EFFECTIVE PROPERTIES OF MARKER'S EXTENSIONS

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Abstract. The paper is devoted to the study of Marker's extensions of sequences of countable structures. In the first part of the paper definability properties of the Marker's extension are investigated. The results demonstrate that for any sequence of structures the Marker's extension codes the elements of the sequence so that the n-th structure of the sequence appears positively at the n-th level of the definability hierarchy. In the second part we study the spectra of the Marker's extensions. The results provide a general method given a sequence of structures, to construct a structure with n-th jump spectrum contained in the spectrum of the n-th member of the sequence. As an application a structure with spectrum consisting of the Turing degrees which are non-low$_n$ for all $n < \omega$ is obtained.

1. Introduction

There is a close parallel between notions of classical computability theory and of the theory of effective definability in abstract structures. For example, the notion of "c.e. in" corresponds to the notion of Σ_1 definability, where a set is Σ_1 definable in a structure $\mathfrak{A}$ if it is definable by means of some computable Σ_1 infinitary formula with finitely many parameters. More generally, for all n the Σ^0_{n+1} sets correspond to the sets definable by means of computable Σ_{n+1} formulae. This correspondence is made explicit by the external characterization of the Σ_{n+1} definable sets obtained independently in [2] and [4] which states that a set R is Σ_{n+1} definable in a structure $\mathfrak{A}$ if for every copy $\mathfrak{B}$ of $\mathfrak{A}$ on the natural numbers the respective image of R is Σ^0_{n+1} in the diagram of $\mathfrak{B}$. The last result provides a way to study other classical computability notions in an abstract setting. A natural choice is the notion of enumeration reducibility which can be considered as a generalization of the relation "c.e. in". Say that a set R is *enumeration reducible* to a structure $\mathfrak{A}$ if for each copy $\mathfrak{B}$ of $\mathfrak{A}$ on the natural numbers the respective image of R is enumeration reducible to the positive diagram of $\mathfrak{B}$. Using the same methods as in [2] and [4] one can show that R is enumeration reducible to $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ by means of some positive computable Σ_1 formula with finitely many parameters.

The introduction of the new notion of enumeration reducibility of a set to a structure, gives rise to the question about the novelty of this notion. More precisely it is natural to ask the following:

Question 1. *Given a structure $\mathfrak{A}$, does there exist a structure $\mathfrak{M}$ such that the enumeration reducible to $\mathfrak{A}$ sets coincide with the Σ_1 definable in $\mathfrak{M}$ sets?*

1991 *Mathematics Subject Classification.* 03D45.

Key words and phrases. Marker's extensions, enumeration degrees, spectra of structures.

This research was partially supported by Sofia University Science Fund and a NSF grant DMS-1101123.

In the classical case the answer of the respective question is negative. There exist sets A such that for no set M the sets enumeration reducible to A coincide with the sets c.e. in M.

Next step is to consider sequences of sets. The guiding idea in the following definitions is to think of a sequence of sets as enumerated so that each member of the sequence appears at a level in the arithmetical hierarchy strictly higher than the levels of its predecessors. From this point of view the simplest sequences are generated by a single set and are of the form $\{B^{(n)}\}$, where $B \subseteq \mathbb{N}$. Such sequences are identified with the set which generates them.

1.1. Definition. A sequence of $\mathcal{X} = \{X_n\}_{n<\omega}$ of sets of natural numbers is c.e. in the set B if $(\forall n)(X_n$ is c.e. in $B^{(n)}$ uniformly in $n)$.

Given a sequence $\mathcal{R} = \{R_n\}$ of subsets of the domain of a structure $\mathfrak{A}$, say that $\mathcal{R}$ *is c.e. in* $\mathfrak{A}$ if for each copy $\mathfrak{B}$ of $\mathfrak{A}$ on the natural numbers the respective image of the sequence $\mathcal{R}$ is c.e. in the diagram of $\mathfrak{B}$. Using again the methods from [2, 4] one can show that $\mathcal{R}$ is c.e.in $\mathfrak{A}$ if and only if there exist a computable sequence $\{F_n\}$ of computable Σ_{n+1} formulae and parameters $t_1, \ldots, t_m$ such that for all n, R_n is definable in $\mathfrak{A}$ by means of the formula F_n with parameters $t_1, \ldots, t_m$.

Enumeration reducibility is further generalized in [17, 3, 1] to a notion of enumeration reducibility of sets to sequences of sets and to a notion of enumeration reducibility of sequences of sets to sequences of sets. The starting point of these generalizations is Selman's Theorem [17] which states that the set X is enumeration reducible to the set Y if for all sets B, Y is c.e. in B implies X is c.e. in B. The following definition in a different notation is given by Ash in [1]:

1.2. Definition.

(i) Given a set X of natural numbers and a sequence $\mathcal{Y}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets B, $\mathcal{Y}$ is c.e. in B implies X is Σ^0_{n+1} in B;

(ii) Given sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers, say that $\mathcal{X}$ is ω-enumeration reducible to $\mathcal{Y}$ ($\mathcal{X} \leq_\omega \mathcal{Y}$) if for all sets B, $\mathcal{Y}$ is c.e. in B implies $\mathcal{X}$ is c.e. in B.

In [1] Ash presents a characterization of "$\leq_n$" and "$\leq_\omega$" using computable infinitary propositional formulae. Another characterization in terms of enumeration reducibility is obtained in [19] and [23].

To transfer the above reducibilities to the abstract case consider a sequence of countable structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; P^n_1, \ldots, P^n_{m_n})$. We shall consider here only sequences of structures of first order languages L_n which are computable uniformly in n.

Set $A = \bigcup_n A_n$.

1.3. Definition. An enumeration of A is a bijective mapping f of the set of the natural numbers $\mathbb{N}$ onto A. Given $R \subseteq A^m$ and enumeration f of A, set

$$f^{-1}(R) = \{\langle x_1, \ldots, x_m\rangle | (f(x_1), \ldots, f(x_m)) \in R\}$$

For every enumeration f of A, let

$$f^{-1}(\mathfrak{A}_n) = f^{-1}(A_n) \oplus f^{-1}(P^n_1) \oplus \cdots \oplus f^{-1}(P^n_{m_n}).$$

The set $f^{-1}(\mathfrak{A}_n)$ is actually the positive diagram of the copy of $\mathfrak{A}_n$ under the isomorphism f^{-1}. Notice that if D is the diagram of this copy then $D \equiv_T f^{-1}(\mathfrak{A}_n)$.

Thus every enumeration f of A determines a sequence $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}$ of sets of natural numbers.

Given a subset R of A, say that $R \leq_n \vec{\mathfrak{A}}$ if for every enumeration f of A, $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$. This reducibility has been studied in [22] where it is shown that a set $R \leq_n \vec{\mathfrak{A}}$ if and only if R is definable by means of a positive computable Σ_{n+1} formula built up from the predicates of the structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ and such that the predicates of the structure $\mathfrak{A}_k$ appear for the first time at $k+1$-th level. Such formulae are called Σ^+_{n+1}. Again it is natural to ask the following:

Question 2. *Given a sequence of countable structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$ such that the Σ_{n+1} definable in $\mathfrak{M}$ sets coincide with sets $R \leq_n \vec{\mathfrak{A}}$?*

The respective question about the reducibility $\leq_n$ on the natural numbers has obviously a negative answer.

Finally say that a sequence $\mathcal{R} = \{R_n\}$ of subsets of A is *ω-enumeration reducible* to the sequence of structures $\vec{\mathfrak{A}}$ if for every enumeration f of A, $f^{-1}(\mathcal{R}) \leq_\omega f^{-1}(\vec{\mathfrak{A}})$. Using the methods from [22] one can show that $\mathcal{R} \leq_\omega \vec{\mathfrak{A}}$ if and only if for all n, R_n is Σ^+_{n+1} definable uniformly in n. Now we can state our third question:

Question 3. *Given a sequence $\vec{\mathfrak{A}}$ of structures, does there exist a structure $\mathfrak{M}$ such that the sequences which are ω-enumeration reducible to $\vec{\mathfrak{A}}$ coincide with the c.e.in $\mathfrak{M}$ sequences?*

In the case of computability on the natural numbers the third question could be stated as follows. Given a sequence of sets of natural numbers $\mathcal{A}$, does there exist a set M such that for all sequences $\mathcal{Y}$, $\mathcal{Y} \leq_\omega \mathcal{A} \iff \mathcal{Y} \leq_{r.e.} M$? Again the answer of the last question is negative in the general case.

Contrary to the situation in the case of computability on the natural numbers all the three questions have positive answers. Actually there is a relatively simple model-theoretic construction that does the job in all the three cases. The construction is based on the so called Marker's extensions defined by D. Marker in [14]. In the context of effective model theory Marker's extensions are introduced by Goncharov and Khoussainov in [10] and recently used in the proofs of jump inversion theorems for degree spectra of structures in [26, 27] and in the study of the degrees of categoricity in [7].

In the first part of the paper we define the Marker's extension corresponding to a sequence $\vec{\mathfrak{A}}$ and show that it gives a positive answer to the three questions.

The second part of the paper is devoted to the study of the spectra of Marker's extensions. Here we shall adopt the following definition of spectrum of a structure, which is equivalent to the usual one in the non-trivial cases but has the advantage that the spectra are always upwards closed sets of degrees.

1.4. Definition. Let $\mathfrak{A}$ be a countable structure. *The spectrum of* $\mathfrak{A}$ is the set of Turing degrees $Sp(\mathfrak{A}) = \{\mathbf{a} | \mathbf{a}$ computes the diagram of an isomorphic copy of $\mathfrak{A}\}$.

One may generalize the notion of spectrum of a structure to the more general case of sequences of structures in at least two ways:

1.5. Definition. Given a sequence $\vec{\mathfrak{A}}$ of countable structures,

(i) set $A = \bigcup |\mathfrak{A}_n|$ and let *the relative spectrum of* $\vec{\mathfrak{A}}$ to be the set of sequences of sets of natural numbers $Rsp(\vec{\mathfrak{A}}) = \{\{f^{-1}(\mathfrak{A}_n)\} | f$ is an enumeration of $A\}$;

(ii) *let the joint spectrum of* $\vec{\mathfrak{A}}$ to be the set of sequences of sets of natural numbers $Jsp(\vec{\mathfrak{A}}) = \{\{f_n^{-1}(\mathfrak{A}_n)\} | (\forall n)(f_n$ is an enumeration of $|\mathfrak{A}_n|)\}$.

Our main result states that for every sequence $\vec{\mathfrak{A}}$ of structures one can define structures $\mathfrak{M}_R$ and $\mathfrak{M}_J$ such that $Sp(\mathfrak{M}_R) = \{d_T(B) | (\exists \mathcal{Y} \in Rsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B)\}$ and $Sp(\mathfrak{M}_J) = \{d_T(B) | (\exists \mathcal{Y} \in Jsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B)\}$

We present several applications of the last result including an example of a structure $\mathfrak{M}$ such that $Sp(\mathfrak{M}) = \{\mathbf{a} | (\forall n)(\mathbf{0}^{(n)} \neq \mathbf{a}^{(n)})\}$.

2. Preliminaries

We shall assume fixed a Gödel enumeration $W_0, \dots, W_a, \dots$ of the computably enumerable sets. By D_v we shall denote the finite set with canonical code v. Each c.e. set W_a determines an enumeration operator $W_a : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$ defined as follows.

2.1. Definition. Given two sets X and Y of natural numbers, let

$$X = W_a(Y) \iff (\forall x)(x \in X \iff (\exists v)(\langle x, v\rangle \in W_a \wedge D_v \subseteq Y)).$$

The enumeration operators are closed with respect to composition. Moreover there exists a computable function $\lambda(a, b)$ such that for all $Y \subseteq \mathbb{N}$, $W_a(W_b(Y)) = W_{\lambda(a,b)}(Y)$.

For $A, B \subseteq \mathbb{N}$, $A \leq_e B$ (*A is enumeration reducible to B*) if there exists a c.e. set W such that $A = W(B)$. Let $A \equiv_e B \iff A \leq_e B \;\&\; B \leq_e A$. The relation $\equiv_e$ is an equivalence relation and the respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult [6].

For every set A of natural numbers let $A^+ = A \oplus (\mathbb{N} \setminus A)$. Clearly a set B is c.e. in A if and only if $B \leq_e A^+$. Moreover there exist computable functions λ and μ such that for all $a \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $W_a^A = W_{\lambda(a)}(A^+)$ and $W_a(A^+) = W^A_{\mu(a)}$. A set A is *total* if $A \equiv_e A^+$. Clearly for total sets A, the sets enumeration reducible to A coincide with the c.e. in A sets.

The enumeration jump operator is defined in [5] and further studied in [15]. Here we shall use the following definition of the enumeration jump which is m-equivalent to the original one, see [15].

2.2. Definition. Given a set A of natural numbers, set $L_A = \{\langle a, x\rangle : x \in W_a(A)\}$ and let the *enumeration jump* of A be the set L_A^+.

Here we shall prove some simple properties of the enumeration jump which will be used in the rest of the paper. To avoid any misunderstanding from now on we shall use the notation A' only to denote the enumeration jump of A.

For every set A let $\langle \chi_A \rangle = \{\langle x, 1\rangle : x \in A\} \cup \{\langle x, 0\rangle : x \in \mathbb{N} \setminus A\}$.

It is easy to see that A^+ is uniformly enumeration equivalent to $\langle \chi_A\rangle$, i.e. there exists enumeration operators Φ_1 and Φ_2 such that $A^+ = \Phi_1(\langle \chi_A\rangle)$ and $\langle \chi_A\rangle = \Phi_2(A^+)$.

The next proposition shows that $\langle \chi_A\rangle$ and A^+ are uniformly e-reducible to A'.

2.3. Proposition. *There exists an enumeration operator W_a such that $\langle \chi_A\rangle = W_a(A')$ for all $A \subseteq \mathbb{N}$.*

Proof. Let a_0 be an index of the c.e. set $\{\langle x, \{x\}\rangle : x \in \mathbb{N}\}$. Clearly $W_{a_0}(A) = A$ for every $A \subseteq \mathbb{N}$. Then

$$2\langle a_0, x\rangle \in A' \iff x \in W_{a_0}(A) \iff x \in A \text{ and}$$
$$2\langle a_0, x\rangle + 1 \in A' \iff x \notin W_{a_0}(A) \iff x \notin A.$$

Let a be an index of the r.e set

$$\{\langle\langle x,1\rangle,\{2\langle a_0,x\rangle\}\rangle : x\in\mathbb{N}\}\cup\{\langle\langle x,0\rangle,\{2\langle a_0,x\rangle+1\}\rangle : x\in\mathbb{N}\}.$$

Clearly $W_a(A') = \langle\chi_A\rangle$. □

Given a set A of natural numbers, denote by $J_T(A) = K^A$ the Turing jump of A. Recall that

$$K^A = \{\langle e,x\rangle : \{e\}^A(x) \text{ is defined}\}.$$

2.4. Proposition. *There exists an enumeration operator Φ such that $K^A = \Phi(A')$ for all $A\subseteq\mathbb{N}$.*

Proof. Rewriting the definition of K^A we get

$$K^A = \{\langle e,x\rangle : (\exists \text{ finite function } \theta)(\{e\}^\theta(x) \text{ is defined and } \theta\subseteq\chi_A)\}.$$

From here we get immediately that there exists an enumeration operator Φ_0 such that for all $A\subseteq\mathbb{N}$, $K^A = \Phi_0(\langle\chi_A\rangle)$. By Proposition 2.3 there exists an enumeration operator Φ_1 such that for all $A\subseteq\mathbb{N}$, $\langle\chi_A\rangle = \Phi_1(A')$. So we may define $\Phi(A) = \Phi_0(\Phi_1(A))$. □

The following Proposition shows that the jump is preserved under the standard embedding ι of the Turing degrees into the enumeration degrees defined by $\iota(d_T(A)) = d_e(A^+)$.

2.5. Proposition. *For all $A\subseteq\mathbb{N}$, $(A^+)'\equiv_e K_A^+$ uniformly in A.*

Proof. Let us fix computable functions λ,μ such that for all $a\in\mathbb{N}$ and $A\subseteq\mathbb{N}$, $W_a^A = W_{\lambda(a)}(A^+)$ and $W_a(A^+) = W^A_{\mu(a)}$. Now we have

$$\langle a,x\rangle\in L_{A^+}\iff x\in W_a(A^+)\iff x\in W^A_{\mu(a)}\iff\langle\mu(a),x\rangle\in K_A.$$

From here one can easily construct an enumeration operator Φ_T such that $(A^+)' = L^+_{A^+} = \Phi_T(K_A^+)$ for all $A\subseteq\mathbb{N}$.

