A note on ω-jump inversion of degree spectra of structures

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Abstract. In this note we provide a negative solution to the ω-jump inversion problem for degree spectra of structures.

1 Introduction

Let $\mathfrak{A}$ be a countable structure. The spectrum of $\mathfrak{A}$ is the set of Turing degrees

$$Sp(\mathfrak{A}) = \{\mathbf{a}|\ \mathbf{a} \text{ computes an isomorphic copy of } \mathfrak{A}\}$$

For $\alpha < \omega_1^{CK}$ the α-th jump spectrum of $\mathfrak{A}$ is the set $Sp_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)}|\mathbf{a} \in Sp(\mathfrak{A})\}$.

The jump inversion problem for degree spectra of structures can be stated as follows. Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $Sp(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$. Does there exist a structure $\mathfrak{M}$ such that $Sp_\alpha(\mathfrak{M}) = Sp(\mathfrak{A})$?

A positive solution to the problem for successor ordinals can be found in [1]. Though the problem is not explicitly stated there, the solution is a byproduct of the construction for all successor ordinals $\alpha < \omega_1^{CK}$ of α-categorical structures which are not relatively α-categorical.

Another solution for finite ordinals based on Marker's extensions is given in [2].

In this note we shall show that in the general case the ω-jump inversion problem for degree spectra has a negative solution. The proof can be easily adapted for all recursive limit ordinals.

In what follows we shall define a structure $\mathfrak{A}$ such that $Sp(\mathfrak{A}) \subseteq \{\mathbf{a}|\mathbf{0}^{(\omega)} \leq \mathbf{a}\}$ and for all countable structures $\mathfrak{M}$, $Sp_\omega(\mathfrak{M}) \neq Sp(\mathfrak{A})$. The definition of $\mathfrak{A}$ is based on a special property of the ω co-spectra which can be expressed in terms of enumeration reducibility. Namely we shall show that if $\mathfrak{M}$ is a countable structure then every enumeration degree in the ω co-spectrum of $\mathfrak{M}$ is bounded by a total enumeration degree which also belongs to the ω co-spectrum of $\mathfrak{M}$.

The basic facts about enumeration reducibility and co-spectra of structures needed for the presentation are summarized below.

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2 Preliminaries

2.1 Enumeration reducibility

Definition 1. Given two sets of natural numbers X and Y, say that X is enumeration reducible to Y ($X \leq_e Y$) if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle x, v\rangle \in W_e \wedge D_v \subseteq Y)).$$

Let $X \equiv_e Y$ if $X \leq_e Y$ and $Y \leq_e X$. The enumeration degree $d_e(X)$ of the set X consists of all sets Y which are enumeration equivalent to X. By $\mathcal{D}_e$ we shall denote the set of all enumeration degrees.

Given a set $X \subseteq \mathbb{N}$, by X^+ we shall denote $X \oplus (\mathbb{N} \setminus X)$. We have for all $X, Y \subseteq \mathbb{N}$ that $X \leq_{c.e.} Y \iff X \leq_e Y^+$ and $X \leq_T Y \iff X^+ \leq_e Y^+$.

Definition 2. A set X of natural numbers is called *total* if $X \equiv_e X^+$.

The enumeration jump is defined by Cooper in [3]:

Definition 3. Let $X \subseteq \mathbb{N}$. Set $J_e(X) = \{\langle e, x\rangle | x \in W_e(X)\}$. The *enumeration jump X' of X* is the set $J_e(X)^+$.

Clearly for all $X \subseteq \mathbb{N}$, X' is a total set. The enumeration jump of X is somewhat weaker than the Turing jump $J_T(X)$ of X. Namely $(\forall X \subseteq \mathbb{N})(J_T(X)^+ \equiv_e (X^+)')$ hence $X' \leq_T (X^+)' \leq_T J_T(X)$. Of course for total sets X, $X' \equiv_T J_T(X)$.

