INTRINSICALLY Π_1^1 RELATIONS

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ABSTRACT. An external characterization of the inductive sets on countable abstract structures is presented. The main result is an abstract version of the classical Suslin-Kleene characterization of the hyperarithmetical sets.

CONTENTS

1. INTRODUCTION

The external approach in the abstract recursion theory can be roughly described as follows. Consider a countable structure $\mathfrak{A}$ and a subset A of the domain of $\mathfrak{A}$. Call A relatively intrinsically "definable" on $\mathfrak{A}$ iff for every enumeration f of $\mathfrak{A}$, the pullback $f^{-1}(A)$ of A is "definable" relative to the diagram of the structure $f^{-1}(\mathfrak{A})$. The main problem of this approach is to obtain an explicit (internal) characterization of the relatively intrinsically definable sets. There are several known results of that type. In [Mos69a] Moschovakis proves that the relatively intrinsically recursively enumerable sets coincide with the semi-computable in the sense of [Mos69b] sets. This result is generalized in the papers [AKMS89] and [Chi90] where it is shown that for each constructive ordinal α, the relatively intrinsically Σ_α^0 sets are exactly those definable by

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means of recursive Σ^0_α formulae of the language $L\omega_1\omega$. Finally, in [Gri72] it is proved that on each acceptable structure $\mathfrak{A}$ the relatively intrinsically hyperarithmetical sets coincide with the hyperelementary, i.e. inductive and coinductive, on $\mathfrak{A}$ sets.

Here we continue this line of investigations by proving that the intrinsically relatively Π^1_1 sets on each countable structure $\mathfrak{A}$ coincide with the sets which are inductively definable on the least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$.

The external approach to the definition of the inductive sets leads very fast to some of the central results of the theory presented in [Mos74] as, for example, the Abstract Kleene Theorem, the Perfect Set Theorem, the Normal Form Theorem, etc. Along with this it allows also to transfer some results of the classical recursion theory to the abstract case. This possibility is used in the last two sections of our paper where a notation system for the ordinals of the inductive sets is constructed and a hierarchy for the hyperelementary sets similar to the classical Suslin-Kleene hierarchy is obtained. The last result answers at least partially the respective question posed in [Mos74]. Another hierarchy based on second order exsistentional definitions is contained in [Mos74].

The paper is organized as follows. In section 3 we introduce the so called enumeration structures and prove a general normal form theorem for the relatively intrinsically definable sets which is used intensively in the rest of the paper. Section 4 contains some preliminary facts about the Semi-computable sets. In section 5 we present the internal characterization of the relatively intrinsically Π^1_1 sets. In section 6 we define a set of indices of the hyperelementary sets and show that it is complete with respect to all inductive sets. Section 7 contains the abstract version of the Suslin-Kleene Theorem.

Almost all of the arguments use set theoretic forcing, an idea which we owe to [AKMS89] and [Chi90]. The main technical problem here is the lack of a suitable class of formulae to be forced. We avoid this obstacle by using forcing of appropriate inductive definitions.

2. Preliminaries

Throughout the paper we shall suppose fixed a countable structure $\mathfrak{A} = (B;\ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$, where each Σ_j is an a_j-ary predicate on B. We shall assume that the reader is familiar with the basic notions of the theory of the positive elementary induction on $\mathfrak{A}$ as presented in [Mos74].

The least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ is defined as follows.

Let 0 be an object which does not belong to B and $\langle .,. \rangle$ be a pairing operation chosen so that neither 0 nor any element of B is an ordered pair. Let B^* be the least set containing all elements of $B_0 = B \cup \{0\}$ and closed under the operation $\langle .,. \rangle$.

We associate an element n^* of B^* with each integer n by the inductive definition:

$$\begin{aligned} 0^* &= 0 \\ (n+1)^* &= \langle 0, n^* \rangle \end{aligned}$$

and put $N^* = \{0^*, 1^*, 2^*, \dots\}$.

Let $\mathfrak{A}^*$ be the structure $(B^*; B_0,\ \Sigma_1, \Sigma_2, \dots, \Sigma_k,\ N^*, G_{\langle,\rangle})$, where $G_{\langle,\rangle}$ is the graph of the pairing function.

As pointed out in [Mos74], the structure $\mathfrak{A}^*$ is an acceptable extension of $\mathfrak{A}$ and in some sense it is the least acceptable extension. Namely, the following is true.

2.1. Proposition. *Let $A \subseteq B^n$. Then A is inductive (hyperelementary) on all acceptable extensions of $\mathfrak{A}$ iff A is inductive (hyperelementary) on $\mathfrak{A}^*$.*

From now on we shall suppose fixed a least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$.

An one to one mapping f of the set of the natural numbers N onto B is called *enumeration* of $\mathfrak{A}$.

Clearly each enumeration f of $\mathfrak{A}$ determines a unique structure $\mathfrak{B}_f = (N;\ \sigma_1,\ \sigma_2, \dots, \sigma_k)$ where

$$\sigma_j(x_1, \dots, x_{a_j}) = \Sigma_j(f(x_1), \dots, f(x_{a_j}))$$

for all $x_1, \dots, x_{a_j} \in N$.

By $D(\mathfrak{B}_f)$ we shall denote the set of all Gödel numbers of the elements of the diagram of $\mathfrak{B}_f$.

2.2. Definition. Let $A \subseteq B^n$. The set A is *relatively intrinsically* Π_1^1 (HYP, recursively enumerable) on $\mathfrak{A}$ if for each enumeration f of $\mathfrak{A}$, there exists a Π_1^1 (hyperarithmetical, r. e.) relative to $D(\mathfrak{B}_f)$ subset W of N^n, such that for all $x_1, \dots, x_n \in N$,

$$(x_1, \dots, x_n) \in W \iff (f(x_1), \dots, f(x_n)) \in A.$$

In particular the relatively intrinsically r. e. sets are studied in [Lac64, Mos69a, AKMS89, Chi90].

The relatively intrinsically HYP sets are studied in [Gri72], where is proved that a set A is intrinsically HYP on $\mathfrak{A}$ if and only if it is hyperelementary on $\mathfrak{A}$, provided that $\mathfrak{A}$ is acceptable.

The following simple proposition justifies the use of the least acceptable extension $\mathfrak{A}^*$.

2.3. Proposition. *Let $A \subseteq B^n$ and let f be an enumeration of $\mathfrak{A}$. Denote by W the subset of N^n, defined by*

$$(x_1, \dots, x_n) \in W \iff (f(x_1), \dots, f(x_n)) \in A.$$

There exists an enumeration f^* of $\mathfrak{A}^*$ such that $D(\mathfrak{B}_{f^*}) \leqq_T D(\mathfrak{B}_f)$ and such that if W^* is the pullback of A with respect to f^*, then $W \leqq_m W^*$.

Proof. To simplify the notation, suppose that $n = 1$.

Let $J(x, y) = 2^{x+1}.(2y + 1)$ be an effective coding of the ordered pairs of natural numbers. Define f^* by means of the following inductive definition:

$$\begin{aligned} f^*(0) &= 0^*, \\ f^*(2x+1) &= f(x), \\ f^*(J(x,y)) &= \langle f^*(x), f^*(y)\rangle. \end{aligned}$$

Obviously, f^* is an enumeration of $\mathfrak{A}^*$ and $D(\mathfrak{B}_{f^*}) \leqq_T D(\mathfrak{B}_f)$.

Now, we have

$$x \in W \iff 2x + 1 \in W^*.$$

Indeed,

$$x \in W \iff f(x) \in A \iff f^*(2x+1) \in A \iff 2x+1 \in W^*.$$

2.4. Corollary. *If for every enumeration f^* of $\mathfrak{A}^*$ the pullback of A is Π^1_1(HYP, r.e.) relative to $D(\mathfrak{B}_{f^*})$, then A is intrinsically Π^1_1(HYP, r.e.) on $\mathfrak{A}$.*

2.5. Corollary. *Let $A \subseteq B^n$ and suppose that A is inductive (hyperelementary) on $\mathfrak{A}^*$. Then A is relatively intrinsically Π^1_1(HYP) on $\mathfrak{A}$.*

Proof. Let f^* be an enumeration of $\mathfrak{A}^*$. Then the pullback W^* of A has an inductive definition on the structure $\mathfrak{B}_{f^*}$, and hence, W^* is Π^1_1 relative to $D(\mathfrak{B}_{f^*})$. □

3. Enumeration Structures

In this section we shall describe a general way of obtaining normal form of the intrinsically "definable" sets on $\mathfrak{A}$. This construction will help us to avoid the repetition of the same argument in the rest of the paper.

Denote by $\mathcal{E}$ the set of all enumerations of $\mathfrak{A}$ and by Δ the set of all finite injective mappings of N into B. We shall call the elements of Δ *finite parts* and use $\delta, \tau, \rho, \mu, \nu, \ldots$ to denote arbitrary elements of Δ. By f we shall denote elements of $\mathcal{E}$.