Similarly, we have that for all $a\in\mathbb{N}$ and for all $A\subseteq\mathbb{N}$,

$$\langle a,x\rangle\in K_A\iff x\in W_a^A\iff x\in W_{\lambda(a)}(A^+)\iff\langle\lambda(a),x\rangle\in L_{A^+}.$$

and hence there exists an enumeration operator Φ_E such that $K_A^+ = \Phi_E((A^+)')$ for all $A\subseteq\mathbb{N}$. □

2.6. Proposition. *There exists a computable function g such that for all $e\in\mathbb{N}$ and $B\subseteq\mathbb{N}$, $W_e(B)' = W_{g(e)}(B')$.*

Proof. Consider a computable function λ such that for every a and e, $W_a(W_e(B)) = W_{\lambda(a,e)}(B)$. Then

$$2\langle a,x\rangle\in W_e(B')\iff 2\langle\lambda(a,e),x\rangle\in B' \text{ and}$$
$$2\langle a,x\rangle+1\in W_e(B)'\iff 2\langle\lambda(a,e),x\rangle+1\in B'.$$

Let g be a computable function yielding for every e an index of the r.e set

$$\{\langle 2\langle a,x\rangle,\{2\langle\lambda(a,e),x\rangle\}\rangle : a,x\in\mathbb{N}\}\cup\{\langle 2\langle a,x\rangle+1,\{2\langle\lambda(a,e),x\rangle+1\}\rangle : a,x\in\mathbb{N}\}.$$

Then for all e, $W_e(B)' = W_{g(e)}(B')$. □

2.7. Definition. Let $\mathcal{Y} = \{Y_n\}_{n<\omega}$ and $\mathcal{Z} = \{Z_n\}_{n<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{Y}$ *is enumeration reducible to* $\mathcal{Z}$ ($\mathcal{Y} \leq_e \mathcal{Z}$) if for all n, $Y_n \leq_e Z_n$ uniformly in n i.e. there exists a computable function γ such that $(\forall n)(Y_n = W_{\gamma(n)}(Z_n))$.

Notice that if $\mathcal{Y} \leq_e \mathcal{Z}$ then we may assume that there is a primitive recursive function γ such that for all n, $Y_n = W_{\gamma(n)}(Z_n)$.

Clearly "$\leq_e$" is a reflexive and transitive relation on the sequences of sets of natural numbers. Set $\mathcal{Y} \equiv_e \mathcal{Z} \iff \mathcal{Y} \leq_e \mathcal{Z} \wedge \mathcal{Z} \leq_e \mathcal{Y}$.

Let $J_T^0(A) = A$ and $J_T^{n+1}(A) = J_T(J_T^n(A))$.

2.8. Proposition. *For every* $A \subseteq \mathbb{N}$, $\{J_T^n(A)^+\} \equiv_e \{(A^+)^{(n)}\}$.

Proof. Let g be the computable function defined in Proposition 2.6 and fix two enumeration operators W_T and W_E such that for all sets X, $W_T(J_T(X)^+) = (A^+)'$ and $W_E((X^+)') = J_T(X)^+$. Let λ be a computable function such that for a and b and $X \subseteq \mathbb{N}$, $W_a(W_b(X)) = W_{\lambda(a,b)}(X)$.

We have that $J_T^0(A)^+ = A^+ = (A^+)^{(0)}$. Suppose that $J_T^n(A)^+ = W_a((A^+)^{(n)})$ and $(A^+)^{(n)} = W_b(J_n^T(A)^+)$. Then

$$J_T^{n+1}(A)^+ = J_T(J_T^n(A))^+ = W_E((J_T^n(A)^+)') = W_E(W_a((A^+)^{(n)})') = W_E(W_{g(a)}((A^+)^{(n+1)})) = W_{\lambda(E,g(a))}((A^+)^{(n+1)}) \text{ and}$$

$$(A^+)^{(n+1)} = [(A^+)^{(n)}]' = [W_b(J_T^n(A)^+)]' = W_{g(b)}([J_T^n(A)^+]') = W_{g(b)}(W_T(J_T(J_T^n(A))^+) = W_{\lambda(g(b),T)}(J_T^{n+1}(A)^+).$$

□

For every set A define the ω enumeration jump of A to be the set $A^{(\omega)} = \{\langle n, x\rangle | x \in A^{(n)}\}$. Let $J_T^\omega(A)$ to be the ω Turing jump of A.

2.9. Corollary. *For every set* A, $A^{(\omega)} \equiv_e (A^+)^{(\omega)} \equiv_e J_T^\omega(A)^+$.

Proof. The equivalence $(A^+)^{(\omega)} \equiv_e J_T^\omega(A)^+$ follows directly from the proposition above. Since $A \leq A^+$, we have also that $A^{(\omega)} \leq_e (A^+)^{(\omega)}$. Finally, notice that $A^+ \leq_e A'$ and hence $(A^+)^{(\omega)} \leq_e A^{(\omega)}$. □

Since the set $A^{(\omega)}$ is total, we have also that $A^{(\omega)} \equiv_T J_T^\omega(A)$.

2.10. Corollary. *For every sequence* $\mathcal{X} = \{X_n\}$ *of sets of natural numbers and* $B \subseteq \mathbb{N}$, $\mathcal{X}$ *is c.e. in* B *if and only if* $\mathcal{X} \leq_e \{(B^+)^{(n)}\}$.

Proof. Let λ and μ be computable functions such that for all $A \subseteq \mathbb{N}$, $W_a^A = W_{\lambda(a)}(A^+)$ and $W_a(A^+) = W_{\mu(a)}^A$.

Suppose that $\mathcal{X}$ is c.e. in B. Let γ be a computable function such that $(\forall n)(X_n = W_{\gamma(n)}^{J_T^n(B)})$. Then $(\forall n)(X_n = W_{\lambda(\gamma(n))}(J_T^n(B)^+))$. Hence $\mathcal{X} \leq_e \{J_T^n(B)^+\} \leq_e \{(B^+)^{(n)}\}$. Suppose now that $\mathcal{X} \leq_e \{(B^+)^{(n)}\}$. Then $\mathcal{X} \leq_e \{J_T^n(B)^+\}$. Hence there exists a computable function δ such that $(\forall n)(X_n = W_{\delta(n)}(J_n^T(B)^+))$. Then $(\forall n)(X_n = W_{\mu(\delta(n))}^{J_T^n(B)})$. Therefore $\mathcal{X}$ is c.e. in B. □

2.11. Definition. Let $\mathcal{Y} = \{Y_n\}$ be a sequence of sets of natural numbers. The *jump sequence* $\mathcal{P}(\mathcal{Y}) = \{\mathcal{P}_n(\mathcal{Y})\}$ of $\mathcal{Y}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{Y}) = Y_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{Y}) = \mathcal{P}_n(\mathcal{Y})' \oplus Y_{n+1}$.

By $\mathcal{P}_\omega(\mathcal{Y})$ we shall denote the set $\bigoplus_n \mathcal{P}_n(\mathcal{Y})$. Notice that this set is always total.

Recall that we have defined the ω-enumeration reducibility of sequences of sets to sequences of sets by setting

$$\mathcal{X} \leq_\omega \mathcal{Y} \iff (\forall B \subseteq \mathbb{N})(\mathcal{Y} \text{ is c.e. in } B \Rightarrow \mathcal{X} \text{ is c.e. in } B).$$

Next theorem proved in [23] gives a characterization of "$\leq_\omega$" in terms of enumeration reducibility:

2.12. Theorem. *For every two sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers*

$$\mathcal{X} \leq_\omega \mathcal{Y} \iff \mathcal{X} \leq_e \mathcal{P}(\mathcal{Y}).$$

2.13. Corollary. *Let $X \subseteq \mathbb{N}$ and let $\mathcal{Y}$ be a sequence of sets of natural numbers. Then $X \leq_n \mathcal{Y} \iff X \leq_e \mathcal{P}_n(\mathcal{Y})$.*

Proof. Let $\mathcal{X} = \{X_n\}$ be a sequence of natural numbers such that $X_k = \emptyset$ if $k \neq n$ and $X_n = X$. We have that $X \leq_e \mathcal{P}_n(\mathcal{Y}) \iff \mathcal{X} \leq_e \mathcal{P}(\mathcal{Y}) \iff \mathcal{X} \leq_\omega \mathcal{Y}$. Therefore

$$X \leq_e \mathcal{P}_n(\mathcal{Y}) \iff (\forall B \subseteq \mathbb{N})(\mathcal{Y} \text{ is c.e. in } B \Rightarrow \mathcal{X} \text{ is c.e. in } B).$$

Finally notice that $\mathcal{X}$ is c.e.in a set B if and only if $X = X_n$ is Σ^0_{n+1} in B. □

One can easily check that the following assertions hold for all sequences $\mathcal{Y}$ and $\mathcal{Z}$ of sets of natural numbers:

(1) $\mathcal{Y} \leq_e \mathcal{P}(\mathcal{Y})$.
(2) $\mathcal{P}(\mathcal{P}(\mathcal{Y})) \leq_e \mathcal{P}(\mathcal{Y})$.
(3) $\mathcal{Y} \equiv_\omega \mathcal{P}(\mathcal{Y})$.
(4) $\mathcal{Y} \leq_e \mathcal{Z} \Rightarrow \mathcal{Y} \leq_\omega \mathcal{Z}$.

We conclude this section by adopting the following notation:

Notation. Given a function h, by G_h we shall denote the graph of h, i.e. the set $\{(x, y) : h(x) \simeq y\}$, by $\mathrm{dom}(h)$ we shall denote the domain of h and by $\mathrm{ran}(h)$ we shall denote the range of h.

3. Marker's extensions of sequences of structures

Let us fix a sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ of countable structures. We shall assume that $\mathfrak{A}_n = (A_n; P^n_1, \dots, P^n_{m_n})$, where each P^n_k is an infinite subset of $A_n^{r^n_k}$, $1 \leq k \leq m_n$, and that there exists a computable function ρ such that for all n, $\rho(n) = \langle r^n_1, \dots, r^n_{m_n} \rangle$.

Set $A = \bigcup_n A_n$. Fix a subset R of A^r. For every $n \geq 0$ we define the n-th Marker's extension $\mathfrak{M}_n(R)$ of R as follows. Fix $n+1$ new countable and disjoint sets $X^R_0, \dots, X^R_n$ and define the functions $h^R_0, \dots, h^R_n$ so that

- h^R_0 is a bijective mapping of R onto X^R_0.
- h^R_1 is a bijective mapping of $(A^r \times X^R_0) \setminus G_{h^R_0}$ onto X^R_1.
- ...
- h^R_n is a bijective mapping of $(A^r \times X^R_0 \times \cdots \times X^R_{n-1}) \setminus G_{h^R_{n-1}}$ onto X^R_n.

Set $M^R_k = G_{h^R_k}$, $k \leq n$, and let

$$\mathfrak{M}_n(R) = (A \cup \bigcup_{k \leq n} X^R_k; M^R_n, X^R_0, \dots, X^R_n).$$

The sets $X_0^R, \dots, X_n^R$ are called *companions* of the extension $\mathfrak{M}_n(R)$.

Notice that R is Σ_{n+1} definable in $\mathfrak{M}_n(R)$ in a uniform way. Indeed, clearly for all $\bar{a} \in A^r$,

$$R(\bar{a}) \iff (\exists x_0 \in X_0^R)(M_0^R(\bar{a}, x_0))$$

and for all $k < n, x_0 \in X_0^R, \dots, x_k \in X_k^R$,

$$M_k^R(\bar{a}, x_0, \dots, x_k) \iff (\forall x_{k+1} \in X_{k+1}^R)(\neg M_{k+1}^R(\bar{a}, x_0, \dots, x_k, x_{k+1})).$$

To define the n-th Marker' s extension of the structure $\mathfrak{A}_n$ construct the structures $\mathfrak{M}_n(A_n), \mathfrak{M}_n(P_1^n), \dots, \mathfrak{M}_n(P_{m_n}^n)$ with disjoint companions and let

$$\mathfrak{M}_n(\mathfrak{A}_n) = (|\mathfrak{M}_n(A_n)| \cup \bigcup_{1 \le k \le m_n} |\mathfrak{M}_n(P_k^n)|; M_n^{A_n}, M_n^{P_1}, \dots, M_n^{P_{m_n}^n}, X_0^{A_n}, \dots, X_n^{P_{m_n}^n}).$$

Finally, construct for each $n \ge 0$ the n-th Marker's extension $\mathfrak{M}_n(\mathfrak{A}_n)$ of $\mathfrak{A}_n$ so that all companions are disjoint and let $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ be the structure with domain equal to the union of the domains of the structures $\mathfrak{M}_n(\mathfrak{A}_n)$, $n < \omega$, and with a set of predicates consisting of A and all predicates of the structures $\mathfrak{M}_n(\mathfrak{A}_n)$, $n < \omega$.

Despite the fact that $\mathfrak{M}$ has infinitely many predicates it is a structure of a computable first order language. Notice also that for each n the domain A_n and the predicates $P_1^n, \dots, P_{m_n}^n$ of the structure $\mathfrak{A}_n$ are Σ_{n+1} definable in $\mathfrak{M}$ uniformly in n.

We intend to show that $\mathfrak{M}(\vec{\mathfrak{A}})$ is a structure which answers positively the questions from the introduction. For we need to study the copies of $\mathfrak{M}(\vec{\mathfrak{A}})$ on the natural numbers.

Let f be a one to one mapping of $\mathbb{N}$ onto $|\mathfrak{M}(\vec{\mathfrak{A}})|$. Set

$$\begin{aligned} f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}})) =& f^{-1}(A) \oplus \bigoplus_n [f^{-1}(M_n^{A_n}) \oplus f^{-1}(M_n^{P_1^n}) \oplus \dots \oplus f^{-1}(M_n^{P_{m_n}^n})] \oplus \\ & \bigoplus_n [f^{-1}(X_0^{A_n}) \oplus \dots \oplus f^{-1}(X_n^{P_{m_n}^n})]. \end{aligned}$$

Clearly $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$ is Turing equivalent to the diagram of the copy of $\mathfrak{M}(\vec{\mathfrak{A}})$ under the isomorphism f^{-1}.

The following theorem provides the main tool for proving definability properties of $\mathfrak{M}(\vec{\mathfrak{A}})$.

3.1. Theorem. *Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures and $A = \bigcup_n |\mathfrak{A}_n|$. Let g be an enumeration of A and $\mathcal{Y} = \{Y_n\}$ be a sequence of sets of natural numbers such that $\mathcal{Y} \not\le_\omega \{g^{-1}(\mathfrak{A}_n)\}$.*

Then there exists an enumeration f of $|\mathfrak{M}(\vec{\mathfrak{A}})|$ satisfying the following conditions:

(1) *The set $\{\langle i, j\rangle | f(i) = g(j)\}$ is computable.*
(2) $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^{(\omega)} \equiv_e \mathcal{P}_\omega(g^{-1}(\vec{\mathfrak{A}}))$.
(3) *$\mathcal{Y}$ is not c.e. in $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.*

4. Proof of Theorem 3.1

Let $\mathfrak{A}_n = (A_n, P_1^n, \dots, P_{m_n}^n)$. Let $A = \bigcup_n A_n$. Recall that given an enumeration g of A, by $g^{-1}(\mathfrak{A}_n)$ we denote the set $g^{-1}(A_n) \oplus g^{-1}(P_1^n) \oplus g^{-1}(P_{m_n}^n)$. Thus the domain A_n and the predicates $P_1^n, \dots, P_{m_n}^n$ of $\mathfrak{A}_n$ are equally treated as subsets of A. To simplify the notation we shall assume that for all n, $m_n = 0$, i.e. the structure

$\mathfrak{A}_n$ has no predicates. Denote by $\mathcal{A}$ the sequence $\{A_n\}$. Then $(\forall n)(g^{-1}(\mathfrak{A}_n) = g^{-1}(A_n))$ and hence $g^{-1}(\vec{\mathfrak{A}}) = g^{-1}(\mathcal{A})$.

For $n < \omega$, set $M_n = M_n^{A_n}$ and for $i \leq n$ set $X_{n,i} = X_i^{A_n}$ and $h_{n,i} = h_i^{A_n}$. Set $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$. Then $\mathfrak{M} = (|\mathfrak{M}|; A, \{M_n, X_{n,0}, \ldots, X_{n,n}\}_{n<\omega})$.

Let $f_0(2j) = g(j)$. Notice that if $f \supseteq f_0$ is an enumeration of $|\mathfrak{M}|$ then f satisfies the condition (1).

We continue with the definition of f on the odd numbers. To achieve the satisfaction of (2) and (3), we shall use forcing.

Denote by B the set of all even numbers. Fix a sequence $Z_{n,i}$, $n < \omega, i \leq n$, of sets of odd numbers so that $Z_{n,i}$ are infinite, disjoint, computable uniformly in n and i and $\bigcup_{n,i\leq n} Z_{n,i}$ is equal to the set of all odd numbers.

Call an enumeration f of $|\mathfrak{M}|$ *regular* if $f_0 \subseteq f$ and for all n and $i \leq n$, $f^{-1}(X_{n,i}) = Z_{n,i}$.

Clearly if f is a regular enumeration then $f^{-1}(A) = B$ and $\bigoplus_{i\leq n,n<\omega} f^{-1}(X_{n,i})$ are computable and hence

$$f^{-1}(\mathfrak{M}) \equiv_e \bigoplus_n f^{-1}(M_n) \text{ and } f^{-1}(\mathfrak{M})^+ \equiv_e \bigoplus_n f^{-1}(M_n)^+.$$

For regular enumerations f the sequence $f^{-1}(\mathcal{A}) = f_0^{-1}(\mathcal{A})$ is enumeration equivalent to the sequence $g^{-1}(\mathcal{A})$. Hence the sequences $\mathcal{P}(f^{-1}(\mathcal{A}))$ and $\mathcal{P}(g^{-1}(\mathcal{A}))$ are enumeration equivalent. Therefore $\mathcal{P}_\omega(f^{-1}(\mathcal{A})) \equiv_e \mathcal{P}_\omega(g^{-1}(\mathcal{A}))$. We shall assume fixed a computable in $\mathcal{P}_\omega(g^{-1}(\mathcal{A}))$ function $\sigma(n,k)$ such that for each n the function $\lambda k.\sigma(n,k)$ enumerates the set $f_0^{-1}(A_n)$ without repetitions.

