now on we shall omit the subscript e in the notation of the enumeration jump. So for every recursive ordinal α by $A^{(\alpha)}$ we shall denote the α-th enumeration jump of A.

For every function f and every recursive ordinal α by f^α we shall denote the α-th jump of the graph G_f of f. It is easy to see that there exists a recursive function $e(z, \alpha)$ such that if f is a total function and α is a recursive ordinal, then

$$W_z^{f^{(\alpha)}} = \Phi_{e(z,\alpha)}(f^{(\alpha)}).$$

Hence for jumps of total functions the relations "r.e." in and "$\leq_e$" are uniformly equivalent.

2.3. **The jump set of a sequence of sets.** Let ζ be a recursive ordinal and let $\{B_\alpha\}_{\alpha\leq\zeta}$ be a sequence of sets of natural numbers. For every recursive ordinal α we define the *jump set $\mathcal{P}_\alpha$ of the sequence* $\{B_\alpha\}$ by means of transfinite recursion on α:

2.4. Definition.

(i) $\mathcal{P}_0 = B_0$.

(ii) Let $\alpha = \beta + 1$. Then let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}'_\beta \oplus B_\alpha & \text{if } \alpha \leq \zeta, \\ \mathcal{P}'_\beta & \text{otherwise.} \end{cases}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}_{\alpha(p)}\}$ and let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}_{<\alpha} \oplus B_\alpha & \text{if } \alpha \leq \zeta, \\ \mathcal{P}_{<\alpha} & \text{otherwise.} \end{cases}$$

Notice that if the sequence $\{B_\alpha\}$ contains only one member, i.e $\zeta = 0$, then for every recursive α, $\mathcal{P}_\alpha = B_0^{(\alpha)}$.

The properties of the jump sets $\mathcal{P}_\alpha$ are similar to the properties of the enumeration jumps. Again we have that if α_1 and α_2 are two notations of the same recursive ordinal, then $\mathcal{P}_{\alpha_1} \equiv_e \mathcal{P}_{\alpha_2}$. We shall omit the proof since it is very close to the proof of the corresponding result for the H^A_α sets mentioned above.

We shall use the following properties of the jump sets which follow easily from the definition:

($\mathcal{P}$1) If $\beta \leq \alpha$, then $\mathcal{P}_\beta \leq_e \mathcal{P}_\alpha$ uniformly in β and α.
($\mathcal{P}$2) If $\alpha \leq \zeta$, then $B_\alpha \leq_e \mathcal{P}_\alpha$ uniformly in α.
($\mathcal{P}$3) Let $(\forall \alpha \leq \zeta)(B_\alpha \leq_e A^{(\alpha)}$ uniformly in $\alpha)$. Then $\mathcal{P}_\zeta \leq_e A^{(\zeta)}$.
($\mathcal{P}$4) If α is a limit ordinal, then the set $\mathcal{P}_{<\alpha}$ is total.
($\mathcal{P}$5) If $\zeta < \alpha$, then the set $\mathcal{P}_\alpha$ is total.

2.4. **The Results.** Let us fix a recursive ordinal ζ and a sequence $\{B_\alpha\}_{\alpha \leq \zeta}$ of subsets of $\mathbb{N}$. For every recursive ordinal α denote by $\mathcal{P}_\alpha$ the α-th jump set of the sequence $\{B_\alpha\}_{\alpha \leq \zeta}$.

The following version of Ash's Theorem [1] is proved in [10]:

2.5. Theorem. *Let α be a recursive ordinal and $A \subseteq \mathbb{N}$. Suppose that for all total sets X such that $(\forall \gamma \leq \zeta)(B_\gamma \leq_e X^{(\gamma)})$ uniformly in γ we have that $A \leq_e X^{(\alpha)}$. Then $A \leq_e \mathcal{P}_\alpha$.*

In particular if we take $\zeta = \alpha = 0$, then from Theorem 2.5 we get Selman's Theorem [8] which describes the enumeration reducibility in terms of the relation "r. e. in":

2.6. Theorem.(Selman) Let A and B be sets of natural numbers. Then $A \leq_e B$ if and only if $(\forall X)(B$ is r.e. in $X \Rightarrow$ A is r.e. in $X)$.

Another direct consequence of Theorem 2.5 is the following Theorem of Case [2]:

2.7. Theorem.(Case) Let A and B be sets of natural numbers and $n < \omega$. Then $A \leq_e B \oplus \emptyset^{(n)}$ if and only if $(\forall X)(B$ is $\Sigma^X_n \Rightarrow A$ is $\Sigma^X_n)$.

So, while the theorems of Selman and Case describe a kind of positive reducibilities between sets of natural numbers in terms of the classical relation "Σ^0_n in", Theorem 2.5 gives a description of a positive reducibility of a transfinite sequence of sets to a set. A natural further step is to generalize Theorem 2.5 in order to obtain a reducibility of transfinite sequences of sets to transfinite sequences of sets.

2.8. Definition. Let ζ be a recursive ordinal and let $\mathcal{A} = \{A_\alpha\}_{\alpha \leq \zeta}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha \leq \zeta}$ be two sequences of sets of natural numbers. Then $\mathcal{A}$ is *uniformly reducible* to $\mathcal{B}$ ($\mathcal{A} \leq_u \mathcal{B}$) if there exists a recursive function g such that

$$(\forall \alpha \leq \zeta)(A_\alpha = \Phi_{g(\alpha)}(\mathcal{P}_\alpha(\mathcal{B}))),$$

where $\mathcal{P}_\alpha(\mathcal{B})$ denotes the α-th jump set of the sequence $\mathcal{B}$.

The following theorem is our first result:

2.9. Theorem. *Let ζ be a recursive ordinal and let $\mathcal{A} = \{A_\alpha\}_{\alpha \leq \zeta}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha \leq \zeta}$ be two sequences of sets of natural numbers. Then*

$$\mathcal{A} \leq_u \mathcal{B} \iff (\forall \textit{ total } X)[(\forall \alpha \leq \zeta)(B_\alpha \leq_e X^{(\alpha)} \textit{ uniformly in } \alpha) \Rightarrow$$
$$(\forall \alpha \leq \zeta)(A_\alpha \leq_e X^{(\alpha)} \textit{ uniformly in } \alpha)].$$

The uniform operators are introduced in [9]. Let us fix natural numbers $k_0, \dots, k_r$ and k.

2.10. Definition. A mapping $\Gamma : \mathcal{P}(\mathbb{N})^r \to \mathcal{P}(\mathbb{N})$ is a *uniform operator* of type $(k_0, \dots, k_r \to k)$ if there exists a function g on the natural numbers such that for all $X \subseteq \mathbb{N}$ and for all $b_0, \dots, b_r \in \mathbb{N}$,

$$\Gamma(W_{b_0}^{X^{(k_0)}}, \dots, W_{b_r}^{X^{(k_r)}}) = W_{g(b_0,\dots,b_r)}^{X^{(k)}}.$$

Given sets $B_0, \dots, B_r$ of natural numbers, denote by $\mathcal{P}^k_{k_0,\dots,k_r}(B_0, \dots, B_r)$ the k-th jump set of the sequence $\{A_l\}_{l \le k}$, where

$$A_l = \begin{cases} \emptyset, & \text{if } l \notin \{k_0, \dots, k_r\}, \\ B_m, & \text{if } l = k_m,\ 0 \le m \le r. \end{cases}$$

2.11. Theorem. ([9]) A mapping $\Gamma : \mathcal{P}(\mathbb{N})^r \to \mathcal{P}(\mathbb{N})$ is a uniform operator of type $(k_0, \dots, k_r \to k)$ if and only if there exists an enumeration operator Φ such that for all sets $B_0, \dots, B_r$,

$$\Gamma(B_0, \dots, B_r) = \Phi(\mathcal{P}^k_{k_0,\dots,k_r}(B_0, \dots, B_r)).$$

Combining this result with the theorems of Selman and Case, we obtain the following corollary:

2.12. Corollary. *Let $n \ge 1$ and A and B be sets of natural numbers. Then $(\forall X)(B$ is $\Sigma^X_n \Rightarrow A$ is $\Sigma^X_n)$ if and only if there exists a uniform operator Γ of type $(n \to n)$ such that $A = \Gamma(B)$.*

Here we are going to study uniform operators on transfinite sequences of sets. Let ζ be a recursive ordinal. Denote by $\mathcal{S}_\zeta$ the set of all sequences $\{A_\alpha\}_{\alpha \le \zeta}$ of sets of natural numbers.

2.13. Definition. A mapping $\Gamma : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$ is *uniform operator* if there exists a function g on the natural numbers such that if a is an index of a recursive function φ, then $g(a)$ is an index of a recursive function ψ such that for all $X \subseteq \mathbb{N}$,

$$\Gamma(\{W^{X^{(\alpha)}}_{\varphi(\alpha)}\}_{\alpha \le \zeta}) = \{W^{X^{(\alpha)}}_{\psi(\alpha)}\}_{\alpha \le \zeta}.$$

Our second result is the following generalization of Theorem 2.11.

Given a sequence $\mathcal{B} \in \mathcal{S}_\zeta$, denote by $\mathcal{P}_\alpha(\mathcal{B})$ the α-th jump set of $\mathcal{B}$.

2.14. Theorem. *A mapping $\Gamma : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$ is a uniform operator if and only if there exists a recursive function h such that for every sequence $\mathcal{B} \in \mathcal{S}_\zeta$,*

$$\Gamma(\mathcal{B}) = \{\Phi_{h(\alpha)}(\mathcal{P}_\alpha(\mathcal{B}))\}.$$

Combining Theorem 2.9 and Theorem 2.14 we obtain and the following

2.15. Corollary. *Let $\mathcal{A} = \{A_\alpha\}$ and $\mathcal{B} = \{B_\alpha\}$ be elements of $\mathcal{S}_\zeta$. Then the following assertions are equivalent:*

(1) $\mathcal{A} \le_u \mathcal{B}$.

(2)

$$(\forall \textit{ total } X)[(\forall \alpha \le \zeta)(B_\alpha \le_e X^{(\alpha)} \textit{ uniformly in } \alpha) \Rightarrow$$
$$(\forall \alpha \le \zeta)(A_\alpha \le_e X^{(\alpha)} \textit{ uniformly in } \alpha)].$$

(3) *There exists a uniform operator $\Gamma : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$ such that $\mathcal{A} = \Gamma(\mathcal{B})$.*

In the rest of the paper we shall introduce the reader to he technique of the uniform regular enumerations which will be used in the proofs of Theorem 2.9 and Theorem 2.14.

3. Ordinal approximations

3.1. Definition. Given an ordinal $\alpha > 0$, an *ordinal approximation of* α is a finite sequence $\overline{\alpha} = \alpha_1 < \alpha_2 < \dots \alpha_n < \alpha$ of ordinals, where $n \geq 1$ and $\alpha_1 = 0$.

The only ordinal approximation of 0 is 0.