The set theoretic inclusion "$\subseteq$" induces a natural ordering between finite parts and between finite parts and enumerations. We shall list the properties of $\mathcal{E}$,Δ and $\subseteq$ needed for the proof of the normal form theorem.

E1. *The set Δ is countable and non-empty.*

E2. (i) $\delta \subseteq \delta$,
(ii) $\delta \subseteq \tau \;\&\; \tau \subseteq \rho \Longrightarrow \delta \subseteq \rho$.

E3. $\delta \subseteq \tau \;\&\; \tau \subseteq f \Longrightarrow \delta \subseteq f$.

E4. *If $\delta \subseteq f$ and $\tau \subseteq f$, then there exists a $\rho \subseteq f$ such that $\delta \subseteq \rho$ and $\tau \subseteq \rho$.*

3.1. Definition. Let $X \subseteq \Delta$ and $f \in \mathcal{E}$. The enumeration f *meets* X if for some $\delta \in X$, $\delta \subseteq f$.

3.2. Definition. A subset $X \subseteq \Delta$ is *dense in the enumeration* f if

$$\forall \delta \subseteq f\ \exists \tau \in X(\delta \subseteq \tau).$$

3.3. Definition. Let $\mathcal{F}$ be a family of subsets of Δ. An enumeration f is $\mathcal{F}$*-generic* if whenever $X \in \mathcal{F}$ and X is dense in f, then f meets X.

E5. *For every countable family $\mathcal{F}$ of subsets of Δ and every $\delta \in \Delta$ there exists a $\mathcal{F}$-generic enumeration $f \supseteq \delta$.*

Comment. It is obvious that **E1** $-$ **E5** hold for $\mathcal{E}$, Δ and $\subseteq$ defined above. Most of the theorems which follow can be formulated and proved in the context of arbitrary structures of the form $\langle \mathcal{E}, \Delta, \subseteq \rangle$ satisfying **E1** $-$ **E5**. We call such structures *enumeration structures*. A detailed treatment of the enumeration structures will be presented in a forthcoming paper.

Let us fix a denumerable family $\mathcal{F}$ of subsets of Δ.

3.4. Definition. A sequence $X_0, X_1, \dots$ of subsets of Δ is *dense* if $X_0 \neq \emptyset$ and if $\delta_k \in X_k$ and $\tau \supseteq \delta_k$, then there exists a $\delta_{k+1} \in X_{k+1}$ s. t. $\delta_{k+1} \supseteq \tau$.

3.5. Lemma. *Let $\{X_n\}$ be a dense sequence. There exists a $\mathcal{F}$-generic enumeration f which meets all sets X_n, $n = 0, 1, \dots$*

Proof. Consider the family $\mathcal{F}_1 = \mathcal{F} \cup \{X_0, X_1, \dots\}$. Let $\delta_0 \in X_0$ and let f be a $\mathcal{F}_1$-generic enumeration which extends δ_0. Using induction on n we can show that f meets all X_n, $n = 0, 1, \dots$

Indeed, suppose that f meets X_n. We shall show that X_{n+1} is dense in f. Let $\delta \subseteq f$. By the induction hypothesis, there exists a $\delta_n \in X_n$ s. t. $\delta_n \subseteq f$. Take a $\tau \subseteq f$ s.t. $\delta_n \subseteq \tau$ and $\delta \subseteq \tau$. Since the sequence $\{X_n\}$ is dense, there exists a δ_{n+1} in X_{n+1} such that $\tau \subseteq \delta_{n+1}$.

Now, since f is $\mathcal{F}_1$-generic, we have that f meets X_{n+1}. $\square$

3.6. Definition. A subset Q of $N \times \mathcal{E}$ is *complete* (with resect to the family $\mathcal{F}$) if for each $n \in N$ and each $\delta \in \Delta$, there exists a $\tau \supseteq \delta$ such that if f is $\mathcal{F}$-generic and $f \supseteq \tau$, then $(n, f) \in Q$.

3.7. Proposition. *Let Q be a complete subset of $N \times \mathcal{E}$. There exists a $\mathcal{F}$-generic enumeration f such that $(n, f) \in Q$, for all $n \in N$.*

Proof. Let $\delta_0, \delta_1, \dots$ be an arbitrary enumeration of all finite parts. We shall construct a dense sequence $X_0, X_1, \dots$ so that if f is a $\mathcal{F}$-generic enumeration and f meets X_n, then $(n, f) \in Q$, and apply Lemma 3.5.

The construction of the sets $X_0, X_1, \ldots$ will be carried out by steps. By X_n^q we shall denote the approximation of X_n obtained at step q. We shall ensure that $X_n^q \subseteq X_n^{q+1}$ and take $X_n = \bigcup_{q=0}^{\infty} X_n^q$.

Step $q = 0$. Let $\tau \supseteq \delta_0$ be such that if f is $\mathfrak{F}$-generic and $f \supseteq \tau$, then $(0, f) \in Q$. Set $X_0^0 = \{\tau\}$ and $X_{n+1}^0 = \emptyset$, $n = 0, 1, \ldots$.

Suppose that X_0^q, $X_1^q, \ldots$ are defined. We shall consider two cases.

(a) $q = \langle i, j, n, r\rangle$, where $\delta_i \in X_n^q$ and $\delta_i \subseteq \delta_j$. (Here the number r is left free in order to insure arbitrary large q with first three components i, j, n).

Let $\tau \supseteq \delta_j$ be such that if $f \supseteq \tau$ and f is $\mathfrak{F}$-generic, then $(n+1, f) \in Q$. Set $X_{n+1}^{q+1} = X_{n+1}^q \cup \{\tau\}$ and $X_k^{q+1} = X_k^q$, for $k \neq n+1$.

(b) Do nothing, otherwise.

It follows easily from the construction of sets $X_0, X_1, \ldots$ that the sequence is dense and if f is $\mathfrak{F}$-generic and f meets X_n then $(n, f) \in Q$. $\square$

Our next step is to give a general definition of intrinsically "definable" sets on the structure $\mathfrak{A}$.

Let $P_0, P_1, \ldots P_l, \ldots$ be a sequence of n – ary predicate letters.

Suppose that a satisfaction relation "$\models$" is given saying for all enumerations f and all n – ary vectors $\overline{x}$ of natural numbers whether $P_e(\overline{x})$ holds on f or not.

For $\overline{x} = x_1, \ldots, x_n$ we shall use $f(\overline{x})$ to denote the vector $(f(x_1), \ldots, f(x_n))$ of elements of B.

3.8. Definition. A set $A \subseteq B^n$ is called *admissible in the enumeration* f (relative to "$\models$") if there exists e such that for all $\overline{x} \in N^n$

$$f \models P_e(\overline{x}) \iff f(\overline{x}) \in A.$$

In such a case we shall call P_e a *f-associate* of A.

3.9. Definition. The set A is *intrinsically definable* (relative to "$\models$") if A is admissible in all enumerations.

Let a *forcing relation* $\delta \Vdash P_e(\overline{x})$ be defined so that the following is true:

(1) $\delta \Vdash P_e(\overline{x})$ & $\delta \subseteq \tau \Longrightarrow \tau \Vdash P_e(\overline{x})$;

(2) There exists a denumerable family $\mathfrak{F}$ – of subsets of Δ s.t. if f is $\mathfrak{F}$-generic then for all $e, \overline{x}$,

$$f \models P_e(\overline{x}) \iff \exists \delta \subseteq f(\delta \Vdash P_e(\overline{x})).$$

Finally, given a finite part δ and a vector $\overline{x}$, denote by $\mathfrak{R}(\delta, \overline{x})$, the set $\{\overline{s} : \overline{s} \in B^n \;\&\; \exists \tau \supseteq \delta(\tau(\overline{x}) = \overline{s})\}$.

Now, we are ready to formulate the normal form theorem for the intrinsically definable sets.

3.10. Theorem.(Normal Form Theorem) *Let $A \subseteq B^n$ be intrinsically definable. There exist finite part δ and e such that for all $\overline{x} \in N$ and all $\overline{s} \in \mathcal{R}(\delta, \overline{x})$,*

$$(*) \qquad \overline{s} \in A \iff \exists \tau \supseteq \delta(\tau(\overline{x}) = \overline{s} \ \& \ \tau \Vdash P_e(\overline{x})).$$

Proof. Towards a contradiction, assume that there do not exist δ and e having the needed properties.