Notice also that for every regular f, $f^{-1}(A_n)$ is uniformly Σ^0_{n+1} in $f^{-1}(\mathfrak{M})$. Hence $g^{-1}(\mathcal{A}) \equiv_e f^{-1}(\mathcal{A}) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$.

Therefore $\mathcal{P}_\omega(g^{-1}(\mathcal{A})) \leq_e (f^{-1}(\mathfrak{M})^+)^{(\omega)} \equiv_e f^{-1}(\mathfrak{M})^{(\omega)}$.

Let f be a regular enumeration of $|\mathfrak{M}|$. Next we define for all natural numbers e, x and n the modeling relation $f \models_n F_e(x)$ whose intended meaning is $x \in W_e((f^{-1}(\mathfrak{M})^+)^{(n)})$.

4.1. Definition. Let f be a regular enumeration of $|\mathfrak{M}|$. Define for all natural numbers e and x the relation $f \models_n F_e(x)$ by induction on n:

(i) Let $u \in \mathbb{N}$. Then $f \models_0 u$ if
  a) $u = \langle 1, n, b, z_0, \ldots, z_n\rangle$, b is even, $z_0 \in Z_{n,0}, \ldots,$ $z_n \in Z_{n,n}$ and $M_n(f(b), f(z_1), \ldots, f(z_n))$;
  b) $u = \langle 0, n, b, z_0, \ldots, z_n\rangle$, b is even, $z_0 \in Z_{n,0}, \ldots,$ $z_n \in Z_{n,n}$ and $\neg M_n(f(b), f(z_1), \ldots, f(z_n))$;

(ii) $f \models_0 D_v \iff (\forall u \in D_v)(f \models_0 u)$;

(iii) $f \models_0 F_e(x) \iff (\exists v)(\langle x, v\rangle \in W_e \wedge f \models_0 D_v)$;

(iv) $f \models_{n+1} D_v \iff (\forall u \in D_v)(u = \langle 1, e, x\rangle \wedge f \models_n F_e(x) \vee u = \langle 0, e, x\rangle \wedge f \not\models_n F_e(x))$;

(v) $f \models_{n+1} F_e(x) \iff (\exists v)(\langle x, v\rangle \in W_e \wedge f \models_{n+1} D_v)$;

Set $f \models_n \neg F_e(x) \iff f \not\models_n F_e(x)$.

4.2. Lemma. *There exists a computable function $\gamma(n,e)$ such that for all $n, e \in \mathbb{N}$, $W_e((f^{-1}(\mathfrak{M})^+)^{(n)}) = \{x | f \models_n F_{\gamma(n,e)}(x)\}$.*

Let $n < \omega$. Call the functions $\kappa_{n,0}, \ldots, \kappa_{n,n}$ a *n-consistent system* if

κ_{n_0} is a bijective mapping of $f_0^{-1}(A_n)$ onto $Z_{n,0}$;

κ_{n_1} is a bijective mapping of $(B \times Z_{n,0}) \setminus G_{\kappa_{n,0}}$ onto $Z_{n,1}$;
...
$\kappa_{n,n}$ is a bijective mapping of $(B \times Z_{n,0} \times \cdots \times Z_{n,n-1}) \setminus G_{\kappa_{n,n-1}}$ onto $Z_{n,n}$.

Given a regular enumeration f, we obtain for each n a n-consistent system by setting for $i \leq n$, $\kappa_{n,i} = f^{-1}(h_{n,i})$, i.e.

$$\kappa_{n,i}(b, z_0, \dots, z_{i-1}) \simeq f^{-1}(h_{n,i}(f(b), f(z_0), \dots, f(z_{i-1}))).$$

On the other hand, given n-consistent systems $\kappa_{n,0}, \dots, \kappa_{n,n}$ for all $n < \omega$, there is a unique regular enumeration f such that for all n and $i \leq n$, $\kappa_{n,i} = f^{-1}(h_{n,i})$. Clearly for even numbers z, $f(z) = f_0(z)$. Fix n and define the f on $Z_{n,i}$ by induction on i. If $i = 0$ and $z \in Z_{n,0}$ find the unique $b \in f_0^{-1}(A_n)$ such that $\kappa_{n,0}(b) \simeq z$ and set $f(z) = h_{n,0}(f(b))$. Suppose that $i < n$ and f is defined on $Z_{n,j}, j \leq i$. Given a $z \in Z_{n,i+1}$ find the unique $(b, z_0, \dots, z_i)$ such that $\kappa_{n,i+1}(b, z_0, \dots, z_i) \simeq z$. Set $f(z) = h_{n,i+1}(f(b), f(z_0), \dots, f(z_i))$. One can easily check that for all n and $i \leq n$, $f^{-1}(X_{n,i}) = Z_{n,i}$ and $\kappa_{n,i} = f^{-1}(h_{n,i})$.

From the construction it follows that f is unique.

So to construct a regular enumeration f of A is the same as to construct for all n, n-consistent systems of functions $\kappa_{n,0}, \dots, \kappa_{n,n}$. This idea can be combined with an appropriately defined forcing with conditions which are finite parts of n-consistent systems.

4.1. **Conditions.** Let $k \leq n$. A *prime n-condition* of type k is of the form:

$$E = (B^E, Z^E_{n,0}, \dots, Z^E_{n,n}, \kappa^E_{n,n-k}, \dots, \kappa^E_{n,n}),$$

where $B^E \subseteq B$, $Z^E_{n,i} \subseteq Z_{n,i}$ and all sets $B^E, Z^E_{n,i}$ are finite.

Each of $\kappa^E_{n,n-j}$, $j \leq k$, is a partial injective mapping of $B^E \times Z^E_{n,0} \times \cdots \times Z^E_{n,n-j-1}$ **onto** $Z^E_{n,n-j}$ satisfying also the following:

(i) If $n = k$ then $\mathrm{dom}(\kappa^E_{n,0}) \subseteq f_0^{-1}(A_n)$.
(ii) If $j < k$ then $\mathrm{dom}(\kappa^E_{n,n-j}) \cap G_{\kappa^E_{n,n-j-1}} = \emptyset$.

By $\emptyset^n_k$, $k \leq n$, we shall denote the prime n-condition E of type k, such that $B^E = Z^E_{n,0} = \cdots = Z^E_{n,n} = \emptyset$ and all functions $\kappa^E_{n,n-j}$, $j \leq k$, are totally undefined.

4.3. Definition. Let E be a prime n-condition of type k and $j \leq k$. Then

(i) $E \Vdash M_{n,n-j}(b, z_0, \dots, z_{n-j})$ if $\kappa^E_{n,n-j}(b, z_0, \dots, z_{n-j-1}) \simeq z_{n-j}$.
(ii) $E \Vdash \neg M_{n,n-j}(b, z_0, \dots, z_{n-j})$ if $\kappa^E_{n,n-j}(b', z'_0, \dots, z'_{n-j-1}) \simeq z_{n-j}$ for some $(b', z'_0, \dots, z'_{n-j-1}) \neq (b, z_0, \dots, z_{n-j-1})$

Notice that if E is a prime n-condition of type k and $j < k$ then

$$E \Vdash M_{n,n-j}(b, z_0, \dots, z_{n-j-1}, z_{n-j}) \Rightarrow E \Vdash \neg M_{n,n-j-1}(b, z_0, \dots, z_{n-j-1}).$$

Indeed, assume that $\kappa^E_{n,n-j}(b, z_0, \dots, z_{n-j-1}) \simeq z_{n-j}$. Since the range of $\kappa^E_{n,n-j-1}$ is equal to Z^E_{n-j-1}, there exist $b', z'_0, \dots, z'_{n-j-2}$ such that $\kappa^E_{n,n-j-1}(b', z'_0, \dots, z'_{n-j-2}) \simeq z_{n-j-1}$. Since $b, z_0, \dots, z_{n-j-2}, z_{n-j-1} \in \mathrm{dom}(\kappa^E_{n,n-j})$,

$$b', z'_0, \dots, z'_{n-j-2} \neq b, z_0, \dots, z_{n-j-2}.$$

4.4. Definition. A *condition of type* k is a finite sequence $E = (E_0, \dots, E_n)$, where $k \leq n$, for $i \leq k$, E_i is a prime i-condition of type i and if $k \leq i \leq n$, then E_i is a prime i-condition of type k.

Given a condition E of length n and type k, by B^E we shall denote $\bigcup_{i=0}^{n} B^{E_i}$, for $i \leq j \leq n$, by $Z^E_{j,i}$ we shall denote the set $Z^{E_j}_{j,i}$, by $|E|$ we shall denote the length n of the condition and by *type*(E) we shall denote the type k of the condition.

4.2. **Extensions of prime conditions.**

4.5. Definition. Let $k_C, k_D \leq n$ and C and D be prime n-conditions of type k_C and k_D, respectively. Then D *extends* $C (C \subseteq D)$ if $B^C \subseteq B^D$, $(\forall i \leq n)(Z^C_{n,i} \subseteq Z^D_{n,i})$ and

a) $k_C \leq k_D$ and $(\forall j \leq k_C)(\kappa^C_{n,n-j} \subseteq \kappa^D_{n,n-j})$ or
b) $k_D < k_C$, $(\forall j \leq k_D)(\kappa^C_{n,n-j} \subseteq \kappa^D_{n,n-j})$ and $\mathrm{dom}(\kappa^D_{n,n-k_D}) \cap G_{\kappa^C_{n,n-k_D-1}} = \emptyset$.

4.6. Lemma. *Let $C \subseteq D$ be prime n-conditions of type k_C and k_D respectively. Let $j \leq \min(k_C, k_D)$. Then*

$$C \Vdash (\neg)M_{n,n-j}(b, z_0, \dots, z_{n-j}) \Rightarrow D \Vdash (\neg)M_{n,n-j}(b, z_0, \dots, z_{n-j}).$$

The extension relation is reflexive but not always transitive. Below we formulate some conditions under which the transitivity holds:

4.7. Lemma. *Let C, D, E be prime n-conditions of types k_C, k_D and k_E respectively. Let $C \subseteq D$ and $D \subseteq E$. Then*

(1) *If $k_C \leq k_D$, then $C \subseteq E$;*
(2) *If $k_E < k_D$ then $C \subseteq E$.*

Proof. For the proof of (1) notice that if $k_C \leq k_E$ then for all $j \leq k_C$, $\kappa^C_{n,n_j} \subseteq \kappa^D_{n,n-j} \subseteq \kappa^E_{n,n-j}$. If $k_E < k_C$ then $C \subseteq E$ follows from $D \subseteq E$ and $\kappa^C_{n,n-k_E-1} \subseteq \kappa^D_{n,n-k_E-1}$.

Let us turn to the proof of (2). Consider first the case $k_C \leq k_E$. Then $k_C \leq k_D$. Hence $(\forall j \leq k_C)(\kappa^C_{n,n-j} \subseteq \kappa^D_{n,n-j} \subseteq \kappa^E_{n,n-j})$.

Assume that $k_E < k_C$. Clearly if $j \leq k_E$ then $\kappa^C_{n,n-j} \subseteq \kappa^D_{n,n-j} \subseteq \kappa^E_{n,n-j}$. Moreover $\kappa^C_{n,n-k_E-1} \subseteq \kappa^D_{n,n-k_E-1}$ and hence $\mathrm{dom}(\kappa^E_{n,n-k_E}) \cap G_{\kappa^C_{n,n-k_E-1}} = \emptyset$. $\square$

4.8. Definition. Given a prime n-condition of type k and $r \leq k$, by $(C)_r$ we shall denote the condition $(B^C, Z^C_{n,0}, \dots, Z^C_{n,n}, \kappa^C_{n,n-r}, \dots, \kappa^C_{n,n})$

Clearly $(C)_r$ is well defined prime n-condition of type r, $C \subseteq (C)_r$ and $(C)_r \subseteq C$.

As a direct consequence of Lemma 4.7 we get the following:

4.9. Lemma. *Let C and D be prime n-conditions of type k and r respectively and $r < k$. Then $C \subseteq D \iff (C)_{r+1} \subseteq D$.*

4.10. Lemma. *Let $C \subseteq D$ be prime n-conditions of type k_1 and k_2 respectively. Then for all $r \leq k_1$, $(C)_r \subseteq D$.*

4.11. Lemma. *Let $C \subseteq D$ be prime n-conditions of type k and r, respectively, and $r < k$. There exists a prime n-condition E of type $r+1$ such that $C \subseteq E$ and $D \subseteq (E)_r$.*

Proof. Set $E_0 = (B^D, Z^D_{n,0}, \dots, Z^D_{n,n}, \kappa^C_{n,n-r-1}, \kappa^D_{n,n-r}, \dots, \kappa^D_{n,n})$. We have that $C \subseteq E_0$ and $D = (E_0)_r$. The only problem is that the mapping $\kappa^C_{n,n-r-1}$ could be not onto $Z^D_{n,n-r-1}$.

For each element $z_{n-r-1} \in Z^D_{n,n-r-1}$ which does not belong to the range of $\kappa^C_{n,n-r-1}$ get new elements $z_0, \dots, z_{n-r-2}$ of $Z_{n,0}, \dots, Z_{n,n-r-2}$, respectively and a new element b of $f_0^{-1}(A_n)$ if $n-r-1=0$, or an arbitrary new even number b otherwise, and extend $\kappa^C_{n,n-r-1}$ to $\kappa^E_{n,n-r-1}$ so that

$$\kappa^E_{n,n-r-1}(b, z_0, \dots, z_{n-r-2}) \simeq z_{n-r-1}.$$

This way we obtain finite extensions $B^E, Z^E_{n,0}, \dots, Z^E_{n,n-r-2}$ of $B^D, Z^D_{n,0}, \dots, Z^D_{n,n-r-2}$ respectively and a finite extension $\kappa^E_{n,n-r-1}$ of $\kappa^C_{n,n-r-1}$. Set

$$E = (B^E, Z^E_{n,0}, \dots, Z^E_{n,n-r-2}, Z^D_{n,n-r-1}, \dots, Z^D_{n,n}, \kappa^E_{n,n-r-1}, \kappa^D_{n,n-r}, \dots, \kappa^D_{n,n}).$$

□

4.12. Corollary. *Let $C \subseteq D$ be prime n-conditions of type k and r respectively and $r \leq k$. There exists a prime n-condition E of type k such that $C \subseteq E$ and $D \subseteq (E)_r$.*

4.13. Corollary. *Let D be a prime n-condition of type r. There exists a prime n-condition E of type n such that $D \subseteq (E)_r$.*

Proof. Apply the previous corollary with $C = \emptyset^n_n$. □

4.14. Lemma. *Let $C = (B^C, Z^C_{n,0}, \dots, Z^C_{n,n}, \kappa^C_{n,n-k}, \dots, \kappa^C_{n,n})$ be a prime n-condition of type k. Let $D_0, \dots, D_n$ be finite subsets of $Z_{n,0}, \dots, Z_{n,n}$ respectively. There exist a prime n-condition E of type k such that $C \subseteq E$ and for all $i \leq n$, $D_i \subseteq Z^E_{n,i}$.*

Proof. Extend the function $\kappa^C_{n,n}$ to $\kappa^*_{n,n}$ by adding new elements to $B^C \times Z^C_{n,0} \times \dots \times Z^C_{n,n-1}$ so that the range of $\kappa^*_{n,n}$ becomes equal to $Z^C_{n,n} \cup D_n$. Let the respective extension of the set $Z^C_{n,i}$ be $Z^*_{n,i}$, $i \leq n-1$, and the extension of B^C be B^*. Consider the condition

$$D = (B^*, Z^*_{n,0} \cup D_0, \dots, Z^*_{n,n-1} \cup D_{n-1}, Z^C_{n,n} \cup D_n, \kappa^*_{n,n}).$$

Clearly $C \subseteq D$. Using Corollary 4.12 find a condition E of type k such that $C \subseteq E$ and $D \subseteq E$. □

4.15. Lemma. *Let C be a prime n condition of type n. Let $k \leq n$ and $b, z_0, \dots, z_{k-1}$ be elements of $B, Z_{n,0}, \dots, Z_{n,k-1}$ respectively and such that if $k=0$ then $b \in f_0^{-1}(A_n)$ and if $0<k$ then $(b, z_0, \dots, z_{k-1}) \notin G_{\kappa^C_{n,k-1}}$. Then there exists an extension D of C of type n such that $(b, z_0, \dots, z_{k-1}) \in \mathrm{dom}(\kappa^D_{n,k})$.*

Proof. Suppose that $k=0$. Let z be an element of $Z_{n,0} \setminus Z^C_{n,0}$ and let

$$D = (B^C \cup \{b\}, Z^C_{n,0} \cup \{z\}, \dots, Z^C_{n,n}, \kappa^D_{n,0}, \kappa^C_{n,1}, \dots, \kappa^C_{n,n}),$$

where $\kappa^D_{n,0}$ is the least extension of $\kappa^C_{n,0}$ such that $\kappa^D_{n,0}(b) \simeq z$. Clearly D is a well defined condition of type n and $C \subseteq D$.