For every ordinal approximation $\overline{\alpha} = \alpha_1, \alpha_2, \dots, \alpha_n, \alpha$ and every $\beta < \alpha$ we define the *β-predecessor* $\overline{\beta}$ of $\overline{\alpha}$ by means of the following inductive definition:

3.2. Definition.

1) Let $\beta \leq \alpha_n$. Then
   1.1) If $\beta = \alpha_i$ for some $i \in [1, n]$, then $\overline{\beta} = \alpha_1, \dots, \alpha_i$;
   1.2) Otherwise, if α_i is the least element of the sequence $\alpha_1, \dots, \alpha_n$ such that $\beta < \alpha_i$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \dots, \alpha_i$;
2) Let $\alpha_n < \beta < \alpha$. Then
   2.1) If $\alpha = \delta + 1$ and $\beta = \delta$, then $\overline{\beta} = \alpha_1, \dots, \alpha_n, \delta$;
   2.2) If $\alpha = \delta + 1$ and $\beta < \delta$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \dots, \alpha_n, \delta$;
   2.3) If $\alpha = \lim \alpha(p)$, then $\overline{\beta}$ is the β-predecessor of
$$\alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \dots, \alpha(p_1), \text{ where}$$
$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_1 = \mu p[\beta < \alpha(p)].$$

The following simple lemma can be proved by means of transfinite induction on α.

3.3. Lemma. *For every ordinal approximation $\overline{\alpha}$ and every $\beta < \alpha$, there exists exactly one β-predecessor $\overline{\beta}$ of $\overline{\alpha}$.*

From the definition it follows immediately that there exists a recursive function π such that if $\overline{\alpha}$ is an ordinal approximation and $\beta < \alpha$, then $\pi(\overline{\alpha}, \beta)$ yields the β-predecessor of $\overline{\alpha}$.

By $\overline{\beta} \prec \overline{\alpha}$ we shall denote that $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$. As usual $\overline{\beta} \preceq \overline{\alpha}$ will stand for $\overline{\beta} \prec \overline{\alpha}$ or $\overline{\beta} = \overline{\alpha}$.

Let us point out some useful properties of the predecessor relation which follow directly from the definition.

3.4. Lemma. *Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ be an ordinal approximation of α. Then the following assertions hold:*

(1) *If $\beta \leq \alpha_k$, $1 \leq k \leq n$, then $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \dots, \alpha_k$.*
(2) *If for some $k \in [1, n]$, $\alpha_k \leq \beta < \alpha$ and $\beta_1, \dots, \beta_l$ is the β-predecessor of α, then $k \leq l$ and $\alpha_i = \beta_i, i = 1, \dots, k$.*
(3) *Let $\alpha = \delta + 1$, $\alpha_n < \delta$ and $\beta \leq \delta$. Then $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \dots, \alpha_n, \delta$.*
(4) *Let $\alpha = \lim \alpha(p)$ be a limit ordinal and $p_0 = \mu p[\alpha_n < \alpha(p)]$. Let $\beta < \alpha$, $p_1 \geq p_0$ and $\alpha(p_1) \geq \beta$. Then*
$$\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \dots, \alpha(p_1).$$

3.5. Lemma. *Let $\gamma < \beta < \alpha$ be ordinals, $\overline{\gamma} \prec \overline{\beta}$ and $\overline{\beta} \prec \overline{\alpha}$. Then $\overline{\gamma} \prec \overline{\alpha}$.*

Proof. Transfinite induction on α. Suppose that $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$.

Let $\beta \leq \alpha_n$. Then $\overline{\beta} \preceq \alpha_1, \dots, \alpha_n$. By the induction hypothesis, $\overline{\gamma} \prec \alpha_1, \dots, \alpha_n$. Therefore by Lemma 3.4 $\overline{\gamma} \prec \overline{\alpha}$.

Suppose now that $\alpha_n < \beta$. Let $\alpha = \delta + 1$. Set $\overline{\delta} = \alpha_1, \ldots, \alpha_n, \delta$. Since $\beta \leq \delta$, $\alpha_n < \delta$. By Lemma 3.4 $\overline{\beta} \preceq \overline{\delta}$. By the induction hypothesis $\overline{\gamma} \prec \overline{\delta}$. From here again by Lemma 3.4 it follows that $\overline{\gamma} \prec \overline{\alpha}$.

It remains to consider the case $\alpha_n < \beta$ and $\alpha = \lim \alpha(p)$. Let

$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_\beta = \mu p[\beta < \alpha(p)]$$

.

Set $\overline{\alpha(p_\beta)} = \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_\beta)$. Now we have that $\overline{\beta} \prec \overline{\alpha(p_\beta)}$. By induction $\overline{\gamma} \prec \overline{\alpha(p_\beta)}$ and hence by Lemma 3.4 $\overline{\gamma} \prec \overline{\alpha}$. □

From the last lemma it follows that if we fix an ordinal approximation $\overline{\alpha}$ and consider the set of all ordinal approximations $\overline{\beta} \prec \overline{\alpha}$, then this set is well ordered by the relation " $\prec$ " and its order type is α.

4. Regular finite parts

Let us fix a sequence $\{B_\alpha\}_{\alpha \leq \zeta}$, of subsets of $\mathbb{N}$.

For every $\alpha \leq \zeta$ set $B^*_\alpha = \mathbb{N} \oplus B_\alpha$.

In what follows we shall use the term *finite part* for finite mappings of $\mathbb{N}$ into $\mathbb{N}$ defined on finite segments $[0, q-1]$ of $\mathbb{N}$. Finite parts will be denoted by the letters τ, ρ. If $\mathrm{dom}(\tau) = [0, q-1]$, then let $\mathrm{lh}(\tau) = q$.

We shall suppose that an effective coding of all finite sequences and hence of all finite parts is fixed. Given two finite parts τ and ρ we shall say that τ is less than or equal to ρ if the code of τ is less than or equal to the code of ρ. By $\tau \subseteq \rho$ we shall denote that the partial mapping ρ extends τ and say that ρ is an extension of τ. For any τ, by $\tau \upharpoonright n$ we shall denote the restriction of τ on $[0, n-1]$.

Below we define for every $\alpha \leq \zeta$ and every ordinal approximation $\overline{\alpha}$ of α the $\overline{\alpha}$-regular finite parts. The definition is by transfinite recursion on α.

Let $\alpha \leq \zeta$. Suppose that for all $\beta < \alpha$ we have defined the $\overline{\beta}$-regular finite parts and for every $\overline{\beta}$-regular τ we have defined the $\overline{\beta}$-rank $|\tau|_{\overline{\beta}}$ of τ. Suppose also that for all finite parts ρ and for all $e, x \in \mathbb{N}$ we have defined the forcing relations $\rho \Vdash_{\overline{\beta}} F_e(x)$ and $\rho \Vdash_{\overline{\beta}} \neg F_e(x)$

4.1. Definition. Given a $\overline{\beta}$-regular finite part τ, say that $\rho \supseteq \tau$ is a *normal* $\overline{\beta}$-regular extension of τ if $|\rho|_{\overline{\beta}} = |\tau|_{\overline{\beta}} + 1$.

Let us fix an ordinal approximation $\overline{\alpha}$ of α.

1) $\alpha = 0$. Then $\overline{\alpha} = 0$. The 0-regular finite parts are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in B^*_0$.

If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank $|\tau|_0$ of τ is equal to the number $q+1$ of the odd elements of $\mathrm{dom}(\tau)$.

Given a finite part ρ, let

$$\rho \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\rho((u)_0) \simeq (u)_1))$$

$$\rho \Vdash_0 \neg F_e(x) \iff \forall (0\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_0 F_e(x)).$$

2) $\alpha = \beta + 1$. Let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

Set $X^{\overline{\beta}}_p = \{\rho : \rho \text{ is } \overline{\beta}\text{-regular } \&\ \rho \Vdash_{\overline{\beta}} F_{(p)_0}((p)_1)\}$.

Given a finite part τ and a set X of $\overline{\beta}$-regular finite parts, let $\mu_{\overline{\beta}}(\tau, X)$ be the least extension of τ belonging to X if any, and $\mu_{\overline{\beta}}(\tau, X)$ be the least $\overline{\beta}$-regular extension of τ otherwise. We shall assume that $\mu_{\overline{\beta}}(\tau, X)$ is undefined if there is no $\overline{\beta}$-regular extension of τ.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is *$\overline{\alpha}$-regular* of *$\overline{\alpha}$-rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < m_0 < b_0 < n_1 < l_1 < m_1 < b_1 \cdots < n_r < l_r < m_r < b_r < n_{r+1} = q$$

such that $\tau \restriction n_0$ is a $\overline{\beta}$-regular finite part of rank 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

s_a) $\tau \restriction l_j$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction n_j$;

s_b)

$$\tau \restriction m_j = \begin{cases} \mu_{\overline{\beta}}(\tau \restriction (l_j+1), X^{\overline{\beta}}_{\langle e, l_j \rangle}), & \text{if } \tau(n_j) \simeq \langle 0, \beta, e \rangle + 1, \\ \mu_{\overline{\beta}}(\tau \restriction (l_j+1), X^{\overline{\beta}}_{p}), & \text{if } \tau(n_j) \simeq \langle 1, \beta, p \rangle + 1, \\ \text{a normal } \overline{\beta}\text{-regular extension of } \tau \restriction l_j, & \text{otherwise;} \end{cases}$$

s_c) $\tau \restriction b_j$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction m_j$ and $\tau(b_j) \in B^*_\alpha$;

s_d) $\tau \restriction n_{j+1}$ is a normal $\overline{\beta}$-regular extension of $\tau \restriction b_j$.

To conclude with the definition of the $\overline{\alpha}$-regular finite parts in this case, let for every ρ, e and x

$$\rho \Vdash_{\overline{\alpha}} F_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \& \ (\forall u \in D_v)((u = \langle e_u, x_u, 0 \rangle \ \& \ \rho \Vdash_{\overline{\beta}} F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1 \rangle \ \& \ \rho \Vdash_{\overline{\beta}} \neg F_{e_u}(x_u)))).$$

$$\rho \Vdash_{\overline{\alpha}} \neg F_e(x) \iff \forall (\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)).$$

3) $\alpha = \lim \alpha(p)$ is a limit ordinal. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. For every p denote by $\overline{\alpha(p)}$ the $\alpha(p)$-predecessor of $\overline{\alpha}$. Notice that for every $p \geq p_0$

$$\overline{\alpha(p)} = \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p).$$

A finite part τ defined on $[0, q-1]$ is $\overline{\alpha}$-regular with $\overline{\alpha}$-rank $r+1$ if there exists natural numbers

$$0 < n_0 < b_0 < m_0 < n_1 < b_1 < m_1 ... < n_r < b_r < m_r < n_{r+1} = q,$$

such that $\tau \restriction n_0$ is an $\alpha_1, \ldots, \alpha_n$-regular finite part of rank 1 and for all $j, 0 \leq j \leq r$, the following conditions are satisfied:

l_a) $\tau \restriction b_j$ is an $\overline{\alpha(p_0+3j)}$-regular finite part of rank 1 and $\tau(b_j) \in B^*_\alpha$;

l_b) $\tau \restriction m_j$ is an $\overline{\alpha(p_0+3j+1)}$-regular finite part of rank 1;

l_c) $\tau \restriction n_{j+1}$ is an $\overline{\alpha(p_0+3j+2)}$-regular finite part of rank 1.