We shall use the following property of the enumeration jump:

Proposition 4. *There exists a computable function j such that for all $e \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, $W_e(X)' = W_{j(e)}(X')$.*

Proof. Consider a computable function λ such that for every a and e and for all X, $W_a(W_e(X)) = W_{\lambda(a,e)}(X)$. Then

$$2\langle a, x\rangle \in W_e(X)' \iff 2\langle \lambda(a, e), x\rangle \in X' \text{ and}$$
$$2\langle a, x\rangle + 1 \in W_e(X)' \iff 2\langle \lambda(a, e), x\rangle + 1 \in X'.$$

Let j be the computable function yielding for every e an index of the c.e. set

$$\{\langle 2\langle a, x\rangle, \{2\langle \lambda(a, e), x\rangle\}\rangle : a, x \in \mathbb{N}\} \cup \{\langle 2\langle a, x\rangle+1, \{2\langle \lambda(a, e), x\rangle+1\}\rangle : a, x \in \mathbb{N}\}.$$

Then for all e, $W_e(X)' = W_{j(e)}(X')$.

2.2 Enumeration reducibility of sequences of sets

Definition 5. Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{X}$ *is enumeration reducible to* $\mathcal{Y}$ ($\mathcal{X} \leq_e \mathcal{Y}$) if for all n, $X_n \leq_e Y_n$ uniformly in n. In other words, if there exists a computable function μ such that for all n, $X_n = W_{\mu(n)}(Y_n)$.

Definition 6. Let $\mathcal{X} = \{X_n\}$ be a sequence of sets of natural numbers. The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;
(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

By $\mathcal{P}_\omega(\mathcal{X})$ we shall denote the set $\bigoplus_n \mathcal{P}_n(\mathcal{X})$. Clearly $\mathcal{X} \leq_e \mathcal{P}(\mathcal{X})$ and hence $\bigoplus_n X_n \leq_e \mathcal{P}_\omega(\mathcal{X})$.

Proposition 7. *For all sequences $\mathcal{X}$ of sets of natural numbers the set $\mathcal{P}_\omega(\mathcal{X})$ is total.*

Proof. Fix z_0 so that for all sets X, $W_{z_0}(X) = X$. Then

$$\langle n, x\rangle \notin \mathcal{P}_\omega(\mathcal{X}) \iff x \notin \mathcal{P}_n(\mathcal{X}) \iff x \notin W_{z_0}(\mathcal{P}_n(\mathcal{X})) \iff$$
$$2\,\langle z_0, x\rangle + 1 \in \mathcal{P}_n'(\mathcal{X}) \iff 2\,(2\,\langle z_0, x\rangle + 1) \in \mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n' \oplus X_{n+1} \iff$$
$$\langle n+1, 2\,(2\,\langle z_0, x\rangle + 1)\rangle \in \mathcal{P}_\omega(\mathcal{X}).$$

So, $\mathbb{N} \setminus \mathcal{P}_\omega(\mathcal{X}) \leq_e \mathcal{P}_\omega(\mathcal{X})$.

Proposition 8. *Let $\mathcal{X} = \{X_n\}$ be a sequence of sets of natural numbers, $M \subseteq \mathbb{N}$ and $\mathcal{X} \leq_e \{M^{(n)}\}_{n<\omega}$. Then $\mathcal{P}(\mathcal{X}) \leq_e \{M^{(n)}\}_{n<\omega}$.*

Proof. Let $\lambda(a, b)$ be a computable function such that for all $Y \subseteq \mathbb{N}$, $W_a(Y) \oplus W_b(Y) = W_{\lambda(a,b)}(Y)$ and j be the recursive function defined in proposition 4. Suppose that for all n, $X_n = W_{\mu(n)}(M^{(n)})$.

Now $P_0(\mathcal{X}) = X_0 = W_{\mu(0)}(M^{(0)})$. Suppose that $\mathcal{P}_n(\mathcal{X}) = W_a(M^{(n)})$. Then $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1} = W_{j(a)}(M^{(n+1)}) \oplus W_{\mu(n+1)}(M^{(n+1)}) =$ $W_{\lambda(j(a),\mu(n+1))}(M^{(n+1)})$.

2.3 Co-spectra of structures

We shall identify the Turing degrees and the total enumeration degrees, i.e. the enumeration degrees containing a total set. This assumption is safe because of the standard embedding ι of the Turing degrees into the enumeration degrees defined by $\iota(d_T(A)) = d_e(A^+)$ which is known to preserve also the jump operation.