Let $Q \subseteq N \times \mathcal{E}$ be defined by the equivalence:

$$(e, f) \in Q \iff P_e \text{ is not } f\text{-associate of } A.$$

We shall show that the set Q is complete. Let $e \in N$ and $\delta \in \Delta$. By the assumption for some $\overline{x} \in N^n$ and $\overline{s} \in \mathcal{R}(\delta, \overline{x})$ the equivalence $(*)$ fails. We have two cases:

(a) $\overline{s} \in A$ and $\forall \tau \supseteq \delta(\tau(\overline{x}) = \overline{s} \Longrightarrow \tau \nVdash P_e(\overline{x}))$. Let $\tau \supseteq \delta$ and $\tau(\overline{x}) = \overline{s}$. Let f be $\mathcal{F}$-generic and $f \supseteq \tau$. Clearly $f \not\models P_e(\overline{x})$. On the other hand, since $f(\overline{x}) = \tau(\overline{x})$, we have $f(\overline{x}) \in A$. So, P_e is not f-associate of A.

(b) $\overline{s} \notin A$ but for some $\tau \supseteq \delta$, $\tau(\overline{x}) = \overline{s}$ and $\tau \Vdash P_e(\overline{x})$. Let f be $\mathcal{F}$-generic and $f \supseteq \tau$. Then $f \models P_e(\overline{x})$ but $f(\overline{x}) \notin A$. Therefore, $(e, f) \in Q$.

Now, applying Proposition 3.7, we obtain that there exists a $\mathcal{F}$-generic f such that for all e, $(e, f) \in Q$ and hence A is not admissible in f. $\square$

4. The Semi-computable sets

The semi-computable sets on an abstract structure are introduced by Moschovakis in [Mos69b] as a counterpart of the recursively enumerable sets of natural numbers. In [Mos69a] Moschovakis proved that on each countable structure the semi-computable sets coincide with the $\forall$-recursively enumerable sets of Lacombe [Lac64] and, hence, with the intrinsically r. e. sets in our terminology. Since all structures under consideration here are countable, we shall identify the semi-computable sets with the intrinsically r. e. sets.

In this section we are going to applay Theorem 3.10 to get a normal form of the semi-computable sets on $\mathfrak{A}$. The satisfaction and the respective forcing relations for the recursively enumerable sets defined here are used also in the more sophisticated applications of Theorem 3.10 in the rest of the paper.

For the sake of simplicity we shall consider only subsets of B.

Let us fix an effective coding of the finite sets of natural numbers (by E_v we shall denote the finite set with code v). And let $W_0, W_1, \dots, W_e, \dots$ be a standard enumeration of the r. e. subsets of N.

Let $R_0, R_1, \dots, R_e, \dots$ be a sequence of unary predicate symbols. Given an enumeration f of the structure $\mathfrak{A}$, define

$$f \models R_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \& \ E_v \subseteq D(\mathfrak{B}_f)).$$

In other words,

$$f \models R_e(x) \iff x \in \Gamma_e(D(\mathfrak{B}_f)),$$

where Γ_e is the e-th enumeration operator, see [Rog67].

Clearly, the sets $W_e^{D(\mathfrak{B}_f)} = \{x : f \models R_e(x)\}$ coincide with the r.e. relative to $D(\mathfrak{B}_f)$ subsets of N. Hence a set A is intrinsically definable relative to "$\models$" if it is semi-computable on $\mathfrak{A}$.

The definition of the relation $\delta \Vdash R_e(x)$ is a little bit more complicated.

Our starting point is the representation of the diagram $D(\mathfrak{B}_f)$ of the structure $\mathfrak{B}_f = (N;\ \sigma_1,\ \sigma_2,\ \dots, \sigma_k)$. The set $D(\mathfrak{B}_f)$ consists of codes of atoms or negated atoms which are true on $\mathfrak{B}_f$. We shall call these formulae *literals*.

Now given a finite part δ and a natural number u, define $\delta \Vdash u$ if u is code of a literal $L(x_1, \dots, x_a)$, all $x_1, \dots, x_a$ are elements of the domain of δ and

$$\mathfrak{A} \models L(\delta(x_1), \dots, \delta(x_a)).$$

For each finite set $E_v = \{u_1, \dots, u_l\}$, define

$$\delta \Vdash E_v \iff \delta \Vdash u_1 \ \&\ \cdots \ \&\ \delta \Vdash u_l$$

and finally define

$$\delta \Vdash R_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ \delta \Vdash E_v).$$

From the definition it follows immediately that

(1) $\delta \Vdash R_e(x) \ \&\ \delta \subseteq \tau \Longrightarrow \tau \Vdash R_e(x)$;

(2) For any enumeration f,

$$f \models R_e(x) \iff \exists \delta \subseteq f(\delta \Vdash R_e(x)).$$

Now let A be a fixed subset of B and suppose that A is semi-computable. From the Normal Form Theorem it follows that there exist δ and e such that for all x and $s \in \mathcal{R}(\delta, x)$

$$s \in A \iff \exists \tau \supseteq \delta(\tau(x) = s \ \&\ \tau \Vdash R_e(x)). \tag{4.1}$$

Given finite part δ and natural number e, denote by $S_{\delta,e}$ the subset of B defined by the equivalence

$$s \in S_{\delta,e} \iff \exists x \in N \exists \tau \supseteq \delta(\tau(x) = s \ \&\ \tau \Vdash R_e(x)).$$

4.1. Proposition. *A set A is semi-computable on $\mathfrak{A}$ iff there exist finite part δ and natural number e such that $A = S_{\delta,e}$.*

Proof. Let A be semi-computable. Then there exist finite part δ and natural number e such that 4.1 holds for all x and $s \in \mathcal{R}(\delta, x)$. From here we obtain directly that $A = S_{\delta,e}$.

To prove the proposition in the other direction, we represent each finite part as an element of B^*. For example, we can identify the finite part δ mapping $x_1, \dots, x_n$ onto $t_1, \dots, t_n$, respectively, with the element $\langle\langle x_1, t_1\rangle, \dots, \langle x_n, t_n\rangle, 0^*\rangle$ of B^*. Now it is easy to show that given an enumeration f^* of $\mathfrak{A}^*$, the pullback of each set

$S_{\delta,e}$ is r. e. in $D(\mathfrak{B}_{f^*})$. From here using Corollary 2.4 we obtain that each $S_{\delta,e}$ is semi-computable. □

5. Relatively intrinsically Π^1_1 sets.

In this section we shall show that a subset of B^n is intrinsically relatively Π^1_1 if and only if it has an inductive definition on the structure $\mathfrak{A}^*$.

Again for simplicity we shall assume $n = 1$.

By Corollary 2.5, each inductive on $\mathfrak{A}^*$ subset of B is relatively intrinsically Π^1_1.

To prove the converse of Corollary 2.5 we shall apply the Normal Form Theorem. The definitions of the satisfaction and forcing relations used here are inductive, based on the respective satisfaction and forcing relations for the r. e. sets defined in the previous section.

Let $R_0, R_1, \ldots, R_e, \ldots$ be a sequence of predicate symbols which now are supposed to be 2–ary.

Given an enumeration f, define $f \models R_e(a, x)$ using the definition given in the previous section. Clearly the sets $\{(a, x) : f \models R_e(a, x)\}$ coincide with the r. e. relative to $D(\mathfrak{B}_f)$ subsets of N^2.

Now let $P_0, P_1, \ldots, P_e, \ldots$ be a new sequence of 2–ary predicate letters. Let a be a variable with range the (codes of) finite strings of natural numbers. If $a = \langle z_1, \ldots, z_k\rangle$ and $z \in N$, then by $a * z$ we shall denote the string $\langle z_1, \ldots, z_k, z\rangle$. By $\langle\rangle$ we shall denote the empty string.

Let f be an enumeration of $\mathfrak{A}$. The satisfaction relation $f \models P_e(a, x)$ is defined by means of the following inductive definition:

5.1. Definition.

If $f \models R_e(a, x)$, then $f \models P_e(a, x)$;
if $\forall z(f \models P_e(a * z, x))$, then $f \models P_e(a, x)$.

It is natural to associate ordinals with the sequences e, a, x such that $f \models P_e(a, x)$ by the following:

5.2. Definition. Let $f \models P_e(a, x)$, then

$|e, a, x|_f = 0$, if $f \models R_e(a, x)$ and
$|e, a, x|_f = sup(|e, a * z, x|_f + 1 : z \in N)$, otherwise.

The following lemma can be proved by means of the standard argument which shows that the Π^1_1 sets coincide with the inductive sets on the structure of the arithmetic.

5.3. Lemma. *The sets* $Y_e = \{x : f \models P_e(\langle\rangle, x)\}$ *coincide with the* Π^1_1 *relative to* $D(\mathfrak{B}_f)$ *subsets of* N.

Our next task is to define the forcing relation $\delta \Vdash P_e(a,x)$. The definition is inductive following the definition of the relation "$\models$ " and the usual forcing rules for interpretation of the $\forall$-quantifier.

5.4. Definition.

If $\delta \Vdash R_e(a,x)$, then $\delta \Vdash P_e(a,x)$;
If $\forall z \in N \forall \tau \supseteq \delta \exists \rho \supseteq \tau(\rho \Vdash P_e(a*z,x))$, then $\delta \Vdash P_e(a,x)$.