For every finite part ρ and every $e, x \in \mathbb{N}$ set:

$$\rho \Vdash_{\overline{\alpha}} F_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \& \ (\forall u \in D_v)(u = \langle p_u, e_u, x_u \rangle \ \& \ \rho \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u))).$$

$$\rho \Vdash_{\overline{\alpha}} \neg F_e(x) \iff \forall (\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)).$$

The following lemma shows that the $\overline{\alpha}$-rank is well defined.

4.2. Lemma. *Let $\alpha \leq \zeta$ and let τ be an $\overline{\alpha}$-regular finite part. Then the following assertions hold:*

(1) *Suppose that* $\alpha = \beta+1$. *Let* $n'_0, l'_0, m'_0, b'_0, \ldots, n'_p, l'_p, m'_p, n'_{p+1}$ *and* $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ *be two sequences of natural numbers satisfying s_a)– s_d) from the definition above. Then* $r = p, n_{p+1} = n'_{p+1}$ *and for all* $j \le r, n_j = n'_j, l_j = l'_j, m_j = m'_j$ *and* $b_j = a_j$.

(2) *Suppose that* $\alpha = \lim \alpha(p)$ *is a limit ordinal and let* $n'_0, b'_0, m'_0 \ldots, n'_p, b'_p, m'_p, n'_{p+1}$ *and* $n_0, b_0, m_0 \ldots, n_r, b_r, m_r, n_{r+1}$ *be two sequences of natural numbers satisfying the conditions l_a) – l_c). Then* $r = p, n_{p+1} = n'_{p+1}$ *and for all* $j \le r, n'_j = n_j$, $b'_j = b_j$ *and* $m'_j = m_j$.

(3) *If* ρ *is* $\overline{\alpha}$*-regular,* $\tau \subseteq \rho$ *and* $|\tau|_{\overline{\alpha}} = |\rho|_{\overline{\alpha}}$, *then* $\tau = \rho$.

Proof. The proof follows easily from the definition of the $\overline{\alpha}$-regular finite parts by transfinite induction on α. □

4.3. Corollary. *Let* $\alpha = \beta+1$, $\overline{\alpha}$ *be an ordinal approximation of* α *and let* $\overline{\beta}$ *be the* β*-predecessor of* $\overline{\alpha}$. *Then every* $\overline{\alpha}$*-regular finite part* τ *is* $\overline{\beta}$*-regular and* $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$.

We shall state several properties of the regular finite parts omitting the proofs which can be found in [10].

4.4. Lemma. *Let* $1 \le \alpha \le \zeta$ *and* $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$, $1 \le k \le n$. *Then every* $\overline{\alpha}$*-regular finite part* τ *is* $\alpha_1, \ldots, \alpha_k$*-regular and the* $\alpha_1, \ldots, \alpha_k$*-rank of* τ *is strictly greater than* $|\tau|_{\overline{\alpha}}$.

4.5. Lemma. *Let* $\alpha = \lim \alpha(p)$ *be a limit ordinal. Let* $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ *and* $p_0 = \mu p[\alpha(p) > \alpha_n]$. *Suppose that* $p_1 \ge p_0$ *and* τ *is an* $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1)$*-regular finite part of rank 1. Then for every* $\overline{\beta} \prec \overline{\alpha}$ *if* τ *is* $\overline{\beta}$*-regular, then* $\beta \le \alpha(p_1)$.

4.6. Definition. For every finite part τ and every ordinal approximation $\overline{\alpha}$ let

$$Reg(\tau, \overline{\alpha}) = \{\overline{\beta} : \overline{\beta} \preceq \overline{\alpha} \text{ and } \tau \text{ is } \overline{\beta}\text{-regular}\}.$$

4.7. Lemma. *Let* $\alpha \le \zeta$, *let* $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ *be an ordinal approximation of* α *and let* τ *be an* $\overline{\alpha}$*-regular finite part. Then the following assertions are true:*

(1) *If* $\alpha = \delta + 1$ *and* $\overline{\delta}$ *is the* δ*-predecessor of* $\overline{\alpha}$, *then*

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \ \vee\ \overline{\beta} \in Reg(\tau, \overline{\delta}).$$

(2) *Let* $\alpha = \lim \alpha(p)$. *Set* $p_0 = \mu p[\alpha(p) > \alpha_n]$ *and for every* $p \ge p_0$ *let* $\overline{\alpha(p)}$ *be the* $\alpha(p)$*-predecessor of* $\overline{\alpha}$. *Suppose that* $p_1 \ge p_0$ *and* τ *is an* $\overline{\alpha(p_1)}$*-regular finite part of rank 1. Then*

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \ \vee\ \overline{\beta} \in Reg(\tau, \overline{\alpha(p_1)}).$$

4.8. Lemma. *Let* $\alpha \le \zeta$ *and let* $\overline{\alpha}$ *be an approximation of* α. *Suppose that* $\overline{\beta} \preceq \overline{\alpha}$. *Then there exists a natural number* $k(\overline{\alpha}, \overline{\beta})$ *such that every* $\overline{\alpha}$*-regular finite part of rank greater than or equal to* $k(\overline{\alpha}, \overline{\beta})$ *is* $\overline{\beta}$*-regular.*

Proof. We shall use transfinite induction on α. The assertion is obviously true for $\alpha = 0$.

Suppose that $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$. Let $\overline{\beta} \prec \overline{\alpha}$. Then $\overline{\beta} \preceq \overline{\delta}$. By induction every $\overline{\delta}$-regular finite part of rank at least $k(\overline{\delta}, \overline{\beta})$ is $\overline{\beta}$-regular. Set $k(\overline{\alpha}, \overline{\beta}) = k(\overline{\delta}, \overline{\beta}) = k$. Consider an $\overline{\alpha}$-regular finite part τ of rank at least k. Then τ is $\overline{\delta}$-regular of rank greater than k and hence τ is $\overline{\beta}$-regular.

Let $\alpha = \lim \alpha(p)$, $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ and $\overline{\beta} \prec \overline{\alpha}$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. By definition an $\overline{\alpha}$-regular finite part τ is of rank $r+1$ if and only if τ is $\overline{\alpha(p_0+3r+2)}$-regular of rank 1. Let $p_1 = \mu p[p > p_0 \wedge \alpha(p) > \beta]$.

By Lemma 3.4 $\overline{\beta} \preceq \overline{\alpha(p_1)}$. Hence, by induction, every $\overline{\alpha(p_1)}$-regular finite part of rank at least $k = k(\overline{\alpha(p_1)}, \overline{\beta})$ is $\overline{\beta}$-regular.

Now let r be the least natural number such that $p_0+3r+2 > p_1+k$. Consider an $\overline{\alpha}$-regular finite part τ of rank greater than or equal to $r+1$. Then by Lemma 4.4 τ is $\overline{\alpha(p_1+k)}$-regular of rank at least 2 and hence again by Lemma 4.4 τ is $\overline{\alpha(p_1)}$-regular of rank greater than k. So τ is $\overline{\beta}$-regular. $\square$

4.9. Remark. *From the proof above it follows that we may assume that the function k is recursive.*

4.10. Corollary. *Let $\alpha \leq \zeta$, $\overline{\alpha}$ be an ordinal approximation of α and $\overline{\beta} \preceq \overline{\alpha}$. Suppose that τ is an $\overline{\alpha}$-regular finite part of rank greater than $k(\overline{\alpha}, \overline{\beta}) + s$. Then $|\tau|_{\overline{\beta}} > s$.*

Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α.

Denote by $\mathcal{R}_{\overline{\alpha}}$ the set of all $\overline{\alpha}$-regular finite parts.

For every $p \in \mathbb{N}$ let

$$Y_p^{\overline{\alpha}} = \{\tau : \tau \in \mathcal{R}_{\overline{\alpha}} \ \&\ (\exists \rho \supseteq \tau)(\rho \in \mathcal{R}_{\overline{\alpha}} \ \&\ \rho \Vdash_{\overline{\alpha}} F_{(p)_0}((p)_1))\},$$
$$Z_p^{\overline{\alpha}} = \{\tau : \tau \in \mathcal{R}_{\overline{\alpha}} \& \ \tau \Vdash_{\overline{\alpha}} \neg F_{(p)_0}((p)_1)\}.$$

Let $\mu_{\overline{\alpha}}^X(\tau, p) \simeq \mu_{\overline{\alpha}}(\tau, X_p^{\overline{\alpha}})$

4.11. Proposition. *There exist recursive functions $h_1, \dots, h_5$ such that for every sequence $\mathcal{B} \in \mathcal{S}_\zeta$ and for every ordinal approximation $\overline{\alpha}$, $\alpha \leq \zeta$, the following assertions are true:*

1. $\mathcal{R}_{\overline{\alpha}} = \Phi_{h_1(\overline{\alpha})}(\mathcal{P}_\alpha(\mathcal{B}))$.
2. *For every* $p \in \mathbb{N}$, $X_p^{\overline{\alpha}} = \Phi_{h_2(\overline{\alpha},p)}(\mathcal{P}_\alpha(\mathcal{B}))$.
3. *For every* $p \in \mathbb{N}$, $Y_p^{\overline{\alpha}} = \Phi_{h_3(\overline{\alpha},p)}(\mathcal{P}_\alpha(\mathcal{B}))$.
4. *For every* $p \in \mathbb{N}$ *the characteristic function of* $\{Z_p^{\overline{\alpha}}\}$ *is equal to* $\{h_4(\overline{\alpha})\}^{\mathcal{P}'_\alpha(\mathcal{B})}$.
5. $\mu_{\overline{\alpha}}^X = \{h_5(\overline{\alpha})\}^{\mathcal{P}'_\alpha(\mathcal{B})}$.

5. Regular enumerations

For every $\overline{\alpha}$-regular finite part τ of rank $r+1$ we define the subsets $N_{\overline{\alpha}}^\tau$ and $B_{\overline{\alpha}}^\tau$ of $\mathrm{dom}(\tau)$ as follows.

5.1. Definition.

a) If $\alpha = 0$, then let $N_{\overline{\alpha}}^\tau = \{n : n \in \mathrm{dom}(\tau)\ \&\ n \text{ is even}\}$ and $B_{\overline{\alpha}}^\tau = \{b : b \in \mathrm{dom}(\tau)\ \&\ b \text{ is odd}\}$.

b) Let $\alpha = \beta + 1$ and let $n_0, l_0, m_0, b_0, \dots, n_r, l_r, m_r, b_r, n_{r+1}$ satisfy the conditions s_a)–s_d) from the definition of the regular finite parts. Set $N_{\overline{\alpha}}^\tau = \{n_0, \dots, n_r\}$ and $B_{\overline{\alpha}}^\tau = \{b_0, \dots, b_r\}$.

c) Let $\alpha = \lim \alpha(p)$ and $n_0, b_0, m_0 \dots, n_r, b_r, m_r, n_{r+1}$ satisfy the conditions l_a)–l_c) from the definition of the regular finite parts. Set $N_{\overline{\alpha}}^\tau = \{n_0, \dots, n_r\}$ and $B_{\overline{\alpha}}^\tau = \{b_0, \dots, b_r\}$.