Definition 9. Let $\mathfrak{M}$ be a countable structure and $\alpha < \omega_1^{CK}$. The α-th co-spectrum of $\mathfrak{M}$ is the set

$$CoSp_\alpha(\mathfrak{M}) = \{\mathbf{a} | \mathbf{a} \in \mathcal{D}_e \wedge (\forall \mathbf{b} \in Sp_\alpha(\mathfrak{M}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

Definition 10. Let $\alpha < \omega_1^{CK}$. A subset R of $\mathbb{N}$ is Σ_α^c definable in $\mathfrak{M}$ if there exist a computable function γ taking as values codes of computable Σ_α infinitary formulas $F_{\gamma(x)}$ and finitely many parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m).$$

For the definition of the computable Σ_α formulae the reader may consult [4].

Theorem 11. *([5]). Let $\alpha < \omega_1^{CK}$. Then*

1. *If $\alpha < \omega$ then $\mathbf{a} \in CoSp_\alpha(\mathfrak{M})$ if and only if all elements of $\mathbf{a}$ are $\Sigma^c_{\alpha+1}$ definable in $\mathfrak{M}$.*
2. *If $\omega \leq \alpha$ then $\mathbf{a} \in CoSp_\alpha(\mathfrak{M})$ if and only if all elements of $\mathbf{a}$ are Σ^c_α definable in $\mathfrak{M}$.*

3 A property of the ω co-spectra

Theorem 12. *Let $\mathfrak{M}$ be a countable structure and $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$. Then there exists a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq_e \mathbf{b}$ and $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$,*

Proof. Fix an element R of $\mathbf{a}$. The set R is Σ^c_ω definable in $\mathfrak{M}$ and hence there exists a computable function γ and parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m).$$

Since each $F_{\gamma(x)}$ is a computable Σ_ω formula, i.e. a c.e. disjunction of computable Σ_{n+1} formulae, $n < \omega$, we may assume that there exists a computable function $\delta(n, x)$ such that for all n and x, $\delta(n, x)$ yields a code of some computable Σ_{n+1} formula $F_{\delta(n,x)}$ and

$$x \in R \iff (\exists n)(\mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_m)).$$

Let $R_n = \{x | x \in \mathbb{N} \wedge \mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_n)\}$. It is easy to see that if B is the diagram of some isomorphic copy $\mathfrak{B}$ of $\mathfrak{M}$ on the natural numbers then $\{R_n\} \leq_e \{B^{(n)}\}$. Indeed, let κ be an isomorphism from $\mathfrak{M}$ to $\mathfrak{B}$ and $x_1 = \kappa(t_1), \ldots, x_m = \kappa(t_m)$. Then

$$x \in R_n \iff \mathfrak{B} \models F_{\delta(n,x)}(x_1, \ldots, x_m).$$

Clearly the set of all computable Σ_{n+1} formulae $F_{\delta(n,x)}$ with fixed parameters $x_1, \ldots, x_m$ which are satisfied in $\mathfrak{B}$ is uniformly in n enumeration reducible to $B^{(n)}$.

By proposition 8 we have also that $\mathcal{P}(\{R_n\}) \leq_e \{B^{(n)}\}$. Hence $\mathcal{P}_\omega(\{R_n\}) \leq_e B^{(\omega)}$.

Set $\mathbf{b} = d_e(\mathcal{P}_\omega(\{R_n\}))$. As shown above $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$. Notice that $\mathbf{b}$ is a total degree. It remains to see that $\mathbf{a} \leq_e \mathbf{b}$. Indeed, since $x \in R \iff (\exists n)(x \in R_n)$, $R \leq_e \bigoplus_n R_n$. On the other hand $\bigoplus_n R_n \leq_e \mathcal{P}_\omega(\{R_n\})$. Therefore $R \leq_e \mathcal{P}_\omega(\{R_n\})$

Now we are ready to define the structure $\mathfrak{A}$ promised in the introduction. Let Y be a set which is quasi-minimal above $\emptyset^{(\omega)}$. This means that $\emptyset^{(\omega)} <_e Y$ and if X is a total set and $X \leq_e Y$ then $X \leq_e \emptyset^{(\omega)}$. The existence of such sets is well known in the theory of the enumeration degrees. For example, one can take $Y = \emptyset^{(\omega)} \oplus G$, where G is one-generic relatively $\emptyset^{(\omega)}$, see [6]. Notice that the enumeration degree of Y does not contain any total set.