Suppose that $k>0$. Let $z \in Z_{n,k} \setminus Z^C_{n,k}$ and let

$$E = (B^C \cup \{b\}, Z^C_{n,0} \cup \{z_0\}, \dots, Z^C_{n,k-1} \cup \{z_{k-1}\}, Z^C_{n,k} \cup \{z\}, \\ Z^C_{n,k+1} \dots, Z^C_{n,n}, \kappa^E_{n,k}, \kappa^C_{n,k+1}, \dots, \kappa^C_{n,n}),$$

where $\kappa^E_{n,k}$ is the least extension of $\kappa^C_{n,k}$ such that $\kappa^E_{n,k}(b, z_0, \dots, z_{k-1}) \simeq z$. Clearly E is a well defined condition of type k and $C \subseteq E$. Using Corollary 4.12 extend E to a condition D of type n such that $C \subseteq D$. □

4.3. **Extensions of conditions.**

4.16. Definition. Given conditions $E = (E_0, \dots, E_n)$ and $D = (D_0, \dots, D_m)$, say that D extends E ($E \subseteq D$) if $n \leq m$ and $(\forall i \leq n)(E_i \subseteq D_i)$.

4.17. Definition. Let $E = (E_0, \dots, E_r, \dots, E_n)$ be a condition of type k and $r \leq k$. Then by $(E)_r$ we shall denote the condition $(E_0, \dots, E_r, (E_{r+1})_r, \dots, (E_n)_r)$,

Notice that $E \subseteq (E)_r$ and in the same time $(E)_r \subseteq E$.

The following lemma is a direct consequence of Lemma 4.9:

4.18. Lemma. *Let C and D be conditions of types k and r respectively and $r < k$. Then $C \subseteq D \iff (C)_{r+1} \subseteq D$.*

The following properties of the extension relation on conditions follow directly from Lemma 4.11

4.19. Lemma. *Let $C \subseteq D$ be conditions of types k and r respectively and $r < k$. There exists a condition E of type k such that $C \subseteq E$ and $D \subseteq (E)_r$.*

4.20. Lemma. *Let D be a condition of type r and $r \leq k$. There exists a condition E of type k such that $D \subseteq (E)_r$.*

4.4. **The forcing.**

4.21. Definition. Given two conditions C and D, say that $C \leq D$ if $C \subseteq D$ and $type(C) \leq type(D)$.

Obviously " $\leq$ " is a partial order on the set of all conditions.

4.22. Definition. Suppose that $C = (C_0, \dots, C_m)$ is a condition an $i, e, x \in \mathbb{N}$. The forcing relation $C \Vdash_i (\neg)F_e(x)$ is defined by induction on i:

(i) Let $u \in \mathbb{N}$. Then $C \Vdash_0 u$ if
  a) $u = \langle 1, n, b, z_0, \dots, z_n \rangle$, b is even, $z_0 \in Z_{n,0}, \dots,$ $z_n \in Z_{n,n}$, $n \leq |C|$ and $C_n \Vdash M_{n,n}(b, z_0, \dots, z_n)$.
  b) $u = \langle 0, n, b, z_0, \dots, z_n \rangle$, b is even, $z_0 \in Z_{n,0}, \dots,$ $z_n \in Z_{n,n}$, $n \leq |C|$ and $C_n \Vdash \neg M_{n,n}(b, z_0, \dots, z_n)$

(ii) $C \Vdash_0 D_v \iff (\forall u \in D_v)(C \Vdash_0 u)$.

(iii) $C \Vdash_0 F_e(x) \iff (\exists v)(\langle x, v \rangle \in W_e \wedge C \Vdash_0 D_v)$.

(iv) Suppose that $C \Vdash_i F_e(x)$ is defined. Then let

$$C \Vdash_i \neg F_e(x) \iff (\forall D \geq C)(D \nVdash_i F_e(x)).$$

$$C \Vdash_{i+1} D_v \iff (\forall u \in D_v)(u = \langle 1, e, x \rangle \wedge C \Vdash_i F_e(x) \vee u = \langle 0, e, x \rangle \wedge C \Vdash_i \neg F_e(x)).$$

$$C \Vdash_{i+1} F_e(x) \iff (\exists v)(\langle x, v \rangle \in W_e \wedge C \Vdash_{i+1} D_v).$$

4.23. Lemma. *For all conditions C and D and for all $i < \omega$,*

$$C \Vdash_i (\neg)F_e(x) \wedge C \leq D \Rightarrow D \Vdash_i (\neg)F_e(x).$$

4.5. **Genericity.**

4.24. Definition. Let f be a regular enumeration of $|\mathfrak{M}|$ and C be a prime n-condition of type k. Then $C \subseteq f$ if for all $j \leq k$, $\kappa^C_{n,n-j} \subseteq f^{-1}(h_{n,n-j})$, i.e.

$$(\forall b, z_0, \dots, z_{n-j-1} \in \mathrm{dom}(\kappa^C_{n,n-j}))(f(\kappa^C_{n,n-j}(b, z_0, \dots z_{n-j-1})) = h_{n,n-j}(f(b), f(z_0), \dots, f(z_{n-j-1})).$$

4.25. Lemma. *Let f be a regular enumeration of $|\mathfrak{M}|$. Let $\kappa_{n,0},\dots,\kappa_{n,n}$ be finite functions and $\kappa_{n,i} \subseteq f^{-1}(h_{n,i})$, $i = 0,\dots,n$. Then there exists a prime n-condition $C \subseteq f$ of type n such that $\kappa_{n,i} \subseteq \kappa^C_{n,i}$.*

Proof. Set $Z^C_{n,n} = \operatorname{ran}(\kappa_{n,n})$ and $\kappa^C_{n,n} = \kappa_{n,n}$. Suppose that $i < n$ and for all $j \le i$ the sets $Z^C_{n,n-j}$ and the functions $\kappa^C_{n,n-j}$ are defined so that the following conditions hold:

(1) $\operatorname{ran}(\kappa^C_{n,n-j}) = Z^C_{n,n-j}$,
(2) $\kappa_{n,n-j} \subseteq \kappa^C_{n,n-j} \subseteq f^{-1}(h_{n,n-j})$.
(3) If $j < i$ and $(b, z_0,\dots,z_{n-j-1}) \in \operatorname{dom}(\kappa^C_{n,n-j})$ then $z_{n-i} \in Z^C_{n,n-i},\dots, z_{n-j-1} \in Z^C_{n,n-j-1}$.

Let $Z^C_{n,n-i-1}$ be the set of all z which either belong to $\operatorname{ran}(\kappa_{n,n-i-1})$ or for some $j \le i$, z occurs at $n-i-1$-th position in some tuple belonging to $\operatorname{dom}(\kappa^C_{n,n-j})$.

Let $\kappa^C_{n,n-i-1}$ be the least finite function such that

$$\kappa_{n,n-i-1} \subseteq \kappa^C_{n,n-i-1} \subseteq f^{-1}(h_{n,n-i-1}) \wedge \operatorname{ran}(\kappa^C_{n,n-i-1}) = Z^C_{n,n-i-1}.$$

Finally set $B^C = \{b | (\exists i \le n)(\exists \bar{z})((b,\bar{z}) \in dom(\kappa^C_{n,i}))\}$.

Set $C = (B^C, Z^C_{n,0},\dots,Z^C_{n,n},\kappa^C_{n,0},\dots,\kappa^C_{n,n})$. □

4.26. Corollary. *Let f be a regular enumeration. Then*

(1) *For every $n \ge 0$ there exists a prime n-condition $C \subseteq f$ of type n.*
(2) *If $C_1 \subseteq f$ and $C_2 \subseteq f$ are prime n-conditions then here exists a prime n-condition $C \subseteq f$ of type n such that $C_1 \subseteq C$ and $C_2 \subseteq C$.*

4.27. Definition. Let f be a regular enumeration of $|\mathfrak{M}|$ and $C = (C_0,\dots,C_m)$ be a condition. Then $C \subseteq f$ if for all $i \le m$, $C_i \subseteq f$.

The following lemma is immediate:

4.28. Lemma. *Let f be a regular enumeration and $C \subseteq f$ and $D \subseteq f$ be conditions. Then*

(1) *For each $m \ge \max(|C|,|D|)$ there exists a condition $E \subseteq f$ of type m such that $C \subseteq E$ and $D \subseteq E$.*
(2) *There exists a condition $E \subseteq f$ such that $C \le E$ and $D \le E$.*

4.29. Definition. A regular enumeration f *is generic* if for all natural numbers i, e, x, there is a condition $C \subseteq f$ such that $C \Vdash_i F_e(x) \vee C \Vdash_i \neg F_e(x)$.

The proof of the following lemma is standard.

4.30. Lemma. *Let f be a generic enumeration of $|\mathfrak{M}|$. Then $f \models_i (\neg)F_e(x) \iff (\exists C \subseteq f)(C \Vdash_i (\neg)F_e(x))$.*

In the next subsection we show that while checking whether $C \Vdash_i F_e(x)$ we may restrict ourselves to extensions of C which are of type i. To express formally this idea we introduce a modification of the forcing relation.

4.6. **The starred forcing.**

4.31. Definition. Let C be a condition. The relation $C \Vdash^*_i F_e(x)$ is defined by induction on i:

(1) $C \Vdash^*_0 F_e(x) \iff C \Vdash_0 F_e(x)$.

(2) Let $C \Vdash_i^* F_e(x)$ be defined. Then

$$C \Vdash_i^* \neg F_e(x) \iff (\forall D)(C \subseteq D \wedge \mathit{type}(D) = i \Rightarrow D \nVdash_i^* F_e(x)).$$

$$C \Vdash_{i+1}^* D_v \iff (\forall u \in D_v)(u = \langle 1, e, x\rangle \wedge C \Vdash_i^* F_e(x) \vee u = \langle 0, e, x\rangle \wedge C \Vdash_i^* \neg F_e(x)).$$

$$C \Vdash_{i+1}^* F_e(x) \iff (\exists v)(\langle x, v\rangle \in W_e \wedge C \Vdash_{i+1}^* D_v).$$

4.32. Lemma. $C \Vdash_i^* (\neg)F_e(x) \wedge C \leq D \Rightarrow D \Vdash_i^* (\neg)F_e(x)$.

Proof. Induction on i. Let $C \leq D$. Obviously $C \Vdash_0^* F_e(x) \Rightarrow D \Vdash_0^* F_e(x)$. Suppose that $C \Vdash_i^* F_e(x) \Rightarrow D \Vdash_i^* F_e(x)$. Assume that $C \Vdash_i^* \neg F_e(x)$ and for some E of type i, $D \subseteq E$ and $E \Vdash_i^* F_e(x)$. Since $\mathit{type}(C) \leq \mathit{type}(D)$, it follows that $C \subseteq E$. A contradiction. Hence $D \Vdash_i^* \neg F_e(x)$. From here it follows directly that $C \Vdash_{i+1}^* D_v \Rightarrow D \Vdash_{i+1}^* D_v$ and hence $C \Vdash_{i+1}^* F_e(x) \Rightarrow D \Vdash_{i+1}^* F_e(x)$. □

4.33. Lemma. *For all $i \in \mathbb{N}$ and for all conditions C of type k,*

(1) *If $i \leq k$ then $C \Vdash_i^* F_e(x) \iff (C)_i \Vdash_i^* F_e(x)$.*
(2) *If $i < k$, then $C \Vdash_i^* \neg F_e(x) \iff (C)_{i+1} \Vdash_i^* \neg F_e(x)$.*

Proof. Induction on i. The assertion (1) is obviously true for $i = 0$. Assume that (1) is true for some i.

Let C be a condition of type $k \geq i + 1$.

Now suppose that $C \Vdash_i^* \neg F_e(x)$. Assume that there exists a condition D of type i such that $(C)_{i+1} \subseteq D$ and $D \Vdash_i^* F_e(x)$. Then $C \subseteq D$. A contradiction.

Assume that $(C)_{i+1} \Vdash_i^* \neg F_e(x)$. Since $(C)_{i+1} \leq C$, $C \Vdash_i^* \neg F_e(x)$ by the previous lemma. By this (2) is proven for i.

Notice that since $((C)_{i+1})_i = (C)_i$,

$$C \Vdash_i^* F_e(x) \iff (C)_i \Vdash_i^* F_e(x) \iff (C)_{i+1} \Vdash_i^* F_e(x).$$

Hence for all D_v, $C \Vdash_{i+1}^* D_v \iff (C)_{i+1} \Vdash_{i+1}^* D_v$. Therefore $C \Vdash_{i+1}^* F_e(x) \iff (C)_{i+1} \Vdash_{i+1}^* F_e(x)$. □

4.34. Lemma. *For all conditions C and all $i, e, x \in \mathbb{N}$,*

$$C \Vdash_i (\neg)F_e(x) \iff C \Vdash_i^* (\neg)F_e(x).$$

Proof. Induction on i. Clearly $C \Vdash_0 F_e(x) \iff C \Vdash_0^* F_e(x)$. Assume that for some i and for all conditions C, $C \Vdash_i F_e(x) \iff C \Vdash_i^* F_e(x)$.

First we shall show that $C \Vdash_i \neg F_e(x) \iff C \Vdash_i^* \neg F_e(x)$.

Indeed, assume that $C \Vdash_i \neg F_e(x)$. Suppose that there exists a D of type i such that $C \subseteq D$ and $D \Vdash_i^* F_e(x)$. If $\mathit{type}(C) \leq i$ then $C \leq D$ and $D \Vdash_i F_e(x)$. A contradiction. If $i < \mathit{type}(C)$, then there exists an E, $C \leq E$ and such that $D \subseteq (E)_i$ Then $D \leq E$ and hence $E \Vdash_i^* F_e(x)$. Therefore $E \Vdash_i F_e(x)$, a contradiction again.

Suppose that $C \Vdash_i^* \neg F_e(x)$ and for some $D \geq C$, $D \Vdash_i F_e(x)$. Then $D \Vdash_i^* F_e(x)$. Without lost of generality we may assume that $\mathit{type}(D) \geq i$. Hence $C \subseteq (D)_i$ and $(D)_i \Vdash_i^* F_e(x)$. A contradiction.

The equivalence $C \Vdash_{i+1} F_e(x) \iff C \Vdash_{i+1}^* F_e(x)$ follows directly. □

4.7. **Computations.** We shall assume fixed a coding of all prime conditions and of all conditions. Set $\mathcal{P}_n = \mathcal{P}_n(f_0^{-1}(\mathcal{A}))$. By $\mathcal{P}$ we shall denote the sequence $\{\mathcal{P}_n\}$ and by $\mathcal{P}_\omega$ we shall denote the set $\mathcal{P}_\omega(f_0^{-1}(\mathcal{A}))$.

4.35. Lemma. *The sets $\mathcal{C}_i$ of all conditions of type i are uniformly in i enumeration reducible to $\bigoplus_{j\leq i} f_0^{-1}(A_j)$ and hence to $\mathcal{P}_i$.*

4.36. Lemma. *The sets $\mathcal{F}_i = \{(C,e,x) | C \in \mathcal{C}_i \wedge C \Vdash_i^* F_e(x)\}$ are enumeration reducible to $\mathcal{P}_i$ uniformly in i.*

Proof. The e-reducibility of $\mathcal{F}_0$ to $\mathcal{P}_0$ is obvious.

Suppose that $\mathcal{F}_i \leq \mathcal{P}_i$. Then for every condition $C \in \mathcal{C}_{i+1}$ the set $\{(e,x)|(\exists D \supseteq C)(D \in \mathcal{C}_i \wedge D \Vdash_i^* F_e(x))\}$ is enumeration reducible to $\mathcal{P}_i$ uniformly in C. Hence its compliment $\{(e,x)|C \Vdash_i^* \neg F_e(x)\}$ is enumeration reducible to $\mathcal{P}_i'$ uniformly in C. On the other hand for $C \in \mathcal{C}_{i+1}$,

$$\{(e,x)|C \Vdash_i^* F_e(x)\} = \{(e,x)|(C)_i \Vdash_i^* F_e(x)\}.$$

Therefore for all $C \in \mathcal{C}_{i+1}$, $\{(e,x)|C \Vdash_i^* F_e(x)\}$ is enumeration reducible to $\mathcal{P}_i$ and hence to $\mathcal{P}_i'$ uniformly in C. Finally since $\mathcal{C}_{i+1} \leq_e \mathcal{P}_{i+1}$, we get that

$$\{(C,e,x)|C \in \mathcal{C}_{i+1} \wedge (\exists v)(\langle x,v\rangle \in W_e \wedge C \Vdash_i^* D_v)\} \leq_e \mathcal{P}_{i+1}.$$

and hence $\mathcal{F}_{i+1} \leq_e \mathcal{P}_{i+1}$. □

4.8. **The construction of the enumeration** f**.** Fix a sequence $\mathcal{Y} = \{Y_n\}$ of sets of natural numbers such that $\mathcal{Y} \not\leq_\omega g^{-1}(\mathcal{A})$. We are going to construct an enumeration f satisfying the conditions (1) - (3) of Theorem 3.1 under the additional assumption that $\bigoplus_n Y_n^+ \leq_T \mathcal{P}_\omega$.