5.2. Definition. Let $\overline{\zeta}$ be an ordinal approximation of ζ. A a total mapping f of $\mathbb{N}$ in $\mathbb{N}$ is called *regular enumeration* (with respect to $\overline{\zeta}$) if the following two conditions hold:

(i) For every finite part $\rho \subseteq f$, there exists a $\overline{\zeta}$-regular extension τ of ρ such that $\tau \subseteq f$.
(ii) If $\overline{\alpha} \preceq \overline{\zeta}$ and $z \in B^*_\alpha$, then there exists an $\overline{\alpha}$-regular $\tau \subseteq f$, such that $z \in \tau(B^\tau_{\overline{\alpha}})$.
(iii) If $\overline{\alpha+1} \preceq \overline{\zeta}$ and $p \in \mathbb{N}$, then there exists an $\overline{\alpha+1}$-regular $\tau \subseteq f$, such that $\langle 1, \alpha, p\rangle + 1 \in \tau(N^\tau_{\overline{\alpha+1}})$.

Clearly, if f is a regular enumeration and $\overline{\alpha} \preceq \overline{\zeta}$, then for every $\rho \subseteq f$, there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Moreover there exist $\overline{\alpha}$-regular finite parts of f of arbitrary large rank.

Given a regular f and $\overline{\alpha} \preceq \overline{\zeta}$, let $B^f_{\overline{\alpha}} = \{b : (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \& \ b \in B^\tau_{\overline{\alpha}})\}$. Evidently $f(B^f_{\overline{\alpha}}) = B^*_\alpha$.

5.3. Proposition. *Suppose that f is a regular enumeration. Then*

(1) $B_0 \le_e f$.
(2) *If $\alpha = \beta + 1 \le \zeta$, then $B_\alpha \le_e f \oplus \mathcal{P}'_\beta$ uniformly in α.*
(3) *If $\alpha \le \zeta$ is a limit ordinal, then $B_\alpha \le_e f \oplus \mathcal{P}_{<\alpha}$ uniformly in α.*
(4) *$\mathcal{P}_\alpha \le_e f^{(\alpha)}$ uniformly in α.*
(5) *$B_\alpha \le_e f^{(\alpha)}$ uniformly in α.*

Proof. Notice that since $B^*_\alpha = \mathbb{N} \oplus B_\alpha$, we have that for every α,

$$B_\alpha = \{x : 2x + 1 \in B^*_\alpha\}$$

.

Since f is regular, $B^*_0 = f(B^f_0)$. Clearly B^f_0 is equal to the set of all odd natural numbers.

Let us turn to the proof of (2) and (3). We shall describe an effective procedure satisfying the requirements of (2) and (3) by means of effective transfinite recursion on α.

Let $\alpha = \beta+1$. Suppose that $\overline{\alpha}$ is the α-predecessor of $\overline{\zeta}$ and $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$.

Since f is regular, for every finite part ρ of f there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Hence there exist natural numbers

$$0 < n_0 < l_0 < m_0 < b_0 < n_1 < l_1 < m_1 < b_1 < \cdots < n_r < l_r < m_r < b_r < \dots,$$

such that for every $r \ge 0$, the finite part $\tau_r = f \restriction n_{r+1}$ is $\overline{\alpha}$-regular and $n_0, l_0, m_0, b_0, \dots, n_r, l_r, m_r, b_r, n_{r+1}$ are the numbers satisfying the conditions s_a)–s_d) from the definition of the $\overline{\alpha}$-regular finite part τ_r. Clearly $B^f_{\overline{\alpha}} = \{b_0, b_1 \dots\}$. We shall show that there exists a recursive in $f \oplus \mathcal{P}'_\beta$ way to list $n_0, l_0, m_0, b_0, \dots$ in an increasing order.

Clearly $f \restriction n_0$ is $\overline{\beta}$-regular and $|f \restriction n_0|_{\overline{\beta}} = 1$. By Proposition 4.11 $\mathcal{R}_{\overline{\beta}}$ is uniformly recursive in $\mathcal{P}'_\beta$. Using f we can generate consecutively the finite parts $f \restriction q$ for $q = 1, 2 \dots$. By Lemma 4.2 $f \restriction n_0$ is the first element of this sequence which belongs to $\mathcal{R}_{\overline{\beta}}$. Clearly $n_0 = \mathrm{lh}(f \restriction n_0)$.

Suppose that $r \ge -1$ and $n_0, l_0, m_0, b_0, \dots, n_r, l_r, m_r, b_r, n_{r+1}$ have already been listed. Since $f \restriction l_{r+1}$ is a normal $\overline{\beta}$-regular extension of $f \restriction n_{r+1}$, it is the first element of the sequence $f \restriction q$, $q > n_{r+1}$, belonging to $\mathcal{R}_{\overline{\beta}}$. This way we get l_{r+1}. In order to determine m_{r+1} we have to consider the value y of $f(n_{r+1})$:

a) $y = \langle 0, \beta, e\rangle + 1$. Then $m_{r+1} = \mathrm{lh}(\mu^X_{\overline{\beta}}(f \restriction (l_{r+1}+1), \langle e, l_{r+1}\rangle))$.

b) $y = \langle 1, \beta, p\rangle + 1$. Then $m_{r+1} = \mathrm{lh}(\mu^X_{\overline{\beta}}(f \restriction (l_{r+1}+1), p))$.

c) Otherwise, m_{r+1} is equal to the least $q > l_{r+1}$ such that $f \restriction q \in \mathcal{R}_{\overline{\beta}}$.

Since $f \restriction b_{r+1}$ is a normal $\overline{\beta}$-regular extension of $f \restriction m_{r+1}$ and $f \restriction n_{r+2}$ is a normal $\overline{\beta}$-regular extension of $f \restriction b_{r+1}$, we can find b_{r+1} and n_{r+2} in the same way as above.

So $B^f_{\overline{\alpha}}$ is recursive in $f \oplus \mathcal{P}'_\beta$. Hence, since $B^*_\alpha = f(B^f_\alpha)$, $B^*_\alpha \le_e f \oplus \mathcal{P}'_\beta$.

Suppose now that $\alpha = \lim \alpha(p)$ is a limit ordinal. Clearly for every p the set $\mathcal{P}_{\alpha(p)}$ is recursive in $\mathcal{P}_{<\alpha}$ uniformly in p. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$ and for every p let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. We may think that f is an infinite union of $\overline{\alpha}$-regular finite parts. So there exists an infinite sequence of natural numbers

$$n_0 < b_0 < m_0 < n_1 < b_1 < m_1 < \ldots < n_r < b_r < m_r < n_{r+1} \ldots$$

such that for every r the finite part $f \restriction n_{r+1}$ is $\overline{\alpha}$-regular of rank $r+1$ and $n_0, b_0, m_0 \ldots, n_r, b_r, m_r$ are the elements of $\mathrm{dom}(f \restriction n_{r+1})$ satisfying the conditions l_a)–l_c) from the definition of the $\overline{\alpha}$-regular finite part. As in the previous case there exists an recursive in $f \oplus \mathcal{P}_{<\alpha}$ way to list the numbers $n_0, b_0, m_0 \ldots$ in an increasing order. To show this we need to know only that for every p the set $\mathcal{R}_{\overline{\alpha(p)}}$ is uniformly recursive $\mathcal{P}'_{\alpha(p)}$ and hence, it is uniformly recursive in $\mathcal{P}_{<\alpha}$.

The assertions (4) and (5) follow easily from (1), (2) and (3). $\square$

5.4. Remark. *An inspection of the proof above shows there exists a recursive function φ such that for every sequence $\{B_\alpha\}_{\alpha \le \zeta}$ of sets of natural numbers and every regular with respect to this sequence enumeration f, we have that*

$$B_\alpha = W^{f^{(\alpha)}}_{\varphi(\alpha)}.$$

Let f be a total mapping on $\mathbb{N}$. We define for every recursive ordinal α, $e, x \in \mathbb{N}$ the relations $f \models_\alpha F_e(x)$ and $f \models \neg F_e(x)$ by means of transfinite recursion on α:

5.5. Definition.

(i) Let $\alpha = 0$. Then

$$f \models_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(f((u)_0) = (u)_1)).$$

(ii) Let $\alpha = \beta + 1$. Then

$$f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)))).$$

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Then

$$f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(u = \langle p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u))).$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Following the definition of the enumeration jump and the definition above, we can define a recursive function h such that for every recursive ordinal α and every enumeration operator Φ_z the following equivalence is true:

$$x \in \Phi_z(f^{(\alpha)}) \iff f \models_\alpha F_{h(\alpha,z)}(x).$$

Therefore we have the following lemma:

5.6. Lemma. *Let f be a total mapping on $\mathbb{N}$ and let α be a recursive ordinal. Then $A \leq_e f^{(\alpha)}$ iff there exists an e such that for all x, $x \in A \iff f \models_\alpha F_e(x)$.*

Our next goal is the proof of the Truth Lemma. Notice that for all $\overline{\alpha} \preceq \overline{\zeta}$ the relation $\Vdash_{\overline{\alpha}}$ is monotone, i.e. if $\tau \subseteq \rho$ are $\overline{\alpha}$-regular and $\tau \Vdash_{\overline{\alpha}} (\neg)F_e(x)$, then $\rho \Vdash_{\overline{\alpha}} (\neg)F_e(x)$.

Suppose that f is a regular enumeration.

5.7. Lemma. *Let $\alpha < \zeta$ and let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$. Assume also that*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \& \ \tau \Vdash_{\overline{\alpha}} F_e(x)).$$

Then

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \& \ \tau \Vdash_{\overline{\alpha}} \neg F_e(x)).$$

Proof. Assume that $f \models_\alpha \neg F_e(x)$ and for all $\overline{\alpha}$-regular $\tau \subseteq f$, $\tau \nVdash_{\overline{\alpha}} \neg F_e(x)$. Then for all $\overline{\alpha}$-regular finite parts τ of f there exists an $\overline{\alpha}$-regular $\rho \supseteq \tau$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$.

Since f is regular there exists an $\overline{\alpha+1}$-regular finite part $\tau \subseteq f$ such that $\langle 1, \alpha, \langle e, x\rangle\rangle + 1 \in \tau(N^{\tau}_{\overline{\alpha+1}})$. Let $|\tau_{\overline{\alpha+1}}| = r+1$ and let

$$0 < n_0 < l_0 < m_0 < b_0 < \cdots < n_r < l_r < m_r < b_r < n_{r+1}$$

be the natural numbers satisfying the conditions s_a)–s_d) of the definition of the $\overline{\alpha+1}$-regular finite parts. Then $N^{\tau}_{\overline{\alpha+1}} = \{n_0, \dots, n_r\}$. Let $\tau(n_j) \simeq \langle 1, \alpha, \langle e, x\rangle\rangle + 1$. Hence $\tau \restriction b_j = \mu_{\overline{\alpha}}(\tau \restriction (l_j+1), X^{\overline{\alpha}}_{\langle e,x\rangle})$.