We associate ordinals with the tuples (e,δ,a,x) such that $\delta \Vdash P_e(a,x)$ as usual.

5.5. Definition.

$|e,\delta,a,x| = 0$, if $\delta \Vdash R_e(a,x)$,
$|e,\delta,a,x| = sup(min(|e,\rho,a*z,x|+1 : \rho \supseteq \tau) : \tau \supseteq \delta, z \in N)$, if $\delta \nVdash R_e(a,x)$.

The following Lemma is immediate from the Definition 5.4.

5.6. Lemma. *Let δ,τ be finite parts, $\delta \subseteq \tau$ and $\delta \Vdash P_e(a,x)$, then $\tau \Vdash P_e(a,x)$.*

Let $\mathcal{F}_1$ be the family containing all subsets

$$X_{e,\delta,a,x,z} = \{\rho : \rho \Vdash P_e(a*z,x) \,\&|e,\rho,a*z,x| < |e,\delta,a,x|\} \text{ of } \Delta.$$

5.7. Lemma. *Let f be a $\mathcal{F}_1$-generic enumeration, $\delta \subseteq f$ and $\delta \Vdash P_e(a,x)$. Then $f \models P_e(a,x)$.*

Proof. Transfinite induction on $|e,\delta,a,x|$. Skipping the obvious case $f \models R_e(a,x)$, assume $f \not\models R_e(a,x)$. Fix a $z \in N$ and consider the element

$$X = \{\rho : \rho \Vdash P_e(a*z,x) \,\&\, |e,\rho,a*z,x| < |e,\delta,a,x|\}$$

of $\mathcal{F}_1$. We shall show that X is dense in f. Let $\mu \subseteq f$. Take a $\tau \subseteq f$ such that $\mu \subseteq \tau$ and $\delta \subseteq \tau$. Since $f \not\models R_e(a,x), \delta \nVdash R_e(a,x)$ and hence by the definition of "$\Vdash$" there exists a $\rho \supseteq \tau$ which belongs to X. From here, by genericity, there exists a $\rho \subseteq f$ which belongs to X.

Now, we have that $|e,\rho,a*z,x| < |e,\delta,a,x|$ and $\rho \Vdash P_e(a*z,x)$. Hence, by the inductive hypothesis, $f \models P_e(a*z,x)$. From here, it follows that $\forall z(f \models P_e(a*z,x))$, and hence, $f \models P_e(a,x)$. □

Denote by $\mathcal{F}_2$ the family containing all sets $\{\tau : \exists z \forall \rho \supseteq \tau(\rho \nVdash P_e(a*z,x))\}$.

5.8. Lemma. *Let f be $\mathcal{F}_2$-generic and $f \models P_e(a,x)$. Then there exists a $\delta \subseteq f$ such that $\delta \Vdash P_e(a,x)$.*

Proof. Transfinite induction on $|e, a, x|_f$.

Assume that $\forall \delta \subseteq f(\delta \nVdash P_e(a, x))$. Then the set $X = \{\tau : \exists z \forall \rho \supseteq \tau(\rho \nVdash P_e(a * z, x))\}$ is dense in f. Hence there exist a $\tau \subseteq f$ and $z \in N$, such that $\forall \rho \supseteq \tau(\rho \nVdash P_e(a * z, x))$.

On the other hand, $f \models P_e(a, x)$ and $f \not\models R_e(a, x)$. (Otherwise we could find a $\delta \subseteq f$ s.t. $\delta \Vdash R_e(a, x)$). Hence, $f \models P_e(a * z, x)$, and hence, by induction, there exists a $\rho \subseteq f$ s.t. $\rho \supseteq \tau$ and $\rho \Vdash P_e(a * z, x)$. A contradiction. $\square$

5.9. Theorem. *Let $A \subseteq B$. Then A is relatively intrinsically Π^1_1 on $\mathfrak{A}$ iff A is inductive on $\mathfrak{A}^*$.*

Proof. As we have already pointed out, in the one direction the theorem follows from Corollary 2.5.

Suppose now that A is relatively intrinsically Π^1_1 on $\mathfrak{A}$. By the Normal Form Theorem there exist δ and e such that for all $x \in N$ and $s \in \mathfrak{R}(\delta, x)$,

$$s \in A \iff \exists \tau \supseteq \delta(\tau(x) = s \mathbin{\&} \tau \Vdash P_e(\langle\rangle, x)).$$

Let domain of $\delta = \{w_1, \ldots, w_r\}$ and $\delta(w_i) = t_i,\ i = 1, \ldots, r$.

Fix a $x_0 \notin dom(\delta)$. Then we have the following representation of A.

$$\begin{aligned} s \in A \iff & \exists \tau \supseteq \delta(\tau(x_0) = s \mathbin{\&} \tau \Vdash P_e(\langle\rangle, x_0) \text{ or} \\ & s = t_1 \mathbin{\&} \exists \tau \supseteq \delta(\tau \Vdash P_e(\langle\rangle, w_1, y)) \text{ or} \\ & \ldots \\ & s = t_r \mathbin{\&} \exists \tau \supseteq \delta(\tau \Vdash P_e(\langle\rangle, w_r, y)). \end{aligned} \tag{5.1}$$

Now considering the finite parts as elements of B^*, using the fact that the set $\{(\tau, a, x) : \tau \Vdash R_e(a, x)\}$ is semi-computable and hence first order definable on $\mathfrak{A}^*$, see [Mos69a], we obtain easily that the set $\{(\tau, a, x) : \tau \Vdash P_e(a, x)\}$ is inductive on $\mathfrak{A}^*$ and, hence, that A is inductive on $\mathfrak{A}^*$. $\square$

5.10. Corollary. *Let $A \subseteq B$. Then A is relatively intrinsically HYP on $\mathfrak{A}$ iff A is hyperelementary on $\mathfrak{A}^*$.*

Proof. Let A be relatively intrinsically HYP on $\mathfrak{A}$. Then A and the complement $\overline{A}$ of A are relatively intrinsically Π^1_1 and hence both are inductive. $\square$

Comment. The proof of Theorem 5.9 gives in fact more than formulated. The representation (5.1) gives a normal form of the inductive on $\mathfrak{A}^*$ sets. An easy application of this normal form is the characterization of the inductive sets by means of the Game quantifier [Mos74]. Another application of Theorem 5.9 shows that on each countable acceptable structure $\mathfrak{A}$ the Π^1_1 sets on $\mathfrak{A}$ coincide with the inductive ones [Mos74]. Indeed, since the Π^1_1 sets on $\mathfrak{A}$ are defined by means of Π^1_1-formulae in the language of $\mathfrak{A}$, it is obvious that each such set is relatively intrinsically Π^1_1.

6. Ordinal notations

One of the main difficulties in the theory of the inductive (Π^1_1) sets on abstract structures is the lack of a nice notational system for the ordinals of the inductive sets as, for example, the set $\mathcal{O}$ in the classical recursion theory (cf. [Mos69c]). Here we shall obtain notations of the ordinals of the inductive sets on $\mathfrak{A}^*$ by transferring the classical ordinal notational systems via enumerations of $\mathfrak{A}$.

Consider a subset X of N. The set of the indices of the hyperarithmetical sets relative to X is defined by means of the following inductive definition, [Sho67]:

6.1. Definition.

(1) For each e, $\langle 0, e\rangle$ is index;
(2) If e is index then, $\langle 1, e\rangle$ is index;
(3) If all elements of W_e^X are indices, then $\langle 2, e\rangle$ is index.

Here W_e^X denotes the set $\Gamma_e(X)$, where Γ_e is the e-th enumeration operator.

The ordinals associated with the elements of the set Ind_X of all indices relative to X are as follows:

(1) $|\langle 0, e\rangle|_X = 0$, for all $e \in N$;
(2) $|\langle 1, e\rangle|_X = |e|_X + 1$;
(3) $|\langle 2, e\rangle|_X = sup(|z|_X + 1 : z \in W_e^X)$.

The following facts should be well known though the author was not able to find the exact references.

Let ω_1^X be the least ordinal which is not constructive relative to X.

Fact 1. $\omega_1^X = \{|e|_X : e \in Ind_X\}$.

Fact 2. There exists a recursive function h, which does not depend on X, such that if $X \subseteq N$, then

$$a \in \mathcal{O}_X \iff h(a) \in Ind_X.$$

Here $\mathcal{O}_X$ is the set of the Church - Kleene ordinal notations relativized to X, see [Rog67].

We prefer to work with Ind_X instead of $\mathcal{O}_X$ because of the obvious inductive definition of Ind_X. Both sets are very similar though.

Let I be a new unary predicate symbol. For each enumeration f of $\mathfrak{A}$ define $f \models I(u)$, by means of the following repetition of Definition 6.1.