Suppose that $\mathfrak{A}$ is a countable structure with $CoSp(\mathfrak{A}) = \{\mathbf{a}|\mathbf{a} \leq_e d_e(Y)\}$. Clearly $Sp(\mathfrak{A}) \subseteq \{\mathbf{b}|\mathbf{0}^{(\omega)} \leq_T \mathbf{b}\}$. Assume that there exists a countable structure $\mathfrak{M}$ such that $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$. Then $CoSp_\omega(\mathfrak{M}) = CoSp(\mathfrak{A})$ and hence there exists a total degree $\mathbf{b}$ in $CoSp(\mathfrak{A})$ such that $d_e(Y) \leq \mathbf{b}$. Since $d_e(Y)$ is the greatest element of $CoSp(\mathfrak{A})$, $\mathbf{b} = d_e(Y)$. A contradiction.

It remains to see that there exists a countable structure $\mathfrak{A}$ with co-spectrum equal to the set $\{\mathbf{a}|\mathbf{a} \leq_e d_e(Y)\}$. This follows from the fact that every principal ideal of enumeration degrees can be represented as co-spectrum of some subgroup of the additive group Q of the rational numbers, see [7]. To make the presentation self-contained we shall present this short argument here.

Let us fix a non-trivial group $G \subseteq Q$. Let $a \neq 0$ be an element of G. For every prime number p set

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k|a \text{ in } G, \\ \infty & \text{if } p^k|a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \ldots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

It can be easily seen that if a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$. Let $\mathbf{d}_G = d_e(S_a(G))$, where a is some non-zero element of G.

The following proposition follows from results of Coles, Downey and Slaman [8]:

Proposition 13. $Sp(G) = \{\mathbf{b}|\mathbf{b}$ *is total* $\wedge\, \mathbf{d}_G \leq_e \mathbf{b}\}$.

Corollary 14. $CoSp(G) = \{\mathbf{a}|\mathbf{a} \leq_e \mathbf{d}_G\}$.

Proof. Clearly $\mathbf{a} \in CoSp(G)$ if and only if for all total $\mathbf{b}$, $\mathbf{d}_G \leq_e \mathbf{b} \Rightarrow \mathbf{a} \leq_e \mathbf{b}$. According Selman's Theorem [9] the last is equivalent to $\mathbf{a} \leq_e \mathbf{d}_G$.

Now, let $Y \subseteq \mathbb{N}$. Consider the set $S = \{\langle i, j\rangle : (j = 0) \vee (j = 1 \;\&\; i \in Y)\}$.

Clearly $S \equiv_e Y$. Let G be the least subgroup of Q containing the set $\{1/p_i^j : \langle i, j\rangle \in S\}$. Then $1 \in G$ and $S_1(G) = S$. So, $\mathbf{d}_G = d_e(Y)$.

References

1. Goncharov, S., Harizanov, V., Knight, J., McCoy, C., Miller, R., Solomon, R.: Enumerations in computable structure theory. Annals of Pure and Applied Logic **136** (2005) 219–246
2. Soskova, A., Soskov, I.: A jump inversion theorem for the degree spectra. Journal of Logic and Computation **19** (2009) 199–215
3. Cooper, S.: Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense. J. Symbolic Logic **49** (1984) 503–513
4. Ash, C., Knight, J.: Computable Structures and the Hyperarithmetical Hierarchy. Volume 144 of Studies in Logic and the Foundations of Mathematics. North - Holland, Amsterdam-New York-Tokyo (2000)

5. Ash, C., Knight, J., Manasse, M., Slaman, T.: Generic copies of countable structures. Ann. Pure Appl. Logic **42** (1989) 195–205
6. Copestake, K.: 1-Genericity in the enumeration degrees. J. Symbolic Logic **53** (1988) 878–887
7. Soskov, I.: Degree spectra and co-spectra of structures. Ann. Univ. Sofia **96** (2004) 45–68
8. Coles, R., Downey, R., Slaman, T.: Every set has a least jump enumeration. Bulletin London Math. Soc. **62** (2000) 641–649
9. Selman, A.: Arithmetical reducibilities I. Z. Math. Logik Grundlag. Math. **17** (1971) 335–350