Clearly there exists an $\overline{\alpha}$-regular extension τ_1 of τ such that $\tau_1 \subseteq f$. Therefore there exists an $\overline{\alpha}$-regular extension ρ of $\tau \restriction (l_j+1)$ in $X^{\overline{\alpha}}_{\langle e,x\rangle}$. Then $\tau \restriction b_j \in X^{\overline{\alpha}}_{\langle e,x\rangle}$. Clearly $\tau \restriction b_j$ is an $\overline{\alpha}$-regular finite part of f and hence $f \models_\alpha F_e(x)$. A contradiction.

Assume now that $\tau \subseteq f$ is $\overline{\alpha}$-regular, $\tau \Vdash_{\overline{\alpha}} \neg F_e(x)$ and $f \models_\alpha F_e(x)$. Then there exists an $\overline{\alpha}$-regular $\rho \subseteq f$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$. Using the monotonicity of $\Vdash_{\overline{\alpha}}$, we can assume that $\tau \subseteq \rho$ and get a contradiction. □

5.8. Lemma. *Let f be a regular enumeration. Then*

(1) *For all $\overline{\alpha} \preceq \overline{\zeta}$, $f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is $\overline{\alpha}$-regular & $\tau \Vdash_{\overline{\alpha}} F_e(x))$.*
(2) *For all $\overline{\alpha} \prec \overline{\zeta}$, $f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is $\overline{\alpha}$-regular & $\tau \Vdash_{\overline{\alpha}} \neg F_e(x))$.*

Proof. We shall use transfinite induction on α. The condition (1) is obviously true for $\alpha = 0$ and hence according to the Lemma above (2) is also true in this case.

Let $\alpha = \beta+1$. The truth of (1) for α follows easily from the induction hypothesis. The truth of (2) follows from the Lemma above.

Suppose that $\alpha \preceq \zeta$ and $\alpha = \lim \alpha(p)$ is limit ordinal. It is sufficient to show that (1) is true for α. Assume that $f \models_\alpha F_e(x)$. Then there exists a pair $\langle v, x\rangle \in W_e$ such that if $u \in D_v$, then $u = \langle p_u, e_u, x_u\rangle$ and $f \models_{\alpha(p_u)} F_{e_u}(x_u)$. By induction for every $u \in D_v$ there exists a $\overline{\alpha(p_u)}$-regular finite part $\tau_u \subseteq f$ such that $\tau_u \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Clearly there exists a $\overline{\alpha}$-regular finite part τ of f such that for all $u \in D_v$, $\tau_u \subseteq \tau$ and τ is $\overline{\alpha(p_u)}$-regular. Then $\tau \Vdash_{\overline{\alpha}} F_e(x)$.

To prove (1) in the reverse direction assume that $\tau \subseteq f$ and $\tau \Vdash_{\overline{\alpha}} F_e(x)$. Again there exists an element $\langle v, x\rangle$ of W_e such that for all $u \in D_v$, $u = \langle p_u, e_u, x_u\rangle$

and $\tau \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Without loss of generality, we may assume that τ is $\overline{\alpha(p_u)}$-regular for every $u \in D_v$. By induction $f \models_{\alpha(p_u)} F_{e_u}(x_u)$ for all $u \in D_v$. So $f \models_\alpha F_e(x)$. □

6. Regular extensions

Given a finite mapping τ defined on $[0, q-1]$, by $\tau * z$ we shall denote the extension ρ of τ defined on $[0, q]$ and such that $\rho(q) \simeq z$.

6.1. Proposition. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. Then*

(1) *For every $\overline{\alpha}$-regular finite part τ and every $y \in \mathbb{N}$ there exists a normal $\overline{\alpha}$-regular extension ρ of τ such that $\rho(\mathrm{lh}(\tau)) \simeq y$.*
(2) *For every $\overline{\delta} \prec \overline{\alpha}$, every $\overline{\delta}$-regular τ of rank 1 and every $y \in \mathbb{N}$ there exists a $\overline{\delta}, \alpha$-regular extension ρ of τ of rank 1 and such that $\rho(\mathrm{lh}(\tau)) \simeq y$.*

Proof. We shall prove simultaneously (1) and (2) by means of transfinite induction on α.

Notice that since $B^*_\alpha = \mathbb{N} \oplus B_\alpha$, we have that $0 \in B^*_\alpha$.

a) $\alpha = 0$. In this case (2) is trivial. To prove (1) suppose that τ is 0-regular and $y \in \mathbb{N}$. Define ρ as follows

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau), \\ y, & \text{if } x = \mathrm{lh}(\tau), \\ 0, & \text{if } x = \mathrm{lh}(\tau) + 1, \\ \text{undefined}, & \text{if } x > \mathrm{lh}(\tau) + 1. \end{cases}$$

b) Let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

We start with the proof of (1). Suppose that we are given an $\overline{\alpha}$-regular τ and $y \in \mathbb{N}$. Let $\mathrm{dom}(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Set $n_{r+1} = q$. By induction, there exists a $\overline{\beta}$-normal extension ρ_0 of $\tau * y$. Set $l_{r+1} = \mathrm{lh}(\rho_0)$. Clearly there exists a normal $\overline{\beta}$-regular extension μ of $\rho_0 * 0$ and hence for every $p \in \mathbb{N}$ the function $\mu^X_{\overline{\beta}}(\rho_0 * 0, p)$ is defined. Now, let

$$\rho_1 = \begin{cases} \mu^X_{\overline{\beta}}(\rho_0 * 0, \langle e, l_{r+1} \rangle) & \text{if } y = \langle 0, \beta, e \rangle + 1, \\ \mu^X_{\overline{\beta}}(\rho_0 * 0, p) & \text{if } y = \langle 1, \beta, p \rangle + 1, \\ \mu & \text{otherwise.} \end{cases}$$

Set $m_{r+1} = \mathrm{lh}(\rho_1)$. Let ρ_2 be a normal $\overline{\beta}$-regular extension of ρ_1 and ρ be a normal $\overline{\beta}$-regular extension of $\rho_2 * 0$. Clearly ρ is a normal $\overline{\alpha}$-regular extension of τ and $\rho(\mathrm{lh}(\tau)) \simeq y$.

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$.

Suppose that $\delta = \beta$. Then $\overline{\beta} = \overline{\delta}$. Notice that the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\beta}$.

Let $n_0 = \mathrm{lh}(\tau)$. Clearly we can find a normal $\overline{\beta}$-regular extension ρ_0 of $\tau * y$. After that we obtain the $\overline{\beta}$-regular extensions ρ_1, ρ_2 and ρ as above. Then ρ is a $\overline{\beta}, \alpha$-regular extension of τ of rank 1.

Suppose that $\delta < \beta$. Then the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$ and $\overline{\delta} \prec \overline{\beta}$. Using the induction hypothesis, we extend $\tau * y$ to a $\overline{\delta}, \beta$-regular finite part ρ_1 of rank 1. After that we extend ρ_1 to a $\overline{\delta}, \alpha$-regular finite part ρ of rank 1 in the same way as in the previous case.

c) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha_n < \alpha(p)]$. For every p by $\overline{\alpha(p)}$ we shall denote the $\alpha(p)$-predecessor of $\overline{\alpha}$.

To prove (1) suppose that τ is an α-regular finite part of rank $r+1$ and $y \in \mathbb{N}$. Then τ is an $\overline{\alpha(p_0+3r+2)}$-regular finite part of rank 1. By induction there exists an $\overline{\alpha(p_0+3r+2)}, \alpha(p_0+3r+3)$-regular extension ρ_0 of τ of rank 1 and such that $\rho_0(\mathrm{lh}(\tau)) \simeq y$. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. Applying again the induction hypothesis we obtain an $\overline{\alpha(p_0+3r+2)}, \alpha(p_0+3r+3), \alpha(p_0+3r+4)$-regular extension ρ_1 of $\rho_0 * 0$ which is of rank 1. Set $m_{r+1} = \mathrm{lh}(\rho_1)$ and extend ρ_1 to an $\overline{\alpha(p_0+3r+2)}, \alpha(p_0+3r+3), \alpha(p_0+3r+4), \alpha(p_0+3r+5)$-regular finite part of rank 1. Clearly ρ is an $\overline{\alpha}$-regular finite part of rank $r+2$.

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$, let τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$. Let $p_\delta = \mu p[\delta < \alpha(p)]$. By induction there exists a $\overline{\delta}, \alpha(p_\delta)$-regular extension ρ_1 of τ which is of rank 1 and such that $\rho_1(\mathrm{lh}(\tau)) \simeq y$. Set $b_0 = \mathrm{lh}(\rho_1)$. Applying twice the induction hypothesis, we get an $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1), \alpha(p_\delta+2)$-regular extension ρ of $\rho_1 * 0$ which is of rank 1. Clearly ρ is a $\overline{\delta}, \alpha$-regular extension of τ which is of rank 1. □

Let us fix a pair of total functions σ and ν on $\mathbb{N}$ such that for every $\alpha \le \zeta$, $\sigma(\alpha) \in B_\alpha$.

6.2. Definition. Let $\alpha \le \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α. A finite part τ is $\overline{\alpha}$ *complete* (with respect to σ, ν) if

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \Rightarrow \sigma(\beta) \in \tau(B^{\tau}_{\overline{\beta}}) \text{ and}$$
$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \Rightarrow \nu(\beta) \in \tau(N^{\tau}_{\overline{\beta}}).$$

6.3. Proposition. *Let $\alpha \le \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α.*

(1) *For every $\overline{\alpha}$-regular finite part τ there exist a normal $\overline{\alpha}$-regular extension ρ of τ which is $\overline{\alpha}$ complete.*
(2) *For every $\overline{\delta} \prec \overline{\alpha}$ and every $\overline{\delta}$-regular τ of rank 1 there exists a $\overline{\delta}, \alpha$-regular extension ρ of τ which is of rank 1 and $\overline{\delta}, \alpha$ complete.*

Proof. Transfinite induction on α.

a) $\alpha = 0$. Given a 0-regular τ, define ρ as follows

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau), \\ \nu(0), & \text{if } x = \mathrm{lh}(\tau), \\ \sigma(0), & \text{if } x = \mathrm{lh}(\tau)+1, \\ \text{undefined}, & \text{if } x > \mathrm{lh}(\tau)+1. \end{cases}$$

b) $\alpha = \beta + 1$. We start with the proof of (1). Suppose that we are given an $\overline{\alpha}$-regular τ. Let $\mathrm{dom}(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r+1$. Set $n_{r+1} = q$. By the previous proposition there exists a $\overline{\beta}$-normal extension ρ_0 of $\tau * \nu(\alpha)$. Set $l_{r+1} = \mathrm{lh}(\rho_0)$. Let μ be a normal $\overline{\beta}$-regular extension of ρ_0. Now, let

$$\rho_1 = \begin{cases} \mu^X_{\overline{\beta}}(\rho_0 * 0, \langle e, l_{r+1} \rangle) & \text{if } \nu(\alpha) = \langle 0, \beta, e \rangle + 1, \\ \mu^X_{\overline{\beta}}(\rho_0 * 0, p) & \text{if } \nu(\alpha) = \langle 1, \beta, p \rangle + 1, \\ \mu & \text{otherwise.} \end{cases}$$

Set $m_{r+1} = \mathrm{lh}(\rho_1)$. By induction there exists a normal $\overline{\beta}$-regular extension ρ_2 of ρ_1 which is $\overline{\beta}$ complete. Set $b_{r+1} = \mathrm{lh}(\rho_2)$ and let ρ be a normal $\overline{\beta}$-regular extension of $\rho_2 * \sigma(\alpha)$. Clearly ρ is a normal $\overline{\alpha}$-regular extension of τ and ρ is $\overline{\alpha}$ complete.