6.2. Definition.

(1) $f \models I(\langle 0, e\rangle)$, for all $e \in N$;
(2) If $f \models I(e)$, then $f \models I(\langle 1, e\rangle)$;
(3) If $\forall z(f \models R_e(z) \Longrightarrow f \models I(z))$, then $f \models I(\langle 2, e\rangle)$.

Here $R_0, R_1, \dots, R_e, \dots$ are unary predicate symbols and $f \models R_e(z)$ is defined as in Section 4. So, we have that

$$\{z : f \models R_e(z)\} = W_e^{D(\mathfrak{B}_f)}.$$

Obviously, $f \models I(u) \iff u \in Ind_{D(\mathfrak{B}_f)}$.

Set $|f, u|_I = |u|_{D(\mathfrak{B}_f)}$.

Our next task is to define the relation $\delta \Vdash I(u)$. This will be an inductive definition based on the definition of $\delta \Vdash R_e(z)$, given in Section 4.

6.3. Definition.

(1) $\delta \Vdash I(\langle 0, e\rangle)$, for all $e \in N$;
(2) If $\delta \Vdash I(e)$, then $\delta \Vdash I(\langle 1, e\rangle)$;
(3) If $\forall z \forall \tau \supseteq \delta(\tau \Vdash R_e(z) \Longrightarrow \exists \rho \supseteq \tau(\rho \Vdash I(z)))$, then $\delta \Vdash I(\langle 2, e\rangle)$.

The ordinals associated with the pairs δ, u such that $\delta \Vdash I(u)$ are given by

6.4. Definition.

(1) $|\delta, \langle 0, e\rangle|_I = 0$;
(2) $|\delta, \langle 1, e\rangle|_I = |\delta, e|_I + 1$;
(3) $|\delta, \langle 2, e\rangle|_I = sup(\chi(\tau, z) + 1 : \tau \supseteq \delta, z \in N, \tau \Vdash R_e(z))$, where $\chi(\tau, z) = min(|\rho, z|_I : \rho \supseteq \tau)$.

Let $Ind_{\mathfrak{A}} = \{(\delta, u) : \delta \Vdash I(u)\}$. From definition 6.3 it follows immediately that $Ind_{\mathfrak{A}}$ is inductive on $\mathfrak{A}^*$.

Here we shall show two nice properties of the set $Ind_{\mathfrak{A}}$. Namely, that it is a complete inductive set and that the closure ordinal $\kappa^{\mathfrak{A}^*}$ of the structure $\mathfrak{A}^*$ is equal to $\{|\delta, u|_I : \delta \Vdash I(u)\}$.

In the next section we shall use the elements of $Ind_{\mathfrak{A}}$ as indices for the hyperelementary sets in a characterization of those sets similar to the classical Suslin - Kleene characterization of the hyperarithmetical (Borel) sets.

6.5. Lemma. *Let $\delta \Vdash I(u)$ and $\delta \subseteq \delta_1$. Then $\delta_1 \Vdash I(u)$ and $|\delta_1, u|_I \leqq |\delta, u|_I$.*

Proof. Follows directly from the corresponding definitions. $\square$

Let $\mathfrak{F}_1$ be the family of subsets of Δ containing all sets $\{\rho : |\rho, z|_I < |\delta, u|_I\}$, $\delta \in \Delta$ and $z, u \in N$.

6.6. Lemma. *Let f be a $\mathfrak{F}_1$-generic enumeration of $\mathfrak{A}$. Suppose that $\delta \subseteq f$ and $\delta \Vdash I(u)$. Then, $f \Vdash I(u)$ and $|f, u|_I \leqq |\delta, u|_I$.*

Proof. Transfinite induction on $|\delta, u|_I$. Skipping the obvious cases $u = \langle 0, e\rangle$ or $u = \langle 1, e\rangle$, suppose that $u = \langle 2, e\rangle$. Let $z \in N$ and $f \models R_e(z)$. Consider the element X of $\mathfrak{F}_1$ defined by $\rho \in X \iff |\rho, z|_I < |\delta, \langle 2, e\rangle|_I$. We shall show that X is dense in f. Let $\tau \subseteq f$. We can assume that $\tau \Vdash R_e(z)$. Then by Definition 6.4 there exists a $\rho \supseteq \tau$ such that $\rho \in X$. Hence, by genericity, there exists a $\rho \subseteq f$ s.t. $\rho \Vdash I(z)$ & $|\rho, z|_I < |\delta, \langle 2, e\rangle|_I$. By the induction hypothesis, $f \models I(z)$ and $|f, z|_I \leqq |\rho, z|_I < |\delta, \langle 2, e\rangle|_I$. So, $f \models I(\langle 2, e\rangle)$ and

$$|f, \langle 2, e\rangle|_I = sup(|f, z|_I + 1 : f \models R_e(z)) \leqq |\delta, \langle 2, e\rangle|_I.$$

□

Let $\mathfrak{F}_2$ be the family containing all sets $\{\tau : \exists z \in N(\tau \Vdash R_e(z)\ \&\ \forall \rho \supseteq \tau(\rho \nVdash I(z)))\}$.

6.7. Lemma. *Let f be a $\mathfrak{F}_2$-generic enumeration of $\mathfrak{A}$ and $f \models I(u)$. Then there exists a $\delta \subseteq f$ s.t. $\delta \Vdash I(u)$.*

Proof. Transfinite induction on $|f, u|_I$.

Consider the nontrivial case $u = \langle 2, e\rangle$. Assume that $\forall \delta \subseteq f(\delta \nVdash I(\langle 2, e\rangle))$. Then, the set $X = \{\tau : \exists z(\tau \Vdash R_e(z)\ \&\ \forall \rho \supseteq \tau(\rho \nVdash I(z)))\}$ is dense in f. Hence there exist $\tau \subseteq f$ and z such that $\tau \Vdash R_e(z)\ \&\ \forall \rho \supseteq \tau(\rho \nVdash I(z))$. From here it follows that $f \models R_e(z)$ and hence $f \models I(z)$. By the induction hypothesis there exists a $\rho \supseteq \tau$ s.t. $\rho \Vdash I(z)$. A contradiction. □

Combining Lemma 6.6 and Lemma 6.7, we obtain the following:

6.8. Theorem. *There exists a denumerable family $\mathfrak{F}_I$ of subsets of Δ s.t. if f is a $\mathfrak{F}_I$-generic enumeration of $\mathfrak{A}$, then the following is true:*

(i) $f \models I(u) \iff \exists \delta(\delta \subseteq f\ \&\ \delta \Vdash I(u))$;
(ii) $\delta \subseteq f\ \&\ \delta \Vdash I(u) \Longrightarrow |f, u|_I \leqq |\delta, u|_I$.

Proof. Let $\mathfrak{F}_I = \mathfrak{F}_1 \cup \mathfrak{F}_2$. □

Denote by $\kappa^* = \kappa^{\mathfrak{A}^*}$ the closure ordinal of the structure $\mathfrak{A}^*$ [Mos74].

6.9. Theorem. $\kappa^* = \{|\delta, u|_I : (\delta, u) \in Ind_{\mathfrak{A}}\} = min(\omega_1^{D(\mathfrak{B}_f)} : f$ *is enumeration of* $\mathfrak{A})$.

Proof. Denote $\{|\delta, u|_I : (\delta, u) \in Ind_{\mathfrak{A}}\}$ by κ_0 and $min(\omega_1^{D(\mathfrak{B}_f)} : f$ is enumeration of $\mathfrak{A})$ by κ_1.

The ordinal κ_0 is closure of the inductive definition (Definition 6.3) of the set $Ind_{\mathfrak{A}}$ on $\mathfrak{A}^*$ and hence, $\kappa_0 \leqq \kappa^*$.

Let f^* be an enumeration of $\mathfrak{A}^*$. Clearly each inductive definition on $\mathfrak{A}^*$ can be simulated "step by step" on $\mathfrak{B}_{f^*}$, and hence, $\kappa^* \leqq \omega_1^{D(\mathfrak{B}_{f^*})}$. Therefore, $\kappa^* \leqq min(\omega_1^{D(\mathfrak{B}_{f^*})} : f^*$ is enumeration of $\mathfrak{A}^*) \leqq \kappa_1$, by Proposition 2.3.

So, we have $\kappa_0 \leqq \kappa^* \leqq \kappa_1$. To finish the proof it remains to show that $\kappa_0 \geqq \kappa_1$.

Take a $\xi < \kappa_1$. Let f be a $\mathcal{F}_I$-generic enumeration of $\mathfrak{A}$. Clearly, $\xi < \omega_1^{D(\mathfrak{B}_f)}$ and hence there exists an e, such that $f \models I(e)$ and $|f, e|_I = \xi$. By Theorem 6.8, there exists a $\delta \subseteq f$ such that $\delta \Vdash I(e)$ and $\xi \leqq |\delta, e|_I$. So, $\xi < \kappa_0$. $\square$

6.10. Definition. $\delta \Vdash^w I(u)$, if $\forall \tau \supseteq \delta \exists \rho \supseteq \tau(\rho \Vdash I(u))$.