Since $\mathcal{Y} \not\leq_\omega g^{-1}(\mathcal{A})$, $\mathcal{Y} \not\leq_e \mathcal{P}(g^{-1}(\mathcal{A}))$. Hence $\mathcal{Y} \not\leq_e \mathcal{P}$.

We shall define a sequence $\{C_s\}$ of conditions so that the following properties are satisfied.

1. $C_s \leq C_{s+1}$ and $s \leq type(C_s)$.
2. For all n the functions $\kappa_{n,i} = \bigcup_s \kappa_{n,i}^{C_s}$, $i \leq n$, are a n-consistent system.
3. For all $i,e,x \in \mathbb{N}$, there exists a s such that $C_s \Vdash_i F_e(x) \vee C_s \Vdash_i \neg F_e(x)$.
4. For every regular enumeration f, $(\forall s)(C_s \subseteq f) \Rightarrow \mathcal{Y} \not\leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$.

We shall assume that the extension procedures described before could be done deterministically and effectively relative $\mathcal{P}_\omega$. For one can use the enumerations $\lambda k.\sigma(n,k)$ of the sets $f_0^{-1}(A_n)$ fixed in the beginning of the section.

Let λ_e, $e = 0,1,\ldots$ be an effective enumeration of all primitive computable functions.

The sequence $\{C_s\}$ will be defined by recursion on s and effectively relative $\mathcal{P}_\omega$. We shall consider three kinds of stages s. On stages $s = 3q$ we shall ensure that $\{C_s\}$ satisfies (2). On stages $s = 3q+1$ we shall ensure the satisfaction of (3) and on stages $s = 3q+2$ the satisfaction of (4).

Set $C_0 = \emptyset_0^0$. Suppose that C_s is defined. Consider the following cases:

a) $s = 3q$. Let $type(C_s) = k$ and $C_s = (C_0,\ldots,C_k,\ldots,C_m)$ using the extension procedures, we can add finitely many new elements of $Z_{l,i}$ to $Z_{l,i}^{C_s}, l \leq m, i \leq \min(k,l)$. By this we shall ensure that for all n and $i \leq n$, the functions $\kappa_{n,i}$ are onto $Z_{n,i}$.

To achieve that $\kappa_{n,0}$ is defined on all elements of $f_0^{-1}(A_n)$ and for all $i, 1 \leq i \leq n$, $\kappa_{n,i}$ is defined on all elements of $(B \times Z_{n,0} \times \cdots \times Z_{n,i-1}) \setminus G_{\kappa_{n,n-i-1}}$ we use the extension procedure described in Lemma 4.15 to add for each $l \leq k$ consecutively from $i = 0$ to $i = l$ the first available $b \in f_0^{-1}(A_l)$ to the domain of $\kappa_{l,0}^{C_s}$ if $i = 0$, or the first available $(b, z_0, \dots, z_{i-1}) \in (B \times Z_{l,0} \times \dots \times Z_{l,i-1}) \setminus G_{\kappa_{l,i-1}^{C_s}}$ to the domain of $\kappa_{l,i}^{C_s}$, if $0 < i$.

After all extension are done we extend the resulting condition to a condition C_{s+1} of type $k + 1$.

b) $s = 3\langle i, e, x\rangle + 1$. Let $type(C_s) = k$. Check whether there exists a condition $D \supseteq C_s$ of type $\max(i, k) + 1$ such that $(D)_i \Vdash_i^* F_e(x)$. If the answer is positive let C_{s+1} be the least such D. Clearly $C_s \leq C_{s+1}$ and $C_{s+1} \Vdash_i F_e(x)$. In case of negative answer let C_{s+1} be the least extension of C_s of type $k + 1$. Clearly $C_{s+1} \Vdash_i \neg F_e(x)$.

d) $s = 3e + 2$. Let λ_e be the e-th primitive computable function and γ be the computable function defined in Lemma 4.2. Set

$$L_n = \{x | (\exists D \supseteq C_s)(type(D) = n \wedge D \Vdash_n^* F_{\gamma(n,\lambda_e(n))}(x))\}.$$

Clearly $\{L_n\} \leq_e \mathcal{P}$ and hence $\{L_n\} \neq \mathcal{Y}$. Find a n such that $Y_n \neq L_n$ and a x such that $\neg(x \in L_n \iff x \in Y_n)$. Notice that since $\bigoplus_n Y_n^+ \leq_T \mathcal{P}_\omega$, we can find n and x effectively in $\mathcal{P}_\omega$.

If $x \notin L_n$, then let C_{s+1} be the least extension of C_s of type equal to $type(C_s) + 1$. Otherwise, find the least D such that $type(D) = n$, $C_s \subseteq D$ and $D \Vdash_n^* F_e(x)$. Extend D to a condition C_{s+1} such that $C_s \subseteq C_{s+1}$ and $type(C_{s+1}) = \max(n, type(C_s)) + 1$.

4.9. **Verification of the construction.** Let f be the unique enumeration such that for all s, $C_s \subseteq f$. From the construction of f it follows directly that it is generic and hence if $Y \subseteq \mathbb{N}$ and $Y = W_e((f^{-1}(\mathfrak{M})^+)^{(n)})$ then $Y = \{y | ((\exists s)(C_s \Vdash_n F_{\gamma(n,e)}(y)\}$.

First we shall show that $\mathcal{Y} \not\leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$ i.e. that $\mathcal{Y}$ is not c.e.in $f^{-1}(\mathfrak{M})$. Assume otherwise. Then there exists an e such that for all n,

$$Y_n = W_{\lambda_e(n)}((f^{-1}(\mathfrak{M})^+)^{(n)}).$$

Consider the stage $s = 3e + 2$.

According to the construction let n and x be the natural numbers such that $Y_n \neq L_n$ and $\neg(x \in L_n \iff x \in Y_n)$.

Suppose that $x \notin L_n$ and $x \in Y_n$. We shall show that $C_s \Vdash_n \neg F_{\gamma(n,\lambda_e(n))}(x)$. Indeed, assume that there exists $D \geq C_s$ such that $D \Vdash_n F_{\gamma(n,\lambda_e(n))}(x)$. We may assume that $type(D) \geq n$. Then $(D)_n \supseteq C_s$, $D \Vdash_n^* F_{\gamma(n,\lambda_e(n))}(x)$ and hence $(D)_n \Vdash_n^* F_{\gamma(n,\lambda_e(n))}(x)$. Therefore $x \in L_n$. A contradiction. Since $C_s \Vdash_n \neg F_{\gamma(n,\lambda_e(n))}(x)$, $x \notin W_{\lambda_e(n)}((f^{-1}(\mathfrak{M})^+)^{(n)})$ which contradicts the assumption $x \in Y_n$.

Suppose now that x in L_n and $x \notin Y_n$. Then there exists a $D \leq C_{s+1}$ such that $D \Vdash_n^* F_{\gamma(n,\lambda_e(n))}(x)$. Therefore $C_{s+1} \Vdash_n^* F_{\gamma(n,\lambda_e(n))}(x)$ and hence $C_{s+1} \Vdash_n F_{\gamma(n,\lambda_e(n))}(x)$. Therefore $x \in W_{\lambda_e(n)}(f^{-1}(\mathfrak{M})^{(n)})$ which contradicts the assumption $x \notin Y_n$.

It remains to see that $f^{-1}(\mathfrak{M})^{(\omega)} \equiv_e \mathcal{P}_\omega(g^{-1}(\mathcal{A}))$. From the construction it follows that the sequence $\{C_s\}$ is computable in $\mathcal{P}_\omega$. Let us fix an index a such that for all sets Y, $W_a(Y) = Y$. Then

$$(f^{-1}(\mathfrak{M})^+)^{(n)} = W_a((f^{-1}(\mathfrak{M})^+)^{(n)}) = \{y | (\exists s)(type(C_s) = n \wedge C_s \Vdash_n^* F_{\gamma(n,a)}(y))\}.$$

Hence $(f^{-1}(\mathfrak{M})^+)^{(n)}$ is enumeration reducible to $\mathcal{P}_\omega$ uniformly in n. Therefore $(f^{-1}(\mathfrak{M})^+)^{(\omega)} \leq_e \mathcal{P}_\omega \equiv_e \mathcal{P}_\omega(g^{-1}(\mathcal{A}))$. On the other hand, as mentioned in the beginning, $\mathcal{P}_\omega(g^{-1}(\mathcal{A})) \leq_e (f^{-1}(\mathfrak{M})^+)^{(\omega)}$. Hence

$$f^{-1}(\mathfrak{M})^{(\omega)} \equiv_e (f^{-1}(\mathfrak{M})^+)^{(\omega)} \equiv_e \mathcal{P}_\omega(g^{-1}(\mathcal{A})).$$

The last step in the proof of Theorem 3.1 is to remove the additional assumption $\bigoplus_n Y_n \leq_T \mathcal{P}$.

Suppose that $\mathcal{Y} = \{Y_n\}$ is a sequence of sets of natural numbers and $\bigoplus_n Y_n \not\leq_T \mathcal{P}_\omega$. Following the proof above we can construct a regular enumeration f of $|\mathfrak{M}|$ so that $(f^{-1}(\mathfrak{M})^+)^{(\omega)} \equiv_e \mathcal{P}_\omega(g^{-1}(\mathcal{A}))$. Assume that $\mathcal{Y} \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$. Then $\{Y_n^+\} \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n+1)}\}$ and hence

$$\bigoplus_n Y_n^+ \leq_T (f^{-1}(\mathfrak{M})^+)^{(\omega)}.$$

Therefore $\bigoplus_n Y_n^+ \leq_T \mathcal{P}_\omega(g^{-1}(\mathcal{A})) \equiv_T \mathcal{P}_\omega$. A contradiction.

5. Definability in the Marker's extension

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures. Set $A = \bigcup_n |\mathfrak{A}_n|$, denote by $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$ and by M the domain of $\mathfrak{M}$. Recall that for each $n < \omega$ the domain and the predicates of the structure $\mathfrak{A}_n$ are Σ_{n+1} definable in $\mathfrak{M}$ uniformly in n. Hence for each enumeration f of M, the sequence $\{f^{-1}(\mathfrak{A}_n)\}$ is enumeration reducible to the sequence $\{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$.

5.1. Definition. Let $\mathcal{R} = \{R_n\}$ be a sequence of subsets of A.

(i) The sequence $\mathcal{R}$ is *ω-enumeration reducible to* $\vec{\mathfrak{A}}$ ($\mathcal{R} \leq_\omega \vec{\mathfrak{A}}$) if for each enumeration g of A, $g^{-1}(\mathcal{R}) \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.

(ii) The sequence $\mathcal{R}$ is *computablely enumerable in* $\mathfrak{M}$ ($\mathcal{R} \leq_{r.e.} \mathfrak{M}$) if for for each enumeration f of M, $f^{-1}(\mathcal{R})$ is c.e.in $f^{-1}(\mathfrak{M})$, i.e. $(\forall n)(f^{-1}(R_n) \leq_{r.e.} J_T^n(f^{-1}(\mathfrak{M})))$ uniformly in n.

From Corollary 2.10 it follows that $\mathcal{R} \leq_{r.e.} \mathfrak{M}$ if and only if for each enumeration f of M, $f^{-1}(\mathcal{R}) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$.

Next theorem gives a positive answer to Question 3 from the introduction:

5.2. Theorem. *Let $\mathcal{R}$ be a sequence of subsets of A. Then $\mathcal{R} \leq_\omega \vec{\mathfrak{A}} \iff \mathcal{R} \leq_{r.e.} \mathfrak{M}$.*

5.3. Lemma. *Let f be an enumeration of $\mathfrak{M}$. There exists an enumeration g of $\mathfrak{A}$ such that that following assertions hold:*

(1) *The set $E = \{\langle i, j\rangle | f(i) = g(j)\}$ is computable in $f^{-1}(\mathfrak{M})^+$.*

(2) *$\mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_e (f^{-1}(\mathfrak{M})^+)^{(n)}$ uniformly in n.*

(3) *$\mathcal{P}_\omega(g^{-1}(\vec{\mathfrak{A}})) \leq_T (f^{-1}(\mathfrak{M})^+)^{(\omega)}$.*

Proof. Set $B = f^{-1}(A)$. Clearly B is computable in $f^{-1}(\mathfrak{M})^+$. Let λ be a computable in $f^{-1}(\mathfrak{M})^+$ one to one enumeration of B.

Set $g(k) = f(\lambda(k))$. Then $\{\langle i, j\rangle | f(i) = g(j)\} = \{\langle \lambda(k), k\rangle | k \in \mathbb{N}\}$ is computable in $f^{-1}(\mathfrak{M})^+$.

Notice that if $P \subseteq A$, then $g^{-1}(P) = \{k | \lambda(k) \in f^{-1}(P)\}$. From here and from the fact that the sets $f^{-1}(\mathfrak{A}_n)$ are enumeration reducible to $(f^{-1}(\mathfrak{M})^+)^{(n)}$ uniformly in n we get (2). Finally, (3) is a direct consequence of (2). $\square$

Proof of Theorem 5.2. Let $\mathcal{R} = \{R_n\}$ be a sequence of subsets of A. Suppose that $\mathcal{R} \leq_\omega \vec{\mathfrak{A}}$. Let f be an enumeration of M and g be an enumeration of A satisfying the conditions (1) - (3) from the lemma above. Since $\mathcal{R} \leq_\omega \vec{\mathfrak{A}}$, $g^{-1}(\mathcal{R}) \leq_\omega g^{-1}(\vec{\mathfrak{A}})$. Hence $g^{-1}(\mathcal{R}) \leq_e \mathcal{P}(g^{-1}(\vec{\mathfrak{A}})) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$.

Let $R \subseteq A$. Then $f^{-1}(R) = \{i | (\exists j)(\langle i, j\rangle \in E \wedge j \in g^{-1}(R)\}$. Hence $f^{-1}(R) \leq_e g^{-1}(R) \oplus f^{-1}(\mathfrak{M})^+$.

Since $f^{-1}(\mathfrak{M})^+ \leq_e (f^{-1}(\mathfrak{M})^+)^{(n)}$ uniformly in n, we get that

$$f^{-1}(\mathcal{R}) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}.$$

Thus $\mathcal{R} \leq_{r.e.} \mathfrak{M}$.

Suppose now that $\mathcal{R} \leq_{r.e.} \mathfrak{M}$. Assume that $\mathcal{R} \not\leq_\omega \vec{\mathfrak{A}}$. Then there exists an enumeration g of A such that $g^{-1}(\mathcal{R}) \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$. By Theorem 3.1 there exists an enumeration f of M such that $E = \{\langle i, j\rangle | f(i) = g(j)\}$ is computable and $g^{-1}(\mathcal{R}) \not\leq_{r.e.} f^{-1}(\mathfrak{M})$. Clearly $g^{-1}(\mathcal{R}) \leq_e f^{-1}(\mathcal{R})$. Hence $f^{-1}(\mathcal{R}) \not\leq_{r.e.} f^{-1}(\mathfrak{M})$. A contradiction. □

5.4. Definition. Let $R \subseteq A$ and $n \geq 0$.

(i) $R \leq_n \vec{\mathfrak{A}}$ if for each enumeration g of A, $g^{-1}(R) \leq_n g^{-1}(\mathcal{R})$, i.e. $g^{-1}(R) \leq_e \mathcal{P}_n(g^{-1}(\mathcal{R}))$.

(ii) ([2, 4]) R *is relatively intrinsically* Σ_{n+1} *in* $\mathfrak{M}$ if for each enumeration f of M, $f^{-1}(R) \leq_{r.e.} J_T^n(f^{-1}(\mathfrak{M}))$, i.e. $f^{-1}(R) \leq_e (f^{-1}(\mathfrak{M})^+)^{(n)}$.

In [2, 4] it shown that a set R is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$ if and only if R is definable in $\mathfrak{M}$ by means of some computable infinitary Σ_{n+1} formula with finitely many parameters.

Next come the answers to Question 2 and Question 1 from the introduction:

5.5. Corollary. *Let* $R \subseteq A$ *and* $n \geq 0$. *Then* $R \leq_n \vec{\mathfrak{A}} \iff R$ *is relatively intrinsically* Σ_{n+1} *in* $\mathfrak{M}$.

Proof. Set $R_k = \emptyset$ if $k \neq n$ and $R_n = R$. Let $\mathcal{R} = \{R_n\}$. Notice that $\mathcal{R} \leq_\omega \vec{\mathfrak{A}} \iff R \leq_n \vec{\mathfrak{A}}$ and $\mathcal{R} \leq_{r.e.} \mathfrak{M}$ if and only if R is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$. Apply the previous theorem. □

5.6. Definition. Let $\mathfrak{A}$ be a structure and $R \subseteq |\mathfrak{A}|$. Then $R \leq_e \mathfrak{A}$ if for each enumeration g of $|\mathfrak{A}|$, $g^{-1}(R) \leq_e g^{-1}(\mathfrak{A})$.