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1.

Suppose that $\delta = \beta$. Then $\overline{\beta} = \overline{\delta}$ and $\overline{\beta}$ is the β-predecessor of $\overline{\delta}, \alpha$.

Let $n_0 = \mathrm{lh}(\tau)$. Clearly we can find a normal $\overline{\beta}$-regular extension ρ_0 of $\tau * \nu(\alpha)$. After that we obtain consecutively the $\overline{\beta}$-regular extensions ρ_1, ρ_2 and ρ as above. Then ρ is a $\overline{\beta}, \alpha$-regular extension of τ of rank 1 which is $\overline{\alpha}$ complete.

Suppose that $\delta < \beta$. Then the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$ and $\overline{\delta} \prec \overline{\beta}$. Using the induction hypothesis, we extend τ to a $\overline{\delta}, \beta$-regular finite part τ_1 of rank 1. After that we extend τ_1 to a $\overline{\delta}, \alpha$-regular finite part ρ which is of rank 1 and $\overline{\delta}, \alpha$ complete in same way as in the previous case.

c)Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha_n < \alpha(p)]$. For every p by $\overline{\alpha(p)}$ we shall denote the $\alpha(p)$-predecessor of $\overline{\alpha}$.

To prove (1) suppose that τ is an α-regular finite part of rank $r+1$. Then τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. Set $n_{r+1} = \mathrm{lh}(\tau)$. By the previous proposition there exists an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)$-regular extension ρ_0 of $\tau * \nu(\alpha)$ of rank 1. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. There exists an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4)$-regular extension ρ_1 of $\rho_0 * \sigma(\alpha)$ which is of rank 1. Let $m_{r+1} = \mathrm{lh}(\rho_1)$. Finally, by induction, there exists a $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4), \alpha(p_0 + 3r + 5)$-regular extension ρ of ρ_1 which is of rank 1 and $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3), \alpha(p_0 + 3r + 4), \alpha(p_0 + 3r + 5)$ complete. So, we have constructed an $\overline{\alpha(p_0 + 3r + 5)}$ extension ρ of τ of rank 1 which is $\overline{\alpha(p_0 + 3r + 5)}$ complete and such that $\rho(n_{r+1}) \simeq \nu(\alpha)$ and $\rho(b_{r+1}) \simeq \sigma(\alpha)$. Clearly ρ is $\overline{\alpha}$-regular of rank $r+2$. To see that ρ is $\overline{\alpha}$ complete consider an element $\overline{\beta}$ of $Reg(\rho, \alpha)$. Then by Lemma 4.7

$$\overline{\beta} = \overline{\alpha} \vee \overline{\beta} \in Reg(\rho, \overline{\alpha(p_0 + 3r + 5)}).$$

In both cases it follows from the construction of ρ that it satisfies the completeness conditions with respect to $\overline{\beta}$.

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$, let τ be a $\overline{\delta}$-regular finite part of rank 1. Let $p_\delta = \mu p[\delta < \alpha(p)]$. There exists a $\overline{\delta}, \alpha(p_\delta)$-regular extension ρ_1 of τ which is of rank 1 and such that $\rho_1(\mathrm{lh}(\tau)) \simeq \nu(\alpha)$. Set $b_0 = \mathrm{lh}(\rho_1)$. After that we get an $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1)$-regular extension ρ_2 of $\rho_1 * \sigma(\alpha)$ which is of rank 1. Finally we extend ρ_2 to a $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta + 1), \alpha(p_\delta + 2)$-regular finite part of rank 1 which is complete. Clearly ρ is a $\overline{\delta}, \alpha$-regular extension of τ which is of rank 1 which is $\overline{\alpha}$ complete. □

6.4. Proposition. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α.*

(1) *Suppose that τ is an $\overline{\alpha}$-regular finite part, $\overline{\gamma} \in Reg(\tau, \overline{\alpha})$ and μ is a normal $\overline{\gamma}$-regular extension of τ. Then there exists a normal $\overline{\alpha}$-regular extension ρ of τ such that $\mu \subseteq \rho$.*
(2) *Let $\overline{\delta} \prec \overline{\alpha}$, let τ be $\overline{\delta}$-regular of rank 1 and $\overline{\gamma} \in Reg(\tau, \overline{\delta})$. Then every normal $\overline{\gamma}$-regular extension μ of τ can be extended to a $\overline{\delta}, \alpha$-regular extension ρ of τ such that $|\rho|_{\overline{\delta}, \alpha} = 1$.*

Proof. Transfinite induction on α. (1) and (2) are trivial for $\alpha = 0$.

Suppose that $\alpha = \beta + 1$. Let $\overline{\alpha}$ be an approximation of α and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

We start with the proof of (1). Consider an $\overline{\alpha}$-regular finite part τ, let $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\alpha})$ and let μ be a normal $\overline{\gamma}$-regular extension of τ. If $\gamma = \alpha$, then μ is an $\overline{\alpha}$-regular extension of τ. Set $\rho = \mu$. Otherwise, $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\beta})$. By induction μ can be extended to a normal $\overline{\beta}$-regular extension ρ_1 of τ which can be extended to a normal $\overline{\alpha}$-regular extension ρ of τ by means of Proposition 6.1.

The proof of (2) is similar. Let $\overline{\delta} \prec \overline{\alpha}$, let τ be a $\overline{\delta}$-regular finite part of rank 1, let $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\delta})$ and μ be a normal $\overline{\gamma}$-regular extension of τ. Now, we have to consider two cases.

a) $\delta = \beta$. In this case $\overline{\delta} = \overline{\beta}$ and $\overline{\beta}$ is the β predecessor of $\overline{\delta}, \alpha$. By induction we can extend μ to a normal $\overline{\beta}$-regular extension ρ_1 of τ and by Proposition 6.1 we can extend ρ_1 to a $\overline{\delta}, \alpha$-regular finite part of rank 1.

b) $\delta < \beta$. Then $\overline{\delta} \prec \overline{\beta}$ and the β predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$. By induction there exists a $\overline{\delta}, \beta$-regular extension ρ_1 of τ such that $\mu \subseteq \rho_1$. Then we can extend ρ_1 to an $\overline{\delta}, \alpha$-regular finite part of rank 1 by Proposition 6.1.

Suppose now that $\alpha = \lim \alpha(p)$ is a limit ordinal. Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$

To prove (1) let τ be an $\overline{\alpha}$-regular finite part of rank $r+1$, let $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\alpha})$ and let μ be a normal $\overline{\gamma}$-regular extension of τ.

Clearly τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. By Lemma 4.7,

$$\overline{\gamma} = \overline{\alpha} \vee \overline{\gamma} \in \mathit{Reg}(\tau, \overline{\alpha(p_0 + 3r + 2)}).$$

The case $\overline{\gamma} = \overline{\alpha}$ is trivial. Let $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\alpha(p_0 + 3r + 2)})$. By induction we can extend μ to an $\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)$-regular extension ρ_1 of τ which is of rank 1. After that we can extend ρ_1 to a normal $\overline{\alpha}$-regular extension ρ of τ using Proposition 6.1.

It remains to prove (2) in this case. Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be $\overline{\delta}$-regular finite part of rank 1. Let $\overline{\gamma} \in \mathit{Reg}(\tau, \overline{\delta})$ and let μ be a normal $\overline{\gamma}$-regular extension of τ. Set $p_\delta = \mu p[\alpha(p) > \delta]$.

By induction we can extend μ to a $\overline{\delta}, \alpha(p_\delta)$-regular ρ_1 of rank 1 and after that using Proposition 6.1 we can extend ρ_1 to an $\overline{\delta}, \alpha$-regular ρ of rank 1. $\square$

6.5. Definition. Let τ be an $\overline{\alpha}$-regular finite part. An extension ρ of τ is *canonical* if $(\forall x \in \mathrm{dom}(\rho))(x > \mathrm{lh}(\tau) \Rightarrow \rho(x) \simeq 0)$.

6.6. Definition. Given an $\overline{\alpha}$-regular finite part define the $\overline{\alpha}$-characteristic $D^\tau_{\overline{\alpha}}$ to be the set

$$\{\langle \overline{\beta}, |\tau|_{\overline{\beta}} \rangle : \overline{\beta} \in \mathit{Reg}(\tau, \overline{\alpha})\}.$$

Notice that by Lemma 4.7 the set $\mathit{Reg}(\tau, \overline{\alpha})$ is finite and hence the set $D^\tau_{\overline{\alpha}}$ is also finite.

6.7. Definition. Let $\alpha \le \zeta$. A natural number y is *α-nice* if $y = 0$ or $y - 1$ is not of the form $\langle 0, \beta, e \rangle$ or $\langle 1, \beta, p \rangle$ for any $\beta < \alpha$.

Clearly if y is α-nice, then it is β-nice for every $\beta < \alpha$.

6.8. Proposition. *There exist recursive functions ext, ext_l, chs and chs_l such that for every approximation $\overline{\alpha}$ of an ordinal $\alpha \le \zeta$ the following assertions hold:*

(1) *For every $\overline{\alpha}$-regular finite part τ and every α-nice y, $ext(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y)$ is a canonical $\overline{\alpha}$-regular extension of $\tau * y$ with $\overline{\alpha}$-characteristic $chs(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y)$ and $\overline{\alpha}$-rank equal to $|\tau_\alpha| + 1$.*

(2) *For every $\overline{\delta} \prec \overline{\alpha}$, every $\overline{\delta}$-regular finite part τ of rank 1 and every α-nice y, $ext_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y)$ is a $\overline{\delta}, \alpha$-regular and canonical extension of $\tau * y$ of rank 1 with characteristic $chs_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y)$*

Proof. We shall construct the functions ext, chs, ext_l and chs_l simultaneously by means of effective transfinite recursion on α.

Let $\alpha = 0$. Then $\overline{\alpha} = 0$. Let τ be a 0-regular finite part. Using D^{τ}_0 we can determine the 0-rank $r + 1$ of τ. Let

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau), \\ y, & \text{if } x = \mathrm{lh}(\tau), \\ 0, & \text{if } x = \mathrm{lh}(\tau) + 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Set $ext(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = \rho$ and $chs(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = \{\langle 0, r + 2\rangle\}$.

Let $\alpha = \beta + 1$. Given an approximation $\overline{\alpha}$ of α find first the β predecessor $\overline{\beta}$ of $\overline{\alpha}$.