6.11. Proposition. *The following are equivalent:*

(i) $\forall f \supseteq \delta(f$ *is* $\mathcal{F}_I$*-generic* $\implies f \models I(u))$.
(ii) $\delta \Vdash^w I(u)$.
(iii) $\delta \Vdash I(u)$.

Proof. (i) $\longrightarrow$ (ii) and (iii) $\longrightarrow$ (i) follow from Theorem 6.8.

The proof of (ii) $\longrightarrow$ (iii) is by induction on $\|\delta, u\|_I$, defined by

$\|\delta, u\|_I = 0$, if $u = \langle 0, e\rangle$ or $u = \langle 2, e\rangle$.
$\|\delta, \langle 1, e\rangle\|_I = \|\delta, e\|_I + 1$.

Let $\delta \Vdash^w I(u)$. If $u = \langle 0, e\rangle$, then $\delta \Vdash I(u)$ by definition. If $u = \langle 1, e\rangle$, then clearly $\delta \Vdash^w I(e)$ and hence, by the induction hypothesis, $\delta \Vdash I(e)$. So $\delta \Vdash I(\langle 1, e\rangle)$.

Suppose now that $u = \langle 2, e\rangle$. Let $\tau \supseteq \delta$ & $\tau \Vdash R_e(z)$ for some $z \in N$. Let $\rho \supseteq \tau$ and $\rho \Vdash I(\langle 2, e\rangle)$. Clearly $\rho \Vdash R_e(z)$, and hence, there exists a $\mu \supseteq \rho$ such that $\mu \Vdash I(z)$. Obviously $\mu \supseteq \tau$. So, we have that $\forall \tau \supseteq \delta(\tau \Vdash R_e(z) \implies \exists \mu \supseteq \tau(\mu \Vdash I(z)))$. Hence, $\delta \Vdash I(\langle 2, e\rangle)$. $\square$

Our next goal is to show that the set $Ind_{\mathfrak{A}}$ is complete with respect to the relatively intrinsically Π_1^1 sets. As in the previous cases, we shall carry out the proof for $n = 1$. Let f be an enumeration of $\mathfrak{A}$. It is well known that a set W is Π_1^1 in $D(\mathfrak{B}_f)$ iff W is many-one reducible to $\mathcal{O}_{D(\mathfrak{B}_f)}$ and hence, by Fact 2, iff W is many-one reducible to $Ind_{D(\mathfrak{B}_f)}$. So, W is Π_1^1 in $D(\mathfrak{B}_f)$ iff there exists a total recursive function h, such that:

$$x \in W \iff f \models I(h(x)).$$

Given $A \subseteq B$ and an enumeration f of $\mathfrak{A}$, call the recursive function h an *f-associate* of A if for all $x \in N$,

$$f(x) \in A \iff f \models I(h(x)).$$

Given finite part δ, $x \in N$ and $s \in \mathcal{R}(\delta, x)$, denote by $\delta * \langle x, s\rangle$ the finite part δ' such that $dom(\delta') = dom(\delta) \cup \{x\}$ and $\delta'(x) = s$.

6.12. Lemma. *Let $A \subseteq B$ be intrinsically relatively Π_1^1 on $\mathfrak{A}$. Then there exist δ and recursive function h such that for all $x \in N$ and all $s \in \mathcal{R}(\delta, x)$,*

$$s \in A \iff \delta * \langle x, s\rangle \Vdash I(h(x)).$$

Proof. Assume the contrary. Fix an arbitrary enumeration $h_0, h_1, \dots, h_e, \dots$ of the total binary recursive functions and consider the subset Q of $N \times \mathcal{E}$, defined by

$$(e, f) \in Q \iff h_e \text{ is not } f\text{-associate of } A.$$

We shall show, that Q is complete (with respect to the family $\mathcal{F}_I$).

Let $\delta \in \Delta$ and $e \in N$. By assumption, there exist $x \in N$ and $s \in \mathcal{R}(\delta, x)$ such that

$$s \in A \not\Leftrightarrow \delta * \langle x, s \rangle \Vdash I(h_e(x)).$$

We have two cases:

1. $s \in A$ and $\delta * \langle x, s \rangle \nVdash I(h_e(x))$. By Proposition 6.11, $\delta * \langle x, s \rangle \nVdash^w I(h_e(x))$, and, hence, there exists a $\tau \supseteq \delta * \langle x, s \rangle$, such that $\forall \rho \supseteq \tau (\rho \nVdash I(h_e(x))$.

Let f be $\mathcal{F}_I$-generic and $f \supseteq \tau$. Obviously $(e, f) \in Q$.

2. $s \notin A$ and $\delta * \langle x, s \rangle \Vdash I(h_e(x))$. Take $\tau = \delta * \langle x, s \rangle$. Clearly, for all $\mathcal{F}_I$-generic enumerations, if $f \supseteq \tau$ then $(e, f) \in Q$.

By the completeness of Q, there exist a $\mathcal{F}_I$-generic f such that for all e, $(e, f) \in Q$. The last contradicts the fact that A is relatively intrinsically Π^1_1. □

Let A be an relatively intrinsically Π^1_1 subset of B. According to the Lemma above, there exist finite part δ and recursive function h, such that for all $x \in N$, and $s \in \mathcal{R}(\delta, x)$,

$$s \in A \iff \delta * \langle x, s \rangle \Vdash I(h(x)).$$

Let domain of δ be $\{w_1, \dots, w_r\}$ and $\delta(w_i) = t_i$. Fix a $x_0 \notin \{w_1, \dots, w_r\}$. Then

$$\begin{aligned} s \in A \iff & s \neq t_1 \;\&\; \cdots \;\&\; s \neq t_r \;\&\; \delta * \langle x_0, s \rangle \Vdash I(h(x_0)) \\ & \text{or } s = t_1 \;\&\; \delta \Vdash I(h(w_1)) \\ & \dots \\ & \text{or } s = t_r \;\&\; \delta \Vdash I(h(w_r)). \end{aligned} \tag{6.1}$$

Let $i(s)$ be a function on B^* defined by

$$\begin{aligned} i(s) &= \langle \delta * \langle x_0, s \rangle, h(x_0) \rangle, \text{ if } s \neq t_1, \dots, s \neq t_r; \\ i(s) &= \langle \delta, h(w_1) \rangle, \text{ if } s = t_1, \\ &\vdots \\ i(s) &= \langle \delta, h(w_r) \rangle, \text{ if } s = t_r. \end{aligned}$$

Obviously, $s \in A \iff i(s) \in Ind_{\mathfrak{A}}$.

The function i has a very simple definition and it is obviously effective on B^*. Actually, i is Prime Computable on B^* relative to the empty set of givens [Mos69b].

So, we have proved the following theorem:

6.13. Theorem. *Let $A \subseteq B$. Then the following are equivalent:*

(i) *A is relatively intrinsically Π^1_1 on $\mathfrak{A}$;*

(ii) *A is inductive on $\mathfrak{A}^*$;*
(iii) *There exists a prime computable relative to $\{\emptyset\}$ function i such that*

$$s \in A \iff i(s) \in Ind_{\mathfrak{A}}.$$

7. A hierarchy for the Hyperelementary sets.

One of the main results in the classical theory of the hyperarithmetical sets is the Suslin-Kleene theorem which allows us to construct the Δ^1_1-sets starting from the r.e. sets and iterating the operations taking the complement and effective union.

As shown in [Mos74], it is not possible to obtain a direct generalization of this theorem on arbitrary denumerable abstract structures.

Here we shall transfer the Suslin-Kleene theorem as far as possible using enumerations.

First we shall attach to each index $(\delta, u) \in Ind_{\mathfrak{A}}$ a set $H_{\delta,u}$ in a way very similar to that used in the proof of the Suslin-Kleene theorem given in [Sho67].

After that we shall show that the sets $H_{\delta,u}$ coincide with the hyperelementary ones. To avoid some technical complications, we shall formulate and prove our results only for subsets of B. However, all results can be easily generalized for subsets of B^n, $n \geqq 1$.

Recall the definition of the forcing $\tau \Vdash R_e(y)$ from section 4. Clearly the set $G_{\delta,e} = \{(\tau, y) : \tau \supseteq \delta \ \&\ \tau \Vdash R_e(y)\}$ is semi-computable on $\mathfrak{A}^*$.

Denote by $S_{\delta,e}$ the set $\{s : \exists x \exists \tau \supseteq \delta(\tau(x) = s \ \&\ \tau \Vdash R_e(x))\}$. From the characterization of the semi-computable sets in section 4., it follows that the sets $S_{\delta,e}$ coincide with the semi-computable subsets of B.

The sets $H_{\delta,u}$ are defined for all δ, u s.t. $\delta \Vdash I(u)$ by means of induction on $|\delta, u|_I$.