5.7. Corollary. *For every structure* $\mathfrak{A}$ *there exists a structure* $\mathfrak{M}$ *such that* $|\mathfrak{A}| \subseteq |\mathfrak{M}|$ *and for all* $R \subseteq |\mathfrak{A}|$, $R \leq_e \mathfrak{A}$ *if and only if* R *is relatively intrinsically* Σ_1 *in* $\mathfrak{M}$.

Proof. Consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ such that $\mathfrak{A}_0 = \mathfrak{A}$. Clearly if $R \subseteq |\mathfrak{A}|$ then $R \leq_0 \vec{\mathfrak{A}} \iff R \leq_e \mathfrak{A}$. Let $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$. □

6. Co-spectra of Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures. Set $A = \bigcup_n |\mathfrak{A}_n|$, denote by $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$ and by M the domain of $\mathfrak{M}$.

Recall that the spectrum of $\mathfrak{M}$ is the set

$$\{\mathbf{a} | \mathbf{a} \in \mathcal{D}_T \wedge \ \mathbf{a} \text{ computes the diagram of an isomorphic copy of } \mathfrak{M}\}.$$

In other words,

$$Sp(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \in \mathcal{D}_T \wedge (\exists f)(f \text{ is an enumeration of } M \wedge d_T(f^{-1}(\mathfrak{M})) \leq_T \mathbf{a})\}.$$

We are going to study the properties of $Sp(\mathfrak{M})$ in the next section. Here we shall prove some facts about the co-spectrum of $\mathfrak{M}$.

Given a set X of natural numbers set $d_e(X) = \{Y|Y \equiv_e X\}$. The set $d_e(X)$ is *the enumeration degree of* X. Set $\mathcal{D}_e = \{d_e(X)|X \subseteq \mathbb{N}\}$ and define the partial ordering "$\leq_e$" on $\mathcal{D}_e$ by letting $d_e(X) \leq_e d_e(Y) \iff X \leq_e Y$. The ordering "$\leq_e$" has a least element $\mathbf{0}_e = d_e(\emptyset)$ which consists of all c.e. sets. As usual, set $d_e(X)' = d_e(X')$.

There is a natural embedding ι of the Turing degrees into the enumeration degrees defined by $\iota(d_T(X)) = d_e(X^+)$ which preserves the ordering and the jump operation. Hence we may assume that the Turing degrees are a subset of the enumeration degrees. Actually this subset is equal to the set TOT of all total enumeration degrees, i.e. of all enumeration degrees which contain a total set.

Given a Turing degree $\mathbf{x} = d_T(X)$ and $n \geq 0$, by $\mathbf{x}^{(n)}$ we shall denote the $n-th$ Turing jump of $\mathbf{x}$. Clearly $\mathbf{x}^{(n)} = d_e((X^+)^{(n)})$. Notice that for total sets X, $(X^+)^{(n)} \equiv_e X^{(n)}$. In particular $(\emptyset^+)^{(n)} \equiv_e \emptyset^{(n)}$.

6.1. Definition. Let $n \geq 0$. The *n-th jump spectrum* of $\mathfrak{M}$ is the set

$$Sp_n(\mathfrak{M}) = \{\mathbf{x}^{(n)}|\mathbf{x} \in Sp(\mathfrak{M})\}.$$

6.2. Definition. Let $n \geq 0$. The *n-th co-spectrum of* $\mathfrak{M}$ is the set of enumeration degrees

$$CoSp_n(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \in \mathcal{D}_e \wedge (\forall \mathbf{x} \in Sp_n(\mathfrak{M}))(\mathbf{a} \leq_e \mathbf{x})\}.$$

Using the pre-ordering "$\leq_\omega$" on the sequences of sets of natural numbers, one can define an extension of the enumeration degrees which consists of the so called *ω-enumeration degrees*. Given a sequence $\mathcal{X}$ of sets of natural numbers let $d_\omega(\mathcal{X}) = \{\mathcal{Y}|\mathcal{Y} \equiv_\omega \mathcal{X}\}$ and $\mathcal{D}_\omega = \{d_\omega(\mathcal{X})|\mathcal{X} \in \mathcal{P}(\mathbb{N})^\omega\}$ and define the partial ordering "$\leq_\omega$" on $\mathcal{D}_\omega$ by $d_\omega(\mathcal{X}) \leq_e d_\omega(\mathcal{Y}) \iff \mathcal{X} \leq_\omega \mathcal{Y}$. The ordering "$\leq_\omega$" has a least element $\mathbf{0}_\omega = d_\omega(\{\emptyset^{(n)}\})$. For an introduction to the ω-enumeration degrees the reader may consult [21] and [8].

Again there is a natural embedding κ of the enumeration degrees into the ω-enumeration degrees. Given $X \subseteq \mathbb{N}$, set $X \uparrow = X, \emptyset, \ldots, \emptyset, \ldots$. On can easily see that $X \leq_e Y \iff X \uparrow \leq_\omega Y \uparrow$. Set $\kappa(d_e(X)) = d_\omega(X \uparrow)$.

Combining the embeddings ι and κ we obtain an embedding λ of $\mathcal{D}_T$ into $\mathcal{D}_\omega$, where $\lambda(d_T(X)) = d_\omega((X^+) \uparrow)$. Notice that if $\mathcal{X}$ is a sequence of sets of natural numbers and $B \subseteq \mathbb{N}$ then $d_\omega(\mathcal{X}) \leq_\omega \lambda(d_T(B))$ if and only if $\mathcal{X} \leq_\omega (B^+) \uparrow$ if and only if $\mathcal{X} \leq_e \{(B^+)^{(n)}\}$ if and only if $\mathcal{X}$ is c.e.in B.

6.3. Definition. The ω-enumeration co-spectrum of $\mathfrak{M}$ is the set of ω-enumeration degrees

$$Ocsp(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \in \mathcal{D}_\omega \wedge (\forall \mathbf{x} \in Sp(\mathfrak{M}))(\mathbf{a} \leq_\omega \lambda(\mathbf{x}))\}$$

6.4. Theorem.

(1) $Ocsp(\mathfrak{M}) = \{d_\omega(\mathcal{Y})|$*for all enumerations g of A,* $\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})\}$.
(2) $CoSp_n(\mathfrak{M}) = \{d_e(Y)|$*for all enumerations g of A,* $Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))\}$.

Proof. Let $\mathcal{Y}$ be a sequence of sets of natural numbers.

Suppose that $d_\omega(\mathcal{Y}) \in Ocsp(\mathfrak{M})$. Assume that for some enumeration g of A, $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$. Using Theorem 3.1 define an enumeration f of M such that $\mathcal{Y}$ is not c.e. in $f^{-1}(\mathfrak{M})$. Then $d_\omega(\mathcal{Y}) \not\leq_\omega \lambda(d_T(f^{-1}(\mathfrak{M}))$. A contradiction.

Suppose that for all enumerations g of A, $\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$. Consider an enumeration f of M. By Lemma 5.3 there exists an enumeration g of A such that $\mathcal{P}(g^{-1}(\vec{\mathfrak{A}})) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$ uniformly in n. Since $\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$, $\mathcal{Y} \leq_e \mathcal{P}(g^{-1}(\vec{\mathfrak{A}}))$. Therefore $\mathcal{Y}$ is c.e. in $f^{-1}(\mathfrak{M})$.

To prove (2) fix $Y \subseteq \mathbb{N}$ and consider the sequence $\{Y_k\}$, where $Y_k = \emptyset$ if $k \neq n$ and $Y_n = Y$. Then for each Turing degree $\mathbf{x}$, $d_e(Y) \leq_e \mathbf{x}^{(n)} \iff d_\omega(\mathcal{Y}) \leq_\omega \lambda(\mathbf{x})$. Hence $d_\omega(\mathcal{Y}) \in Ocsp(\mathfrak{M}) \iff d_e(Y) \in CoSp_n(\mathfrak{M})$. Finally, notice that for each enumeration g of A, $\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}}) \iff Y_n \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))$. $\square$

6.1. **Examples.** Let $\mathcal{R} = \{R_n\}$ be a sequence of sets of natural numbers. Set $\mathfrak{A}_0 = (\mathbb{N}; G_S, R_0)$ and $\mathfrak{A}_{n+1} = (\mathbb{N}; R_{n+1})$. Here by G_S we denote the graph of the successor function $\lambda x.x + 1$. Set $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

6.5. Proposition. *For each enumeration g of $\mathbb{N}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.*

Proof. First of all notice that for each enumeration g of $\mathbb{N}$ the mapping g is computable in $g^{-1}(G_S)$. Indeed, we have for all n that $g^{-1}(n+1) = \mu x[\langle g^{-1}(n), x\rangle \in g^{-1}(G_S)]$. This shows that g^{-1} is computable in $g^{-1}(G_S)$ and hence g is also computable in $g^{-1}(G_S)$. Now, consider an enumeration g of $\mathbb{N}$. Then $(\forall n)(R_n = \{g(x) | x \in g^{-1}(R_n)\})$. Hence $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$. $\square$

6.6. Corollary.

(1) $Ocsp(\mathfrak{M}) = \{d_\omega(\mathcal{Y}) | \mathcal{Y} \leq_\omega \mathcal{R}\}$.
(2) $(\forall n)(CoSp_n(\mathfrak{M}) = \{\mathbf{a} | \mathbf{a} \leq_e d_e(\mathcal{P}_n(\mathcal{R}))\})$.

Proof. Suppose that $d_\omega(\mathcal{Y}) \in Ocsp(\mathfrak{M})$. Let $g = \lambda x.x$. Then $g^{-1}(\vec{\mathfrak{A}}) \leq_\omega \mathcal{R}$. By Theorem 6.4, $\mathcal{Y} \leq_\omega \mathcal{R}$. Suppose now, that for $\mathcal{Y} \leq_\omega \mathcal{R}$. Then for each enumeration g of $\mathbb{N}$, $\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$. Hence $d_\omega(\mathcal{Y}) \in Ocsp(\mathfrak{M})$.

The proof of (2) is similar. $\square$

6.7. Definition. Call $\mathbf{a}$ *n-th co-degree of* $\mathfrak{M}$ if $\mathbf{a}$ is the greatest element of $CoSp_n(\mathfrak{M})$.

Clearly if $\mathbf{a}$ is the least element of $Sp_n(\mathfrak{M})$, i.e. $\mathbf{a}$ is *n-th jump degree of* $\mathfrak{M}$, then $\mathbf{a}$ is also n-th co-degree of $\mathfrak{M}$. It is tempting to conjecture that a structure $\mathfrak{M}$ has n-th co-degree if and only if $\mathfrak{M}$ has n-th jump degree. The failure of this conjecture follows from a result of Richter [16] which states that every linear ordering has co-degree $\mathbf{0}$ but there exist linear orderings without a degree.

Next comes an example of a simple structure $\mathfrak{M}$ such that for all n the n-th co-degree of $\mathfrak{M}$ is $\mathbf{0}^{(n)}$ but $\mathfrak{M}$ has no n-th jump degree.

Consider a sequence $\mathcal{R}$ of sets of natural numbers having the the following properties:

(i) $(\forall n)(\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)})$.
(ii) $\mathcal{R} \not\leq_e \{\emptyset^{(n)}\}$.
(iii) $\mathcal{R} \leq_e \{\emptyset^{(n+1)}\}$.

The existence of such sequences is shown in [8], where these sequences are called *almost zero*.

By the previous proposition there exists a structure $\mathfrak{M}$ such that $(\forall n)(CoSp_n(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \le_e d_e(\mathcal{P}_n(\mathcal{R}))\})$. Clearly for all n, $\mathbf{0}^{(n)}$ is the greatest element of $CoSp_n(\mathfrak{M})$ and hence $\mathbf{0}^{(n)}$ is the n-th co-degree of $\mathfrak{M}$. Assume that for some n, $Sp_n(\mathfrak{M})$ contains a least degree $\mathbf{b}$. Then $\mathbf{b} = \mathbf{0}^{(n)}$. Let f be an enumeration of $|\mathfrak{M}|$ such that $\emptyset^{(n)} \equiv_T (f^{-1}(\mathfrak{M})^+)^{(n)}$. By Lemma 5.3, there exists an enumeration g of $\mathbb{N}$ such that for all k, $R_k \le_e \mathcal{P}_k(g^{-1}(\mathcal{R})) \le_e (f^{-1}(\mathfrak{M})^+)^{(k)}$ uniformly in k. Then for all $k \ge n$, $R_k \le_e \emptyset^{(k)}$ uniformly in k. From here by (i) $\mathcal{R} \le_e \{\emptyset^{(n)}\}$. A contradiction.

Using (iii) we obtain the $\mathbf{0}^{(\omega)}$ is the least element of $\{\mathbf{a}^{(\omega)}|\mathbf{a} \in Sp(\mathfrak{M})\}$, i.e. that $\mathbf{0}^{(\omega)}$ is the ω jump degree of $\mathfrak{M}$. Indeed, by Theorem 3.1 there exists an enumeration f of $|\mathfrak{M}|$ such that $(f^{-1}(\mathfrak{M})^+)^{(\omega)} \equiv_e \mathcal{P}_\omega(\mathcal{R})$. Since $\mathcal{R} \le_e \{\emptyset^{(n+1)}\}$, $\mathcal{P}(\mathcal{R}) \le_e \{\emptyset^{(n+1)}\}$. Hence $\mathcal{P}_\omega(\mathcal{R}) \equiv_e \emptyset^{(\omega)}$.

In [11] it is shown that if $\mathfrak{B}$ is a Boolean algebra then for each $n < \omega$, $\mathfrak{B}$ has n-th co-degree $\mathbf{0}^{(n)}$ and for each Turing degree d, such that $\mathbf{0}^{(\omega)} \le_T d$ there exists a Boolean algebra with ω jump degree d.

7. Spectra of Marker's extensions

7.1. **The Goncharov-Khoussainov's lemma.** The main tool used in the applications of the Marker's extensions in [10, 26, 27] and [7] is the following lemma from [10]:

7.1. Lemma. *Let R be a co-infinite Σ^0_2-set that possesses an infinite computable subset S such that $R \setminus S$ is infinite. Then there exists a computable predicate Q satisfying the following conditions:*

(1) *for each $n \in \omega$, $(\exists a)(\forall b)Q(n,a,b) \iff n \in R$;*
(2) *for each $n \in \omega$, $(\exists a)(\forall b)Q(n,a,b) \iff (\exists^{=1} a)(\forall b)Q(n,a,b)$;*
(3) *for every b, there is a unique pair $\langle n,a\rangle$ such that $\neg Q(n,a,b)$;*
(4) *for every pair $\langle n,a\rangle$, either $(\exists^{=1} b)\neg Q(n,a,b)$ or $(\forall b)Q(n,a,b)$;*
(5) *for every a, there exists a unique n such that $(\forall b)Q(n,a,b)$.*

Here $(\exists^{=1} x)P(x)$ means that there exists a unique x satisfying P .

By setting $\kappa_0(n) \simeq a \iff (\forall b)Q(n,a,b)$ and $\kappa_1(n,a) \simeq b \iff \neg Q(n,a,b)$ we obtain the following reformulation of the lemma above.

7.2. Lemma. *Let R be a co-infinite Σ^0_2 set that possesses an infinite computable subset S such that $R \setminus S$ is infinite. Then there exist functions κ_0 and κ_1 such that κ_0 is a bijective mapping of R onto $\mathbb{N}$ and κ_1 is a bijective mapping of $\mathbb{N}^2 \setminus G_{\kappa_0}$ onto $\mathbb{N}$ and the graph of κ_1 is computable.*

The second formulation provides a clue how to generalize the lemma for all levels of the arithmetical hierarchy.

7.3. Proposition. *Let $n \ge 0$ and R be a Σ^0_{n+1} set possessing an infinite computable subset S. Then there exist functions $\kappa_0, \dots, \kappa_n$ such that the graph of κ_n is computable and*

κ_0 is a bijective mapping of R onto $\mathbb{N}$;
κ_1 is a bijective mapping of $\mathbb{N}^2 \setminus G_{\kappa_0}$ onto $\mathbb{N}$;
$\dots$
κ_n is a bijective mapping of $\mathbb{N}^{n+1} \setminus G_{\kappa_{n-1}}$ onto $\mathbb{N}$.

Notice that for $n = 1$ the proposition is slightly stronger than Goncharov-Khoussainov's lemma since we do not require the set R to be co-infinite.

The idea of the proof of the following lemma belongs to Mariya Soskova.

7.4. Lemma. *Let $X \subseteq \mathbb{N}$. Let R be a Σ^0_2 in X set that contains an infinite computable subset S. There exists a one to one mapping κ of R onto $\mathbb{N}$ such that G_κ is Π^0_1 in X and $\mathbb{N}^2 \setminus G_\kappa$ contains an infinite computable subset T.*

Proof. Let us fix a Σ^0_2 approximation $\{R_x\}$ of R relatively X. This means that $\{R_x\}$ is a computable in X sequence of finite sets $R_0 = \emptyset$ and

$$a \in R \iff (\exists x)(\forall y > x)(a \in R_y).$$

Say that x is a witness for $a \in R$ if $a \notin R_x$ and for all $y > x$, $a \in R_y$. Clearly $a \in R$ if and only if there exists a witness x for $a \in R$ and this witness is unique.