Let τ be an $\overline{\alpha}$-regular finite part and y be α-nice. Using the $\overline{\alpha}$-characteristic $D^{\tau}_{\overline{\alpha}}$ of τ we can find the $\overline{\beta}$-characteristic $D^{\tau}_{\overline{\beta}}$ of τ and the $\overline{\alpha}$-rank $r + 1$ of τ. Now let

$$\rho_0 = ext(\overline{\beta}, \tau, D^{\tau}_{\overline{\beta}}, y), \rho_1 = ext(\overline{\beta}, \rho, D^{\rho_0}_{\overline{\beta}}, 0), \ldots, \rho_3 = ext(\overline{\beta}, \rho_2, D^{\rho_2}_{\overline{\beta}}, 0).$$

Set $ext(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = \rho_3$ and $chs(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = D^{\rho_3}_{\overline{\beta}} \cup \{\langle \overline{\alpha}, r + 2\rangle\}$.

Suppose now that $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1 and let y be α-nice.

a) $\overline{\delta} = \overline{\beta}$. Set $\rho_0 = ext(\overline{\beta}, \tau, D^{\tau}_{\overline{\beta}}, y)$, ...,$\rho_3 = ext(\overline{\beta}, \rho_2, D^{\rho_2}_{\overline{\beta}}, 0)$.

Let $ext_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y) = \rho_3$ and $chs_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y) = D^{\rho_3}_{\overline{\beta}} \cup \{\langle (\overline{\delta}, \alpha), 1\rangle\}$.

b) $\overline{\delta} \prec \overline{\beta}$. Set $\tau' = ext_l((\overline{\delta}, \beta), \tau, D^{\tau}_{\overline{\delta}}, y)$,

$$\rho_0 = ext((\overline{\delta}, \beta), \tau', D^{\tau'}_{\overline{\delta}, \beta}, 0), \ldots, \rho_3 = ext((\overline{\delta}, \beta), \rho_2, D^{\rho_2}_{\overline{\delta}, \beta}, 0).$$

Let $ext_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}) = \rho_3$ and $chs_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}) = D^{\rho_3}_{\overline{\delta}, \beta} \cup \{\langle (\overline{\delta}, \alpha), 1\rangle\}$.

Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Consider an approximation $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ of α. Let $p_0 = \mu p[\alpha(p) > \alpha_n]$. For every $p \geq p_0$ let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$.

Let τ be an $\overline{\alpha}$-regular finite part with characteristic $D^{\tau}_{\overline{\alpha}}$ and let y be α-nice. Let $|\tau|_{\overline{\alpha}} = r + 1$. Then τ is an $\overline{\alpha(p_0 + 3r + 2)}$-regular finite part of rank 1. By Lemma 4.7

$$D = D^{\tau}_{\overline{\alpha(p_0+3r+2)}} = \{\langle \overline{\beta}, k\rangle : \overline{\beta} \preceq \overline{\alpha(p_0 + 3r + 2)} \;\&\; \langle \overline{\beta}, k\rangle \in D^{\tau}_{\overline{\alpha}}\}.$$

Set

$$\begin{aligned} &\rho_0 = ext_l((\overline{\alpha(p_0 + 3r + 2)}, \alpha(p_0 + 3r + 3)), \tau, D, y), \ldots, \\ &\rho_2 = ext_l((\overline{\alpha(p_0 + 3r + 4), 0}, \alpha(p_0 + 3r + 5)), \rho_1, D^{\rho_1}_{\overline{\alpha(p_0+3r+4)}}, 0). \end{aligned}$$

Let $ext(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = \rho_2$ and $chs(\overline{\alpha}, \tau, D^{\tau}_{\overline{\alpha}}, y) = D^{\rho_2}_{\overline{\alpha(p_0+3r+5)}} \cup \{\langle \overline{\alpha}, r + 2\rangle\}$.

Now suppose that $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1. Set $l = \mu p[\alpha(p) > \delta]$. Let

$\rho_0 = ext_l((\overline{\delta}, \alpha(l)), \tau, D^{\tau}_{\overline{\delta}}, y)$, $\rho_1 = ext_l((\overline{\delta}, \alpha(l), \alpha(l+1)), \rho_0, D^{\rho_0}_{\overline{\delta},\alpha(l)}, 0)$ and

$\rho_2 = ext_l((\overline{\delta}, \alpha(l), \alpha(l+1), \alpha(l+2)), \rho_1, D^{\rho_1}_{\overline{\delta},\alpha(l),\alpha(l+1)} 0)$.

Let $ext_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y) = \rho_2$ and

$$chs_l((\overline{\delta}, \alpha), \tau, D^{\tau}_{\overline{\delta}}, y) = D^{\rho_2}_{\overline{\delta},\alpha(l),\alpha(l+1),\alpha(l+2)} \cup \{\langle(\overline{\delta}, \alpha), 1\rangle\}.$$

□

6.9. Corollary. *There exists an effective way to construct for every ordinal approximation $\overline{\alpha}$ and every $r \geq 0$ an $\overline{\alpha}$-regular finite part $\tau_{\overline{\alpha}}$ of rank $r+1$ and such that $(\forall x \in \mathrm{dom}(\tau_{\overline{\alpha}}))(\tau_{\overline{\alpha}}(x) \simeq 0)$.*

7. The proofs

In this section we shall present the proofs of Theorem 2.9 and Theorem 2.14.

We start with the proof of Theorem 2.9. Let two sequences $\mathcal{A} = \{A_\alpha\}_{\alpha\leq\zeta}$ and $\mathcal{B} = \{B_\alpha\}_{\alpha\leq\zeta}$ of sets of natural numbers be given. Suppose that $\mathcal{A} \not\leq_u \mathcal{B}$, i.e. there does not exist a recursive function g such that

$$(\forall\alpha)(A_\alpha = \Phi_{g(\alpha)}(\mathcal{P}_\alpha(\mathcal{B}))).$$

To prove the Theorem it is sufficient to construct a total set X such that

(1) for some recursive h, $(\forall\alpha \leq \zeta)(B_\alpha = \Phi_{h(\alpha)}(X^{(\alpha)}))$ and
(2) there does not exist a recursive function g such that $(\forall\alpha)(A_\alpha = \Phi_{g(\alpha)}(X^{(\alpha)}))$.

If for some $\alpha \leq \zeta$, $A_\alpha \not\leq_e \mathcal{P}_\alpha(\mathcal{B})$, then we can get the desired X by means of Theorem 2.5.

Assume that $(\forall\alpha)(A_\alpha \leq_e \mathcal{P}_\alpha(\mathcal{B}))$.

We shall construct X as a graph of a regular enumeration f. This way, by Theorem 5.3, we will automatically ensure that X satisfies (1).

Let $\overline{\zeta}$ be an ordinal approximation of ζ. For every $\alpha \leq \zeta$ set $B^*_\alpha = \mathbb{N} \oplus B_\alpha$ and let $\mathcal{P}_\alpha$ denote the set $\mathcal{P}_\alpha(\mathcal{B})$. Fix two total functions $\sigma(\alpha, s)$ and $\nu(\alpha, s)$ such that

(i) $(\forall\alpha \leq \zeta)(\lambda s.\sigma(\alpha, s)$ enumerates the set $B^*_\alpha)$.
(ii) $(\forall\alpha \leq \zeta)(\lambda s.\nu(\alpha, s)$ enumerates the set $\mathbb{N})$.

Set for every $s \in \mathbb{N}$, $\sigma_s = \lambda\alpha.\sigma(\alpha, (s)_0)$ and $\nu_s = \lambda\alpha.\nu(\alpha, (s)_0)$.

The construction f will be carried out by steps. At each step s we shall construct a $\overline{\zeta}$-regular finite part τ_s so that $\tau_s \subseteq \tau_{s+1}$ and $|\tau_s|_{\overline{\zeta}} < |\tau_{s+1}|_{\overline{\zeta}}$. We shall ensure that for every s the finite part τ_{s+1} is complete with respect to σ_s, τ_s. After that we shall define $f = \bigcup \tau_s$. The obtained this way f is a regular enumeration. Indeed, since f contains finite parts of arbitrary large rank, for every $\rho \subseteq f$ there exists a $\overline{\zeta}$-regular finite part τ_s of f such that $\rho \subseteq \tau_s$.

Let $\alpha \leq \zeta$ and $z \in B^*_{\overline{\alpha}}$. Let s be so large that $\overline{\alpha} \in Reg(\tau_{s+1}, \overline{\zeta},)$ and such that $\sigma(\alpha, (s)_0) \simeq z$. Then $\sigma_s(\alpha) \simeq z$ and hence since τ_{s+1} is complete with respect to σ_s, ν_s, $z \in \tau_{s+1}(B^{\tau_{s+1}}_{\overline{\alpha}})$.

Let $p \in \mathbb{N}$. Consider an $s_1 > s$ such that $\nu(\alpha, (s_1)_0) \simeq p$. Then $\nu_{s_1}(\alpha) \simeq p$ and hence by the completeness of τ_{s_1+1} with respect to σ_{s_1}, ν_{s_1} we have that $p \in \tau_{s_1+1}(N^{\tau_{s_1+1}}_{\overline{\alpha}})$.

Fix an enumeration $g_0, g_1, \ldots, g_s, \ldots$ of all total recursive functions.

Let τ_0 be a $\overline{\zeta}$-regular finite part of rank 1. Suppose that τ_s is defined and let $D = D_{\zeta}^{\tau_s}$ be the characteristic of τ_s.

Fix an $\overline{\alpha} \prec \overline{\zeta}$ and let $\overline{\alpha+1}$ be the $\alpha+1$-predecessor of $\overline{\zeta}$. Construct an $\overline{\alpha}$-regular extension δ_α of τ_s as follows. First let $k = k(\overline{\zeta}, \overline{\alpha+1})$ be the number such that every $\overline{\zeta}$-regular finite part of rank greater than k is also $\overline{\alpha+1}$-regular. Using the functions *ext* and *chs* from Proposition 6.8 find a canonical $\overline{\zeta}$-regular extension δ_0 of τ_s which is of $\overline{\zeta}$-rank equal to $\max(|\tau_s|_\zeta, k)$. Clearly δ_0 is $\overline{\alpha+1}$-regular and hence it also $\overline{\alpha}$-regular. Let $g_s(\alpha) = e_\alpha$. Using again Proposition 6.8, define recursively a canonical $\overline{\alpha}$-regular extension δ_1 of $\delta_0 * (\langle 0, \alpha, e_\alpha\rangle + 1)$ such that $|\delta_1|_{\overline{\alpha}} = |\delta_0|_{\overline{\alpha}} + 1$. Set $\delta_\alpha = \delta_1$. Let $l_\alpha = \mathrm{lh}(\delta_\alpha)$ and set

$$C_\alpha = \{x : (\exists \rho \supseteq \delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular, } \rho(l_\alpha) \simeq x \text{ and } \rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(l_\alpha))\}.$$

Set $C_\zeta = \emptyset$.