7.1. Definition.

(1) $H_{\delta,\langle 0,e\rangle} = S_{\delta,e}$;

(2) $H_{\delta,\langle 1,e\rangle} = \bigcup\limits_{\tau \supseteq \delta} B \setminus H_{\tau,e}$;

(3) $H_{\delta,\langle 2,e\rangle} = \bigcup\limits_{\substack{(\tau,y)\in G_{\delta,e} \\ |\tau,y|_I < |\delta,\langle 2,e\rangle|_I}} H_{\tau,y}.$

Comment. The problematic clause in the above definition is the third one. We are forced to use the restriction $|\tau, y|_I < |\delta, \langle 2, e\rangle|_I$ because, in general, we can not assert that $\delta \Vdash I(\langle 2, e\rangle) \ \&\ \tau \supseteq \delta \ \&\ \tau \Vdash R_e(y)$, implies $|\tau, y|_I < |\delta, \langle 2, e\rangle|_I$.

So in the third clause, we have a hyperelementary union but not a semi-computable one. The fact that the set $\{(\tau, y) : (\tau, y) \in G_{\delta,e} \ \&\ |\tau, y|_I < |\delta, \langle 2, e\rangle|_I\}$ is hyperelementary follows directly from the Stage comparison theorem [Mos74].

On the other hand, using Proposition 6.11, one can easily show that for all $(\tau, y) \in G_{\delta,e}$, $\tau \Vdash I(y)$ and, as we shall see later, we have for all δ and e,

$$H_{\delta,\langle 2,e\rangle} = \bigcup_{(\tau,y)\in G_{\delta,e}} H_{\tau,y}.$$

Of course, since we want an inductive definition of the sets $H_{\delta,u}$, we can not replace the third clause by the last equality.

Let H be a new binary predicate symbol. Given an enumeration f and natural numbers u, x such that $f \models I(u)$, we define the relation $f \models^1 H(u,x)$ by means of induction on $|f, u|_I$. Here the meaning of $f \models R_e(x)$ is $x \in W_e^{D(\mathfrak{B}_f)}$ as usual.

(1) If $f \models R_e(x)$, then $f \models^1 H(\langle 0, e\rangle, x)$;
(2) If $f \not\models^1 H(e, x)$, then $f \models^1 H(\langle 1, e\rangle, x)$;
(3) If $\exists y(f \models R_e(y)\ \&\ f \models^1 H(y, x))$, then $f \models^1 H(\langle 2, e\rangle, x)$.

Let $f \models H(u,x) \iff f \models I(u)\ \&\ f \models^1 H(u,x)$. Using the Suslin-Kleene theorem in the form given in [Sho67], relativized to $D(\mathfrak{B}_f)$, we obtain that a set W is hyperarithmetical relative to $D(\mathfrak{B}_f)$ iff there exists a u, such that $W = \{x : f \models H(u,x)\}$.

For finite parts, we define the relation $\delta \Vdash^1 H(u,x)$, again for those u such that $\delta \Vdash I(u)$, by induction on $|\delta, u|_I$.

(1) If $\delta \Vdash R_e(x)$, then $\delta \Vdash^1 H(\langle 0, e\rangle, x)$;
(2) If $\forall \rho \supseteq \delta(\rho \nVdash^1 H(e, x))$, then $\delta \Vdash^1 H(\langle 1, e\rangle, x)$;
(3) If $\exists y(\delta \Vdash R_e(y)\ \&\ \forall \tau \supseteq \delta(|\tau, y|_I < |\tau, \langle 2, e\rangle|_I)\ \&\ \delta \Vdash^1 H(y, x))$, then $\delta \Vdash^1 H(\langle 2, e\rangle, x)$.

Let $\delta \Vdash H(u,x) \iff \delta \Vdash I(u)\ \&\ \delta \Vdash^1 H(u,x)$.

The following lemma is obvious:

7.2. Lemma. *If $\delta \Vdash H(u,x)$ and $\delta \subseteq \tau$, then $\tau \Vdash H(u,x)$.*

Let $\mathfrak{F}_H = \mathfrak{F}_I \cup \{\{\rho : \rho \Vdash H(u,x)\} : u, x \in N\}$.

7.3. Lemma. *For each $\mathfrak{F}_H$-generic enumeration f,*

$$f \models H(u,x) \iff \exists \delta \subseteq f(\delta \Vdash H(u,x)). \tag{7.1}$$

Proof. If $f \not\models I(u)$, then (7.1) is obvious.

Let $Ind_f = \{u : f \models I(u)\}$. For each element u of Ind_f set

$$|u| = min(|\delta, u|_I : \delta \subseteq f\ \&\ \delta \Vdash I(u)).$$

We shall prove (7.1) for the elements u of Ind_f by means of transfinite induction on $|u|$.

We have to consider three cases:

(a) $u = \langle 0, e\rangle$. Then $f \models H(u,x) \iff f \models R_e(x)$ and $\delta \Vdash H(u,x) \iff \delta \Vdash R_e(x)$. From here (7.1) follows directly.

(b) $u = \langle 1, e\rangle$. Let $\delta_0 \subseteq f$ be such that $|u| = |\delta_0, u|_I$. We have that $|\delta_0, e|_I < |\delta_0, \langle 1, e\rangle|_I$, and hence, $|e| < |u|$.

Suppose that for some δ, $\delta \subseteq f$ and $\delta \Vdash H(u, x)$. Assume $f \not\models H(u, x)$. Then $f \models H(e, x)$, and hence, by induction there exists $\rho \subseteq f$, s.t. $\rho \Vdash H(e, x)$. We can assume that $\delta \subseteq \rho$. The last contradicts $\delta \Vdash H(u, x)$.

Suppose that $f \models H(u, x)$ but $\forall \delta \subseteq f(\delta \nVdash H(u, x))$. From here it follows that $\{\rho : \rho \Vdash H(e, x)\}$ is dense in f. By genericity there exists $\rho \subseteq f$ such that $\rho \Vdash H(e, x)$. By induction, $f \models H(e, x)$. A contradiction.

(c) $u = \langle 2, e\rangle$. Let $\delta_0 \subseteq f$ be such that $|\langle 2, e\rangle| = |\delta_0, \langle 2, e\rangle|_I$.

Suppose that for some $\delta \subseteq f$, $\delta \Vdash H(u, x)$. Then there exists a y s.t. $\delta \Vdash R_e(y)$, $\delta \Vdash H(y, x)$ and for all $\tau \supseteq \delta$, $|\tau, y|_I < |\tau, \langle 2, e\rangle|_I$.

Take a $\tau \subseteq f$, such that $\delta_0 \subseteq \tau$, and $\delta \subseteq \tau$. Using Lemma 6.5, we obtain $|\tau, y|_I < |\tau, \langle 2, e\rangle|_I \leqq |\delta_0, \langle 2, e\rangle|_I$. So $|y| < |\langle 2, e\rangle|$. By induction, $f \models H(y, x)$. Since $\delta \Vdash R_e(y)$, $f \models R_e(y)$. Therefore, $f \models H(\langle 2, e\rangle, x)$.

Assume now that $f \models H(\langle 2, e\rangle, x)$. Then for some $y \in N$, $f \models R_e(y)$ and $f \models H(y, x)$. Consider the element $X = \{\rho : |\rho, y|_I < |\delta_0, \langle 2, e\rangle|_I\}$. We have that X is dense in f. Indeed, take a $\tau \subseteq f$. We can assume that $\delta_0 \subseteq \tau$ and $\tau \Vdash R_e(y)$. Since $\delta_0 \Vdash I(\langle 2, e\rangle)$, there exists a $\rho \supseteq \tau$ s.t. $\rho \Vdash I(y)$ and $|\rho, y|_I < |\delta_0, \langle 2, e\rangle|_I$.

By genericity, $|y| < |\langle 2, e\rangle|$. Fix a $\rho_0 \subseteq f$ such that $|y| = |\rho_0, y|_I$.

By induction, there exists a $\lambda \subseteq f$, such that $\lambda \Vdash H(y, x)$ and $\lambda \Vdash R_e(y)$. Finally, assume that $\forall \mu \subseteq f \exists \tau \supseteq \mu(|\tau, \langle 2, e\rangle|_I < |\delta_0, \langle 2, e\rangle|_I)$. From here, by genericity, there exists a $\tau \subseteq f$ such that $|\tau, \langle 2, e\rangle|_I < |\delta_0, \langle 2, e\rangle|_I$. The last contradicts the choice of δ_0. So, for some $\mu \subseteq f$, $\forall \tau \supseteq \mu(|\tau, \langle 2, e\rangle|_I \geqq |\delta_0, \langle 2, e\rangle|_I)$.