Set $h_0(a) \simeq \langle a, x+1\rangle \iff a \notin S$ and x is a witness for $a \in R$. Clearly h_0 is a one to one mapping of $R \setminus S$ onto the set $H_0 = \{\langle a, x+1\rangle | a \notin S \wedge x \text{ is a witness for } a \in R\}$. Notice that both H_0 and the graph G_{h_0} of h_0 are Π^0_1 in X. Let $H_1 = \mathbb{N} \setminus H_0$. Then H_1 is an infinite Σ^0_1 in X set. Let h_1 be a bijective mapping of S onto H_1 such that the graph of h_1 is computable in X. Set $\kappa = h_0 \cup h_1$. Evidently κ is a bijective mapping of R onto $\mathbb{N}$ and the graph of κ is Π^0_1 in X.

It remains to see that $\mathbb{N}^2 \setminus G_\kappa$ contains an infinite computable subset. Let s_0 be the least element of S. Since $\langle 0, 0\rangle \in H_1$, we may assume that $h_1(s_0) = \langle 0, 0\rangle$. Then the set $T = \{(s_0, u) | u \neq \langle 0, 0\rangle\}$ is an infinite computable subset of $\mathbb{N}^2 \setminus G_\kappa$. □

Proof of Proposition 7.3. Induction on n. Let $n = 0$ and R be an infinite Σ^0_1 set. Let κ be a computable bijective mapping of $\mathbb{N}$ onto R. Set $\kappa_0 = \kappa^{-1}$. Then $(a, x) \in G_{\kappa_0} \iff \kappa(x) \simeq a$. Thus G_{κ_0} is computable.

Let R be a Σ^0_{n+2} set that contains an infinite computable subset S. Then R is Σ^0_2 in $\emptyset^{(n)}$. By the lemma above there exists a bijective mapping κ_0 of R onto $\mathbb{N}$ such that G_{κ_0} is Π^0_1 in $\emptyset^{(n)}$ and $\mathbb{N}^2 \setminus G_{\kappa_0}$ contains an infinite computable set T. Set $R_1 = \mathbb{N}^2 \setminus G_{\kappa_0}$. Clearly R_1 is Σ^0_1 in $\emptyset^{(n)}$. Hence it is Σ^0_{n+1}. Apply the induction hypothesis. □

The construction of the functions $\kappa_0, \dots, \kappa_n$ in the proof above is uniform in R and X. To see that let us return to Lemma 7.4. Suppose that r is a Σ^0_2 index of R relatively X, i.e. $R = W_r^{J_T(X)}$, where $J_T(X)$ is the Turing jump of X. Suppose also that σ is a program computing the characteristic function of the set S. Following the proof of Lemma 7.4 we shall show that there exists a computable function ρ so that $\rho(r, \sigma)$ yields an ordered pair $\langle r_1, \tau\rangle$ such that $\mathbb{N}^2 \setminus G_\kappa = W^X_{r_1}$ and τ is a program that computes the set T.

First we shall show that there is a uniform way to obtain a Σ^0_2 approximation of R relatively X. Let a be an index such that for all X, $J_T(X) = K^X = W^X_a$. Set $K^X_s = \{n | n < s \wedge n \in W^X_{a,s}\}$. Clearly K^X_s is a c.e. approximation of $J_T(X)$ relatively X. Following the definition of the better approximations from [13], define the sequence of finite characteristic functions $\{\alpha_s\}$ as follows. Set $\alpha_0 = \emptyset$. For $s > 0$ set $m_s = \min(K^X_s \setminus K^X_{s-1})$ if $K^X_s \setminus K^X_{s-1}$ is not empty and $m_s = s$ otherwise and let

$$\alpha_s(n) \simeq \begin{cases} 1 & \text{if } n \in K^X_s, \\ 0 & \text{if } n \leq m_s \wedge n \notin K^X_s. \end{cases}$$

Using the fact that for all s, $K^X_s \subseteq K^X$ and the properties of the better approximations from [13], we obtain that the sequence $\{\alpha_s\}$ satisfies the following conditions:

(1) $(\forall n)(\exists s)(K^X \upharpoonright n \subseteq \alpha_s \subseteq K^X)$;
(2) $(\forall n)(\exists s)(\forall t > s)(K^X \upharpoonright n \subseteq \alpha_t)$.

Set $\alpha_s^+ = \{\langle n,1\rangle | \alpha_s(n) \simeq 1\} \oplus \{\langle n,0\rangle | \alpha_s(n) \simeq 0\}$. Find an index e such that $W_r^{J_T(X)} = W_e(J_T(X)^+)$ and set $R_0 = \emptyset$ and $R_s = W_{e,s}(\alpha_s^+)$ for $s > 0$. It is easy to see that $\{R_s\}$ is a Σ_2^0 approximation of R relatively X.

Now, having the approximation $\{R_s\}$ we obtain in a uniform way c.e. indices for $\mathbb{N}^2 \setminus G_{h_0}$ and H_1 relatively X. Using the program σ we can define a computable strictly monotonically increasing mapping μ_0 of $\mathbb{N}$ onto S. Then $\mu_0(0) \simeq s_0$, where s_0 is the least element of S. Let μ_1 be a bijective computable in X mapping of $\mathbb{N}$ onto H_1 such that $\mu_1(0) \simeq \langle 0,0\rangle$. Set $h_1(s) \simeq \mu_1(\mu_0^{-1}(s))$. Clearly $h_1(s_0) \simeq \langle 0,0\rangle$.

The graph of h_1 has the following definition:

$$(s,n) \in G_{h_1} \iff (\exists m \leq s)(\mu_0(m) \simeq s \wedge \mu_1(m) \simeq n).$$

Thus G_{h_1} is computable in X and we can find a program which computes G_{h_1} relatively X and hence a c.e. index for $\mathbb{N}^2 \setminus G_{h_1}$ relatively X. Notice that that $\mathbb{N}^2 \setminus G_\kappa = \mathbb{N}^2 \setminus G_{h_0} \cap \mathbb{N}^2 \setminus G_{h_1}$ and hence we can find effectively r_1 from the indices of $\mathbb{N}^2 \setminus G_{h_0}$ and $\mathbb{N}^2 \setminus G_{h_1}$.

Finally, let τ be a program computing the set $T = \{(s_0, u) | u \neq \langle 0,0\rangle\}$.

Using the computable function ρ and following the proof of Proposition 7.3, we obtain the following uniform version of Proposition 7.3:

7.5. Proposition. *There exists a computable function $\lambda(n,r,\sigma)$ such that if $R = W_r^{\emptyset^{(n)}}$ and σ is a program that computes an infinite computable subset S of R then there exist functions $\kappa_0,\dots,\kappa_n$ such that the graph of κ_n is computable by the program $\lambda(n,r,\sigma)$ and*

κ_0 is a bijective mapping of R onto $\mathbb{N}$;
κ_1 is a bijective mapping of $\mathbb{N}^2 \setminus G_{\kappa_0}$ onto $\mathbb{N}$;
...
κ_n is a bijective mapping of $\mathbb{N}^{n+1} \setminus G_{\kappa_{n-1}}$ onto $\mathbb{N}$.

Relativizing once more we get also the following:

7.6. Proposition. *There exists a computable function $\lambda(n,r,\sigma)$ such that if $X \subseteq \mathbb{N}$, $R = W_r((X^+)^{(n)})$ and σ is a program that computes relatively X an infinite subset S of R then there exist functions $\kappa_0,\dots,\kappa_n$ such that the graph of κ_n is computable relatively X by the program $\lambda(n,r,\sigma)$ and*

κ_0 is a bijective mapping of R onto $\mathbb{N}$;
κ_1 is a bijective mapping of $\mathbb{N}^2 \setminus G_{\kappa_0}$ onto $\mathbb{N}$;
...
κ_n is a bijective mapping of $\mathbb{N}^{n+1} \setminus G_{\kappa_{n-1}}$ onto $\mathbb{N}$.

7.2. **Relative spectra of sequences of structures.** Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures, where $\mathfrak{A}_n = (A_n; P_1^n, \dots, P_{m_n}^n)$. Let $A = \bigcup_n A_n$. Recall that the the relative spectrum of $\vec{\mathfrak{A}}$ is the set of sequences of sets of natural numbers $Rsp(\vec{\mathfrak{A}}) = \{g^{-1}(\vec{\mathfrak{A}}) | g$ is an enumeration of $A\}$.

Fix an element $\top \notin A$ and set $A_\top = A \cup \{\top\}$. Given $R \subseteq A^r$, set $R^\top = \{(\bar{a},t) | \bar{a} \in R \vee t = \top\}$.

For each n set $\mathfrak{A}_n^\top = (A_\top; A_n^\top, (P_1^n)^\top, \dots, (P_{m_n}^n)^\top)$ and let $\vec{\mathfrak{A}}^\top = \{\mathfrak{A}_n^\top\}$. Denote by $\mathfrak{M}$ the Marker's extension $\mathfrak{M}(\vec{\mathfrak{A}}^\top)$ of the sequence $\vec{\mathfrak{A}}^\top$.

7.7. Theorem. *$Sp(\mathfrak{M}) = \{d_T(B) | (\exists \mathcal{Y} \in Rsp(\vec{\mathfrak{A}}))(\mathcal{Y}$ is c.e. in $B)\}$.*

Proof. As in the proof of Theorem 3.1, to simplify the notation we shall assume that for all n the structure $\mathfrak{A}_n$ has no predicates other than its domain and hence $\mathfrak{A}_n^\top = (A_\top; A_n^\top)$. Denote by M the domain of the structure $\mathfrak{M}$. Notice that

$$\mathfrak{M} = (M; A_\top, \{M_n^{A_n^\top}, X_0^{A_n^\top}, \dots, X_n^{A_n^\top}\}_{n<\omega}).$$

Let $d_T(B) \in Sp(\mathfrak{M})$. Then there exists an enumeration f of M such that $f^{-1}(\mathfrak{M}) \leq_T B$. According Lemma 5.3, there exists an enumeration g_0 of $\mathfrak{A}^\top$ such that $g_0^{-1}(A_n^\top) \leq_e \{(f^{-1}(\mathfrak{M})^+)^{(n)}\}$ uniformly in n. Let $g_0^{-1}(\top) \simeq x_0$ and g be a one to one mapping of $\mathbb{N} \setminus \{x_0\}$ onto A such that $g(x) \simeq g_0(x)$ for $x \neq x_0$. Then for all n, $g^{-1}(A_n) = \{x | (\exists y \neq x_0)(\langle x, y\rangle \in g_0^{-1}(A_n^\top)\}$. Hence $g^{-1}(A_n) \leq_e g_0^{-1}(A_n^\top)$ uniformly in n. Thus $g^{-1}(\vec{\mathfrak{A}})$ is c.e.in $f^{-1}(\mathfrak{M})$ and hence $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B.

Suppose now that g is an enumeration of A and $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B. We shall construct an enumeration f of $|\mathfrak{M}|$ such that $f^{-1}(\mathfrak{M}) \leq_T B$.

Let $g_0(0) \simeq \top$ and $g_0(x+1) \simeq g(x)$. Then g_0 is an enumeration of $A_\top$ and for each subset R of A, $g_0^{-1}(R) = \{x+1 | x \in g^{-1}(R)\}$. Hence for all n, $g_0^{-1}(A_n) \equiv_e g^{-1}(A_n)$ uniformly in n. Since $g_0^{-1}(A_n^\top) = \{\langle x, y\rangle | x \in g_0^{-1}(A_n) \vee y = 0\}$, we have that the sequence $\{g_0^{-1}(A_n^\top)\}$ is r.e in B. Let μ be a computable function such that for all n, $g_0^{-1}(A_n^\top) = W_{\mu(n)}((B^+)^{(n)})$. Notice that for each n the set $S = \{\langle x, 0\rangle | x \in \mathbb{N}\}$ is an infinite computable subset of $g_0^{-1}(A_n^\top)$. According Proposition 7.6 for each n there exists a system of functions $\kappa_{n,0} \dots, \kappa_{n,n}$ such that the graphs of the functions $\kappa_{n,n}$ are computable in B uniformly in n and

$\kappa_{n,0}$ is a bijective mapping of $g_0^{-1}(A_n^\top)$ onto $\mathbb{N}$;
$\kappa_{n,1}$ is a bijective mapping of $\mathbb{N}^2 \setminus G_{\kappa_{n,0}}$ onto $\mathbb{N}$;
...
$\kappa_{n,n}$ is a bijective mapping of $\mathbb{N}^{n+1} \setminus G_{\kappa_{n,n-1}}$ onto $\mathbb{N}$.

Now we are ready to define the enumeration f of $\mathfrak{M}$. Set $f(2n) \simeq g_0(n)$. Then $f^{-1}(A_n^\top) = \{\langle 2x, 2y\rangle | \langle x, y\rangle \in g_0^{-1}(A_n^\top)\}$. As in the proof of Theorem 3.1 divide the odd numbers into infinite and disjoint sets $Z_{n,i}, i \leq n$, which are computable uniformly in n and i. Then we can transform each system $\kappa_{n,0}, \dots, \kappa_{n,n}$ into a system $\kappa^*_{n,0}, \dots, \kappa^*_{n,n}$ so that the graphs of the functions $\kappa^*_{n,n}$ are computable in B uniformly in n and

$\kappa^*_{n,0}$ is a bijective mapping of $\{(x, y) | \langle x, y\rangle \in f^{-1}(A_n^\top)\}$ onto $Z_{n,0}$;
$\kappa^*_{n,1}$ is a bijective mapping of $((2\mathbb{N})^2 \times Z_{n,0}) \setminus G_{\kappa_{n,0}}$ onto $Z_{n,1}$;
...
$\kappa^*_{n,n}$ is a bijective mapping of $((2\mathbb{N})^2 \times Z_{n,0} \times \dots \times Z_{n,n-1}) \setminus G_{\kappa^*_{n,n-1}}$ onto $Z_{n,n}$.

To complete the definition of f we need to define it on all sets $Z_{n,i}, i \leq n$. Fix n and define f on $Z_{n,i}$ by induction on i. Given an element z of $Z_{n,0}$ find the unique $\langle x, y\rangle \in f^{-1}(A_n^\top)$ such that $\kappa^*_{n,0}(x, y) \simeq z$ and let $f(z) \simeq h_0^{A_n^\top}(f(x), f(y))$. Suppose that $i < n$ and f is defined on the sets $Z_{n,0}, \dots, Z_{n,i}$. Let $z \in Z_{n,i+1}$. Then there exists a unique element $(x, y, z_0, \dots, z_i)$ of $(2\mathbb{N})^2 \times Z_{n_0} \times \dots \times Z_{n,i}$ such that $\kappa^*_{n,i+1}(x, y, z_0, \dots, z_i) \simeq z$. Let $f(z) \simeq h_{i+1}^{A_n^\top}(f(x), f(y), f(z_0), \dots, f(z_i))$.

Clearly for each n and $i \leq n$, $f^{-1}(X_i^{A_n^\top}) = Z_{n,i}$ and

$$(x, y, z_0, \dots, z_i) \in G_{\kappa^*_{n,i}} \iff (f(x), f(y), f(z_0), \dots, f(z_i)) \in G_{h_i^{A_n^\top}}.$$

Thus $f^{-1}(\mathfrak{M}) \equiv_T \bigoplus_n f^{-1}(M_n^{A_n^\top}) \equiv_T \bigoplus_n G_{\kappa^*_{n,n}} \leq_T B$. Therefore $d_T(B) \in Sp(\mathfrak{M})$. $\square$

The last theorem can be applied to finite sequences $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ of structures with domains subsets of a countable set A. Indeed, by setting $\mathfrak{A}_k = (A; =)$, $k > n$, we obtain a sequence $\vec{\mathfrak{A}}$ of structures such that for all enumerations g of A and $B \subseteq \mathbb{N}$, $g^{-1}(\vec{\mathfrak{A}})$ is c.e.in B if and only if $(\forall k \leq n)(g^{-1}(\mathfrak{A}_k) \leq_e (B^+)^{(k)})$

Hence we have the following:

7.8. Theorem. *Let $\mathfrak{A}_k = (A_k; P_1^k, \dots, P_{m_k}^k)$, $k = 0, \dots, n$ be a finite sequence of structures and A be a countable set such that $\bigcup_{k\leq n} A_k \subseteq A$. There exists a structure $\mathfrak{M}$ such that*

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists g)(g \text{ is an enumeration of } A \wedge (\forall k \leq n)(g^{-1}(\mathfrak{A}_k) \leq_e (B^+)^{(k)}))\}.$$

7.3. **Joint spectra of sequences of structures.** Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures, where $\mathfrak{A}_n = (A_n; P_1^n, \dots, P_{m_n}^n)$. Recall that the the joint spectrum of $\vec{\mathfrak{A}}$ is the set of sequences of sets of natural numbers

$$Jsp(\vec{\mathfrak{A}}) = \{\{g_n^{-1}(\mathfrak{A}_n)\} | (\forall n)(g_n \text{ is enumeration of } A_n)\}.$$

Notice that the joint spectrum is invariant with respect to isomorphisms, that is if $\{\mathfrak{A}_n^*\}$ is a sequence of structures and $(\forall n)(\mathfrak{A}_n \cong \mathfrak{A}_n^*)$ then $Jsp(\{\mathfrak{A}_n\}) = Jsp(\{\mathfrak{A}_n^*\})$. This property is not true for the relative spectra of a sequences of structures.