From the construction of δ_α and from Proposition 4.11 it follows that there exist a recursive function h such that $(\forall \alpha \le \zeta)(C_\alpha = \Phi_{h(\alpha)}(\mathcal{P}_\alpha))$. Hence there exist an α such that $A_\alpha \neq C_\alpha$. Actually there exist infinitely many such α, otherwise since $(\forall \alpha \le \zeta)(A_\alpha \le_e \mathcal{P}_\alpha)$, we could redefine the function h and obtain that $\mathcal{A} \le_u \mathcal{B}$. So we may assume that there exists an $\alpha < \zeta$ such that $A_\alpha \neq C_\alpha$. Let α_s be the least $\alpha \le \zeta$ such that $A_\alpha \neq C_\alpha$.

Now let x_s be the least natural number such that

$$(x_s \in A_{\alpha_s} \ \&\ x_s \notin C_{\alpha_s}) \vee (x_s \notin A_{\alpha_s} \ \&\ x_s \in C_{\alpha_s}).$$

Set $\delta_2 = \mu_{\overline{\alpha_s}}(\delta_{\alpha_s} * x_s, X^{\overline{\alpha_s}}_{\langle e_{\alpha_s}, l_{\alpha_s}\rangle})$. After that let δ_3 be a normal $\overline{\alpha_s}$-regular extension of δ_2 and δ_4 be a normal $\overline{\alpha_s}$-regular extension of $\delta_3 * 0$. Clearly δ_4 is a normal $\overline{\alpha_s+1}$-regular extension of δ_0. By Proposition 6.4 there exist a normal $\overline{\zeta}$-regular extension τ of δ_0 such that $\delta_4 \subseteq \tau$. Let τ_{s+1} be a normal $\overline{\zeta}$-regular extension of τ which is complete with respect to σ_s, ν_s.

Towards a contradiction assume that there exists a recursive function g such that $(\forall \alpha \le \zeta)(A_\alpha = \Phi_{g(\alpha)}(f^{(\alpha)}))$. Then there exists a recursive function h such that $(\forall \alpha \le \zeta)(f^{-1}(A_\alpha) = \Phi_{h(\alpha)}(f^{(\alpha)}))$ and hence there exist a recursive function g_s such that

$$(\forall \alpha \le \zeta)(f^{-1}(A_\alpha) = \{n : f \models_\alpha F_{g_s(\alpha)}(n)\}).$$

Consider the step s of the construction. Clearly $f(l_{\alpha_s}) \simeq x_s$ and $g_s(\alpha_s) = e_{\alpha_s}$. Hence

$$x_s \in A_{\alpha_s} \iff f \models F_{e_{\alpha_s}}(l_{\alpha_s}).$$

Now, assume that $x_s \in A_{\alpha_s}$. Then $f \models F_{e_{\alpha_s}}(l_{\alpha_s})$. Clearly $\delta_{\alpha_s} \subseteq f$. Hence, by Lemma 5.8 there exists a $\overline{\alpha_s}$-regular $\rho \supseteq \delta_{\alpha_s}$ such that

$$\rho(l_{\alpha_s}) \simeq x_s \ \&\ \rho \Vdash_{\overline{\alpha_s}} F_{e_{\alpha_s}}(l_{\alpha_s}).$$

Therefore $x_s \in C_{\alpha_s}$. A contradiction.

So, $x_s \notin A_{\alpha_s}$. Then $x_s \in C_{\alpha_s}$ and hence there exists an $\overline{\alpha}$-regular extension ρ of δ_{α_s} such that $\rho(l_{\alpha_s}) \simeq x_s$ and $\rho \Vdash_{\overline{\alpha_s}} F_{e_{\alpha_s}}(l_{\alpha_s})$. From here, by the construction, it follows that $\delta_2 \Vdash_{\overline{\alpha}_s} F_{e_{\alpha_s}}(l_{\alpha_s})$ and hence $f \models F_{e_{\alpha_s}}(l_{\alpha_s})$. A contradiction.

So we have proved that there does not exist a recursive function g such that $(\forall \alpha \le \zeta)(A_\alpha = \Phi_{g(\alpha)}(f^{(\alpha)}))$ which concludes the proof Theorem 2.9.

Now we proceed with the proof of Theorem 2.14. Let ζ be a recursive ordinal and let $\Gamma : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$ be a uniform operator. Suppose that g is a function on the natural numbers such that given a recursive function φ with index a, $g(a)$ is an index of a recursive function ψ such that for all $X \subseteq \mathbb{N}$,

$$\Gamma(\{W^{X^{(\alpha)}}_{\varphi(\alpha)}\}_{\alpha\le\zeta}) = \{W^{X^{(\alpha)}}_{\psi(\alpha)}\}_{\alpha\le\zeta}.$$

Given $\mathcal{B} \in \mathcal{S}_\zeta$ and $\alpha \le \zeta$, by $\Gamma(\mathcal{B})_\alpha$ we shall denote the α-th member of the sequence $\Gamma(\mathcal{B})$.

Our goal is to show that there exists a recursive function h such that for every $\mathcal{B} \in \mathcal{S}_\zeta$ and every $\alpha \le \zeta$,

$$\Gamma(\mathcal{B})_\alpha = \Phi_{h(\alpha)}(\mathcal{P}_\alpha(\mathcal{B})).$$

Fix an ordinal approximation $\overline{\zeta+1}$ of $\zeta+1$. For every $\alpha \le \zeta$ let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta+1}$.

From now on, given a sequence $\mathcal{B} \in \mathcal{S}_\zeta$, we shall call f *regular enumeration with respect to* $\mathcal{B}$ if it is regular with respect to $\overline{\zeta+1}$ and the sequence $\mathcal{B}$, extended to a sequence of length $\zeta+1$ by setting $B_{\zeta+1} = \emptyset$.

As remarked after the proof of Proposition 5.3 there exists a recursive function φ such that for every sequence $\mathcal{B} = \{B_\alpha\}_{\alpha\le\zeta}$ and every regular with respect to $\mathcal{B}$ enumeration f,

$$(\forall \alpha \le \zeta)(B_\alpha = W^{f^{(\alpha)}}_{\varphi(\alpha)}).$$

Let a be an index of φ and let $g(a)$ be the index of the recursive function ψ. Then for every sequence $\mathcal{B} \in \mathcal{S}_\zeta$, every $\alpha \le \zeta$ and every regular with respect to $\mathcal{B}$ enumeration f we have that

$$\Gamma(\mathcal{B})_\alpha = W^{f^{(\alpha)}}_{\psi(\alpha)}.$$

Then there exists a recursive function χ such that for every regular sequence $\mathcal{B}$ and every regular with respect to $\mathcal{B}$ enumeration f,

$$(\forall \alpha \le \zeta)(f^{-1}(\Gamma(\mathcal{B})_\alpha) = \{n : f \models_\alpha F_{\chi(\alpha)}(n)\}).$$

Given an $\overline{\alpha} \preceq \overline{\zeta}$ define the $\overline{\alpha}$-regular finite part δ_α as follows. First set $k = k(\overline{\zeta+1}, \overline{\alpha+1})$. Let δ_0 be a $\overline{\zeta+1}$-regular finite part of rank k such that $(\forall x \in \mathrm{dom}(\delta_0))(\delta_0(x) \simeq 0)$. Clearly δ_0 is $\overline{\alpha+1}$-regular and hence it is also $\overline{\alpha}$-regular. Let $\chi(\alpha) = e_\alpha$. Notice that $\langle 0, \alpha, e_\alpha\rangle + 1$ is α-nice. Let δ_1 be canonical $\overline{\alpha}$-regular extension of $\delta_0 * (\langle 0, \alpha, e_\alpha\rangle + 1)$ of $\overline{\alpha}$-rank equal to $|\delta_0|_\alpha + 1$. Set $\delta_\alpha = \delta_1$. By Proposition 6.8. There exists a recursive function yielding for every $\alpha \le \zeta$ the (code of) δ_α.

Let $l_\alpha = \mathrm{lh}(\delta_\alpha)$. We shall show that for every sequence $\mathcal{B} \in \mathcal{S}_\zeta$,

$$\begin{aligned} \Gamma(\mathcal{B})_\alpha = \{x :& (\exists \rho \supseteq \delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular with respect to } \mathcal{B}, \\ & \rho(l_\alpha) \simeq x \text{ and } \rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(l_\alpha))\}. \end{aligned} \tag{1}$$

Indeed, fix a sequence $\mathcal{B} \in \mathcal{S}_\zeta$ and let $\alpha \le \zeta$. Denote by C_α the set

$$\{x : (\exists \rho \supseteq \delta_\alpha)(\rho \text{ is } \overline{\alpha}\text{-regular with respect to } \mathcal{B},\ \rho(l_\alpha) \simeq x \text{ and } \rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(l_\alpha))\}.$$

Consider a $x \in \Gamma(\mathcal{B})_\alpha$. Clearly there exists an $\overline{\alpha+1}$-regular extension δ of $\delta_\alpha * x$ which can be extended to a $\overline{\zeta+1}$-regular finite part τ by Proposition 6.4. After that we can extend τ to a regular with respect to $\mathcal{B}$ enumeration f as in the previous proof. Since $f(l_\alpha) \simeq x$, we have that $f \models_\alpha F_{\chi(\alpha)}(l_\alpha)$, i. e. $f \models_\alpha F_{e_\alpha}(l_\alpha)$. By Lemma 5.8, there exists an $\overline{\alpha}$-regular $\rho \subseteq f$ such that $\rho \Vdash_{\overline{\alpha}} F_{e_\alpha}(l_\alpha)$. We may assume that $\delta_\alpha \subseteq \rho$ and $\rho(l_\alpha) \simeq x$. Thus $x \in C_\alpha$.

Assume that $x \in C_\alpha$. Let $\delta_2 = \mu_{\overline{\alpha}}(\delta_\alpha * x, X^{\overline{\alpha}}_{\langle e_\alpha, l_\alpha\rangle})$. Clearly $\delta_2 \Vdash_{\overline{\alpha}} F_{e_\alpha}(l_\alpha)$. Let δ_3 be a normal $\overline{\alpha}$-regular extension of δ_2 and δ_4 be a normal $\overline{\alpha}$-regular extension of $\delta_2 * 0$. Clearly δ_4 is an $\overline{\alpha+1}$-regular extension of δ_0. Then there exists a $\overline{\zeta+1}$-regular extension τ of δ_0 such that $\delta_4 \subseteq \tau$. Finally, let f be a regular with respect to $\mathcal{B}$ enumeration such that $\tau \subseteq f$. By Lemma 5.8, $f \models_\alpha F_{e_\alpha}(l_\alpha)$ and hence $x = f(l_\alpha) \in \Gamma(B)_\alpha$.

From (1) by Proposition 4.11 it follows that there exists a recursive function h such that for all $\mathcal{B} \in \mathcal{S}_\zeta$,

$$\Gamma(\mathcal{B})_\alpha = \Phi_{h(\alpha)}(P_\alpha(\mathcal{B})).$$

It is a matter of routine computation to show that every operator $\Gamma : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$, for which there exists a recursive function h such that for every sequence $\mathcal{B} \in \mathcal{S}_\zeta$, $\Gamma(\mathcal{B}) = \{\Phi_{h(\alpha)}(\mathcal{P}_\alpha(\mathcal{B}))\}_{\alpha\leq\zeta}$, is uniform.

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