Let $\delta \subseteq f$ and let δ majorize $\delta_0, \rho_0, \lambda, \mu$. We shall show that $\delta \Vdash H(\langle 2, e\rangle, x)$. Since $\delta \supseteq \delta_0$, $\delta \Vdash I(\langle 2, e\rangle)$. Since $\delta \supseteq \lambda$, $\delta \Vdash R_e(y)$ and $\delta \Vdash H(y, x)$. Let $\tau \supseteq \delta$. Then $\tau \supseteq \rho_0$, and $\tau \supseteq \mu$. We have $|\tau, y|_I \leqq |\rho_0, y|_I$ and $|\delta_0, \langle 2, e\rangle|_I \leqq |\tau, \langle 2, e\rangle|_I$. Combining both, we obtain

$$|\tau, y|_I \leqq |\rho_0, y|_I < |\delta_0, \langle 2, e\rangle|_I \leqq |\tau, \langle 2, e\rangle|_I.$$

So, $|\tau, y|_I < |\tau, \langle 2, e\rangle|_I$. $\square$

7.4. Lemma. *Let $A \subseteq B$ be relatively intrinsically HYP on $\mathfrak{A}$. There exist δ and u such that $\delta \Vdash I(u)$ and for all $x \in N$ and $s \in \mathcal{R}(\delta, x)$,*

$$s \in A \iff \exists \tau \supseteq \delta(\tau(x) = s \ \&\ \tau \Vdash H(u, x)). \tag{7.2}$$

Proof. Assume the contrary. Consider

$$Q = \{(u, f) : f \not\models I(u) \text{ or } \exists x(f(x) \in A \not\Leftrightarrow f \models H(u, x))\}.$$

We shall show that Q is complete (with respect to the family $\mathcal{F}_H$).

Let δ and u be given. By assumption $\delta \nVdash I(u)$ or (7.2) fails for some $x \in N$ and some $s \in \mathcal{R}(\delta, x)$.

1. Let $\delta \nVdash I(u)$. Then, by Proposition 6.11, $\delta \nVdash_w I(u)$, and hence, for some $\tau \supseteq \delta$, $\forall \rho \supseteq \tau(\rho \nVdash I(u))$. Let $f \supseteq \tau$ and f be $\mathfrak{F}_H$-generic. Then clearly, $f \not\models I(u)$, and hence, $(u, f) \in Q$.

2. Let $\delta \Vdash I(u)$. We have two subcases.

2.1. $s \in A$ and $\forall \tau \supseteq \delta(\tau(x) = s \Longrightarrow \tau \nVdash H(u, x))$. Take a $\delta' \supseteq \delta$ such that $\delta'(x) = s$. Clearly, if $f \supseteq \delta'$ is $\mathfrak{F}_H$-generic, then $(u, f) \in Q$.

2.2. $s \notin A$ and for some $\tau \supseteq \delta$, $\tau(x) = s$ and $\tau \Vdash H(u, x)$. Let $f \supseteq \tau$ be $\mathfrak{F}_H$-generic. Then obviously, $f \models H(u, x)$ and $f(x) = s \notin A$. So, $(u, f) \in Q$.

By Proposition 3.7, it follows from here that there exists a f s.t. $(u, f) \in Q$, for all u. Clearly A does not have a HYP associate in this f. A contradiction. $\square$

7.5. Definition. Given $(\delta, u) \in Ind_{\mathfrak{A}}$, define the set

$$A_{\delta,u} = \{s : \exists x \exists \tau \supseteq \delta(\tau(x) = s \ \&\ \tau \Vdash H(u, x))\}.$$

7.6. Theorem. *The following are equivalent for all $A \subseteq B$:*

(i) *A is relatively intrinsically HYP on $\mathfrak{A}$;*
(ii) *A is hyperelementary on $\mathfrak{A}^*$;*
(iii) *There exists $(\delta, u) \in Ind_{\mathfrak{A}}$ s.t. $A = A_{\delta,u}$.*

Proof. The equivalence (i) $\longleftrightarrow$ (ii) is proven in Corollary 5.10.

The implication (i) $\longrightarrow$ (iii), and hence, the implication (ii) $\longrightarrow$ (iii) follow from the previous Lemma.

It remains to show that each of the sets $A_{\delta,u}$ is hyperelementary on $\mathfrak{A}^*$. For we can define an inductive on $\mathfrak{A}^*$ relation $R(\delta, u, x, i)$, such that if $\delta \Vdash I(u)$, then $R(\delta, u, x, 0) \Longleftrightarrow \delta \Vdash H(u, x)$ and $R(\delta, u, x, 1) \Longleftrightarrow \delta \nVdash H(u, x)$. The definition of R follows the definition of the relation "$\Vdash^1$". The only nontrivial moment is to translate the part $\forall \tau \supseteq \delta(|\tau, y|_I < |\tau, \langle 2, e\rangle|_I)$ of the third clause of the definition. This can be done by means of the Stage comparison theorem [Mos74].

Now, having the inductive relation R we can define:

$$s \in A_{\delta,u} \Longleftrightarrow \exists x \exists \tau \supseteq \delta(\tau(x) = s \ \&\ R(\tau, u, x, 0))$$

$$s \in B \backslash A_{\delta,u} \Longleftrightarrow \forall x \forall \tau \supseteq \delta(\tau(x) \neq s \ \vee\ R(\tau, u, x, 1)).$$

The last two equivalencies show that both $A_{\delta,u}$ and the complement of $A_{\delta,u}$ are inductive, and hence that $A_{\delta,u}$ is hyperelementary on $\mathfrak{A}^*$. $\square$

7.7. Lemma. *The sets $A_{\delta,\langle 0,e\rangle}$ coincide with the semi-computable on $\mathfrak{A}$ sets.*

Given a subset A of B, denote by $\overline{A}$ the set $B \backslash A$.

7.8. Lemma. *Let $\delta \Vdash I(\langle 1, e\rangle)$. Then $A_{\delta,\langle 1,e\rangle} = \bigcup_{\tau \supseteq \delta} \overline{A}_{\tau,e}$.*

Proof. Let $s \in A_{\delta,\langle 1,e\rangle}$. Then for some $\tau \supseteq \delta$, and some x, $\tau(x) = s$ and $\tau \Vdash H(\langle 1,e\rangle, x)$. Hence, $\forall\rho \supseteq \tau(\rho \nVdash H(e,x))$, and therefore, $s \in \overline{A}_{\tau,e}$.

Suppose now, that $s \in \overline{A}_{\tau,e}$ for some $\tau \supseteq \delta$. Then

$$\forall\rho \supseteq \tau\forall x(\rho(x) = s \Longrightarrow \rho \nVdash H(e,x)).$$

Clearly, there exists $x_0 \in N$ and extension δ' of τ s.t. $\delta'(x_0) = s$. We have that $\delta' \Vdash H(\langle 1,e\rangle, x_0)$. So, $s \in A_{\delta,\langle 1,e\rangle}$. □

7.9. Lemma. *Let $\delta \Vdash I(\langle 2,e\rangle)$. Then*

$$A_{\delta,\langle 2,e\rangle} = \bigcup_{\substack{(\tau,y)\in G_{\delta,e}\\ |\tau,y|_I<|\delta,\langle 2,e\rangle|_I}} A_{\tau,y} = \bigcup_{(\tau,y)\in G_{\delta,e}} A_{\tau,y}.$$

Proof. Suppose that $(\tau,y) \in G_{\delta,e}$, i.e. $\tau \supseteq \delta$ and $\tau \Vdash R_e(y)$. Let $s \in A_{\tau,y}$. Then for some $\rho \supseteq \tau$ and some $x \in N$, $\rho(x) = s$ and $\rho \Vdash H(y,x)$.

Let f be an $\mathfrak{F}_H$-generic enumeration s.t. $f \supseteq \rho$. Then $f \supseteq \delta$, and hence, $f \models I(\langle 2,e\rangle)$. Since $f \supseteq \rho$, we have that $f \models R_e(y)$ and $f \models H(y,x)$. Hence $f \models H(\langle 2,e\rangle, x)$. Then, there exists a μ, s.t. $\rho \subseteq \mu \subseteq f$ and $\mu \models H(\langle 2,e\rangle, x)$. From here it follows that $s \in A_{\delta,\langle 2,e\rangle}$.

Let $s \in A_{\delta,\langle 2,e\rangle}$. Then for some $x \in N$ and some $\tau \supseteq \delta$, $\tau(x) = s \;\&\; \tau \Vdash H(\langle 2,e\rangle, x)$. By the definition of "$\Vdash^1$", there exists y such that $\tau \Vdash H(y,x)$, $\tau \Vdash R_e(y)$ and $|\tau,y|_I < |\tau,\langle 2,e\rangle|_I \leqq |\delta,\langle 2,e\rangle|_I$. Clearly $s \in A_{\tau,y}$. □

7.10. Theorem. *For all $(\delta,u) \in Ind_{\mathfrak{A}}$, $A_{\delta,u} = H_{\delta,u}$.*

Proof. Transfinite induction on $|\delta,u|_I$, using Lemmas 7.7 – 7.9. □

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