7.9. Theorem. *There exists a structure $\mathfrak{M}$ such that*

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists \mathcal{Y} \in Jsp(\bar{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B)\}.$$

Proof. We may assume that the domains A_n of the structures $\mathfrak{A}_n$ are disjoint. Let $A = \bigcup_n A_n$. By Theorem 7.7 there exists a structure $\mathfrak{M}$ such that

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists g)(g \text{ is enumeration of } A \text{ and } g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B)\}.$$

Let $d_T(B) \in Sp(\mathfrak{M})$ and g be an enumeration of A such that $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B. We shall construct a sequence $\mathcal{Y} = \{Y_n\}$ of sets of natural numbers so that $\mathcal{Y} \in Jsp(\vec{\mathfrak{A}})$ and $Y_n \leq_e (B^+)^{(n)}$ uniformly in n. Clearly for all n, $g^{-1}(A_n) \leq_e (B^+)^{(n)}$ uniformly in n. Hence there exist uniformly computable in $(B^+)^{(n)}$ bijections ρ_n of $\mathbb{N}$ onto A_n. Set $g_n(x) \simeq g(\rho_n(x))$. Then g_n is an enumeration of A_n and $g_n^{-1}(\mathfrak{A}_n) \leq_e (B^+)^{(n)}$ uniformly in n. Set $Y_n = g_n^{-1}(\mathfrak{A}_n)$.

Suppose now that $\mathcal{Y} = \{Y_n\} \in Jsp(\vec{\mathfrak{A}})$ and $Y_n \leq_e (B^+)^{(n)}$ uniformly in n. For each $n < \omega$ let f_n be an enumeration of A_n such that $f_n^{-1}(\mathfrak{A}_n) = Y_n$. Set $N_n = \{\langle n, x\rangle | x \in \mathbb{N}\}$ and define the bijection g_n of N_n onto A_n by $g_n(\langle n, x\rangle) \simeq f_n(x)$. Let $g = \bigcup_n g_n$. Then g is an enumeration of A and for all n, $g^{-1}(\mathfrak{A}_n) \leq_e Y_n$ uniformly in n. Hence $(\forall n)(g^{-1}(\mathfrak{A}_n) \leq_e (B^+)^{(n)})$ uniformly in n. Thus $d_T(B) \in Sp(\mathfrak{M})$. $\square$

Using Theorem 7.9 we get the following version for Turing reducibility.

7.10. Theorem. *Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$ be a sequence of structures. There exists a structure $\mathfrak{M}$ such that*

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists \{Y_n\} \in Jsp(\vec{\mathfrak{A}}))((\forall n)(Y_n \leq_T J_T^n(B)) \text{ uniformly in } n)\}.$$

Proof. Let $\mathfrak{A}_n = (A_n; P_1^n, \dots, P_{m_n}^n)$. For $1 \le i \le m_n$ let $\bar{P}_i^n$ be the complement of P_i^n. Denote by $\mathfrak{A}_n^+$ the structure $(A_n; P_1^n, \dots, P_{m_n}^n, \bar{P}_1^n, \dots, \bar{P}_{m_n}^n)$. Notice that for each enumeration f_n of A_n, $f_n^{-1}(\mathfrak{A}_n^+) \equiv_e (f_n^{-1}(\mathfrak{A}_n))^+$ uniformly in n. Let $\vec{\mathfrak{A}}^+ = \{\mathfrak{A}_n^+\}$. By Theorem 7.9 there exists a structure $\mathfrak{M}$ such that

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists \{Z_n\} \in Jsp(\vec{\mathfrak{A}}^+))((\forall n)(Z_n \le_e (B^+)^{(n)}) \text{ uniformly in } n)\}.$$

Suppose that $d_T(B) \in Sp(\mathfrak{M})$ and let $\mathcal{Z}$ be a sequence of sets of natural numbers such that $\mathcal{Z} \in Jsp(\vec{\mathfrak{A}}^+)$ and $(\forall n)(Z_n \le_e (B^+)^{(n)})$. Fix $n \in \omega$. Let f_n be an enumeration of A_n such that $Z_n = f_n^{-1}(\mathfrak{A}_n^+)$. Set $Y_n = f_n^{-1}(\mathfrak{A}_n)$. Then $(Y_n)^+ \equiv_e Z_n$ uniformly in n and hence $(Y_n)^+ \le_e (B^+)^{(n)}$ uniformly in n. Since $(B^+)^{(n)} \equiv_e J_T^n(B)^+$ uniformly in n, $(Y_n)^+ \le_e J_T^n(B)^+$ uniformly in n. Hence $Y_n \le_T J_T^n(B)$ uniformly in n.

The proof in the other direction is similar. □

Again we have the respective versions for finite sequences of structures.

7.11. Theorem. *Let $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ be a sequence of structures. Then*

(1) *There exists a structure $\mathfrak{M}$ such that*

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists f_1 \dots f_n)(\forall i \le n)(f_i \text{ is an enumeration of } |\mathfrak{A}_i| \wedge f_i^{-1}(\mathfrak{A}_i) \le_e (B^+)^{(i)})\}.$$

(2) *There exists a structure $\mathfrak{M}$ such that*

$$Sp(\mathfrak{M}) = \{d_T(B) | (\exists f_1 \dots f_n)(\forall i \le n)(f_i \text{ is an enumeration of } |\mathfrak{A}_i| \wedge f_i^{-1}(\mathfrak{A}_i) \le_T J_T^i(B))\}.$$

8. Applications

8.1. **Enumeration spectra.** In this subsection we shall make a brief overview of some modifications of the notion of degree spectrum of a structure which appeared recently in the literature and show that these modifications are closely connected.

Joint spectra and relative spectra of finite sequences of structures are introduced in [25] and [24]. A direct consequence of Theorem 7.8 is that the relative spectrum of each finite sequence of structures is spectrum of some structure. Using Theorem 7.11 we obtain that the same is true for joint spectra of finite sequences of structures.

The enumeration spectra are first considered in [20]. In [12] Kalimullin extends this notion in the following natural way.

Suppose that A is a countable set. A *partial enumeration of A* is a partial injective mapping φ of $\mathbb{N}$ onto A. Given a subset P of A^m and a partial enumeration φ of A, let $\varphi^{-1}(P) = \{\langle x_1, \dots, x_m \rangle | (\forall i)(1 \le i \le m \Rightarrow x_i \in \mathrm{dom}(\varphi)) \wedge (\varphi(x_1), \dots, \varphi(x_m)) \in P\}$

Let $\mathfrak{A} = (A; P_1, \dots, P_m)$ be a countable structure. For each partial enumeration φ of A, set $\varphi^{-1}(\mathfrak{A}) = \mathrm{dom}(\varphi) \oplus \varphi^{-1}(P_1) \oplus \dots \oplus \varphi^{-1}(P_m)$.

8.1. Definition.([12])The *enumeration spectrum of* $\mathfrak{A}$ is the set of enumeration degrees $Esp(\mathfrak{A}) = \{d_e(\varphi^{-1}(\mathfrak{A})) | \varphi \text{ is a partial enumeration of } A\}$.

Recall that by TOT we denote the set of all total enumeration degrees.

8.2. Theorem. *For every structure* $\mathfrak{A}$ *there exists a structure* $\mathfrak{M}$ *such that* $Sp(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \in TOT \wedge (\exists \mathbf{x} \in Esp(\mathfrak{A}))(\mathbf{x} \leq_e \mathbf{a})\}$.

Proof. Let $\mathfrak{A} = (A; P_1, \dots, P_m)$. Consider a countable set $C \supseteq A$ such that $C \setminus A$ is infinite. Set $\mathfrak{C} = (C; A, P_1, \dots, P_m)$. According Theorem 7.8 there exists a structure $\mathfrak{M}$ such that

$$Sp(\mathfrak{M}) = \{d_T(B)|(\exists f)(f \text{ is enumeration of } C \wedge f^{-1}(\mathfrak{C}) \leq_e B^+)\}.$$

Let $d_T(B) \in Sp(\mathfrak{M})$. Then $d_T(B) = d_e(B^+) \in TOT$ and for some enumeration f of C, $f^{-1}(\mathfrak{C}) \leq_e B^+$. Set $\varphi = f \restriction f^{-1}(A)$. Then φ is a partial enumeration of A and $\varphi^{-1}(\mathfrak{A}) \equiv_e f^{-1}(\mathfrak{C})$. Hence $d_e(f^{-1}(\mathfrak{C})) \in Esp(\mathfrak{A})$.

Suppose now that $d_e(B^+)$ is a total enumeration degree bounding an element $\mathbf{x}$ of $Esp(\mathfrak{A})$. Let φ be a partial enumeration of A such that $\varphi^{-1}(\mathfrak{A}) \in \mathbf{x}$. We may assume that $\mathbb{N} \setminus \mathrm{dom}(\varphi)$ is infinite. Let f be an enumeration of C such that $f \restriction \mathrm{dom}(\varphi) = \varphi$. Clearly $f^{-1}(\mathfrak{C}) \equiv_e \varphi^{-1}(\mathfrak{A})$. Hence $f^{-1}(\mathfrak{C}) \leq_e B^+$. Thus $d_T(B) \in Sp(\mathfrak{M})$. □

In [12] Kalimullin constructed a structure $\mathfrak{A}$ with enumeration spectrum consisting of all enumeration degrees $\mathbf{x}$ such that $\mathbf{0}_e <_e \mathbf{x}$. Applying the last Theorem to Kalimullin's structure we obtain a structure $\mathfrak{M}$ such that $Sp(\mathfrak{M})$ consists of all Turing degrees $\mathbf{a}$ such that $\mathbf{0} <_T \mathbf{a}$. Structures with such spectra are for the first time obtained by Slaman [18] and Wehner [28]. We shall return to Wehner's result in Subsection 8.3.

8.3. Definition. The *enumeration co-spectrum* of a structure $\mathfrak{A}$ is the set of enumeration degrees $CoEsp(\mathfrak{A}) = \{\mathbf{y}|(\forall \mathbf{x} \in Esp(\mathfrak{A}))(\mathbf{y} \leq_e \mathbf{x})\}$.

8.4. Corollary. *For every structure* $\mathfrak{A}$ *there exists a structure* $\mathfrak{M}$ *such that* $CoEsp(\mathfrak{A}) = CoSp(\mathfrak{M})$*,*

Proof. Given a structure $\mathfrak{A}$, consider the structure $\mathfrak{M}$ such that $Sp(\mathfrak{M}) = \{\mathbf{a}|\mathbf{a} \in TOT \wedge (\exists \mathbf{x} \in Esp(\mathfrak{A}))(\mathbf{x} \leq_e \mathbf{a})\}$. We shall show that $CoSp(\mathfrak{M}) = CoEsp(\mathfrak{A})$.

Indeed, let $\mathbf{y} \in CoEsp(\mathfrak{A})$ and $\mathbf{a} \in Sp(\mathfrak{M})$. Then for some element $\mathbf{x}$ of $Esp(\mathfrak{A})$, $\mathbf{y} \leq_e \mathbf{x} \leq_e \mathbf{a}$. Hence $\mathbf{y} \in CoSp(\mathfrak{M})$.

Suppose now that $\mathbf{y} \in CoSp(\mathfrak{M})$ and consider an element $\mathbf{x}$ of $Esp(\mathfrak{A})$. Then $(\forall \mathbf{a} \in TOT)(\mathbf{x} \leq_e \mathbf{a} \Rightarrow \mathbf{y} \leq_e \mathbf{a})$. Hence by Selman's Theorem [17] $\mathbf{y} \leq_e \mathbf{x}$. Thus $\mathbf{y} \in CoEsp(\mathfrak{A})$. □

In [20] it is shown that every countable ideal of enumeration degrees can be represented as an enumeration co-spectrum of some structure $\mathfrak{A}$. Hence we have the following corollary:

8.5. Corollary. *Every countable ideal of enumeration degrees is co-spectrum of some structure* $\mathfrak{M}$*.*

8.2. **Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures.** Let $\mathcal{R} = \{R_n\}$ be a sequence of sets of natural numbers. As in subsection 6.1 set $\mathfrak{A}_0 = (\mathbb{N}; G_S, R_0)$ and $\mathfrak{A}_{n+1} = (\mathbb{N}; R_{n+1})$, where G_S is the graph of the successor function $\lambda x.x+1$. Set $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$. According Theorem 7.7 there exists a structure $\mathfrak{M}_{\mathcal{R}}$ such that

$$Sp(\mathfrak{M}_{\mathcal{R}}) = \{d_T(B)|(\exists \mathcal{Y} \in Rsp(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B)\}.$$

8.6. Proposition. $Sp(\mathfrak{M}_{\mathcal{R}}) = \{d_T(B)|\mathcal{R}$ *is c.e. in* $B\}$.

Proof. Suppose that $d_T(B) \in Sp(\mathfrak{M}_{\mathcal{R}})$. Let g be an enumeration of $\mathbb{N}$ such that $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B. By Lemma 6.5 $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$. Hence $\mathcal{R}$ is c.e. in B.

Suppose now that $\mathcal{R}$ is c.e.in B. Consider the enumeration $g = \lambda x.x$ of $\mathbb{N}$. Clearly $g^{-1}(\vec{\mathfrak{A}}) \leq_e \mathcal{R}$. Hence $g^{-1}(\vec{\mathfrak{A}})$ is c.e. in B. Thus $d_T(B) \in Sp(\mathfrak{M}_{\mathcal{R}})$. □

8.7. Corollary. *For any two sequences $\mathcal{R}$ and $\mathcal{P}$ of sets of natural numbers,*

$$\mathcal{R} \leq_\omega \mathcal{P} \iff Sp(\mathfrak{M}_{\mathcal{P}}) \subseteq Sp(\mathfrak{M}_{\mathcal{R}}).$$

Proof. We have that $\mathcal{R} \leq_\omega \mathcal{P}$ if and only if for each $B \subseteq \mathbb{N}$, $\mathcal{P}$ is c.e. in B implies $\mathcal{R}$ is c.e. in B. Hence $\mathcal{R} \leq_\omega \mathcal{P}$ if and only if $Sp(\mathfrak{M}_{\mathcal{P}}) \subseteq Sp(\mathfrak{M}_{\mathcal{R}})$. □

Now, we may define the embedding μ of $\mathcal{D}_\omega$ into the Muchnik degrees generated by spectra of structures by $\mu(d_\omega(\mathcal{R})) = Sp(\mathfrak{M}_{\mathcal{R}})$.

8.3. **A structure with spectrum consisting of the Turing degrees which are non-low$_n$ for all n.** In [28] Wehner constructed a structure with spectrum consisting of all Turing degrees which are not equal to **0**. This result was further used in [9] in the construction for each n of a structure with spectrum consisting of all non-low$_n$ Turing degrees, i.e. of all degrees **a** such that $\mathbf{0}^{(n)} <_T \mathbf{a}^{(n)}$. Here we shall use again Wehner's construction in combination with Theorem 7.10 to obtain a structure with spectrum consisting of the Turing degrees **a** such that $(\forall n)(\mathbf{0}^{(n)} <_T \mathbf{a}^{(n)})$. To apply Theorem 7.10 we need to reveal the uniformity in Wehner's construction.

Let $\mathcal{F}$ be a countable family of sets of natural numbers. A subset U of $\mathbb{N}^2$ is an enumeration of $\mathcal{F}$ if the following two conditions are satisfied:

(1) For each $F \in \mathcal{F}$ there exists an a such that $F = \{n|(a,n) \in U\}$;
(2) For each $a \in \mathbb{N}$ the set $\{n|(a,n) \in U\}$ belongs to $\mathcal{F}$.

Given a countable family $\mathcal{F}$ of sets of natural numbers, Wehner defines in [28] the structure $\mathfrak{A}_{\mathcal{F}} = (A; S, Z, I)$, where $A = \mathcal{F} \times \mathbb{N} \times \mathbb{N}$, $Z = \{(F,x,0)|F \in \mathcal{F} \wedge x \in \mathbb{N}\}$, $S = \{((F,x,n),(F,x,n+1))|F \in \mathcal{F} \wedge x,n \in \mathbb{N}\}$ and $I = \{(F,x,n)|F \in \mathcal{F} \wedge x \in \mathbb{N} \wedge n \in F\}$.

8.8. Proposition. *(*[28]*)*

(1) *There is an uniform way, given an isomorphic copy $\mathfrak{B}$ of $\mathfrak{A}_{\mathcal{F}}$ on the natural numbers, to compute in the diagram of $\mathfrak{B}$ an enumeration U of $\mathcal{F}$.*
(2) *There is an uniform way, given an enumeration U of $\mathcal{F}$, to compute in U the diagram of an isomorphic copy $\mathfrak{B}$ of $\mathfrak{A}_{\mathcal{F}}$ on the natural numbers.*

Further on Wehner defines a family $\mathcal{F}$ of finite sets such that for every $B \subseteq \mathbb{N}$, $\mathcal{F}$ has an enumeration computable in B if and only if $\emptyset <_T B$. Since we want to relativize Wehner's construction in an uniform way we shall repeat it here. We shall use the following form of Wehner's family proposed by Kalimullin in [12]:

Let $X \subseteq \mathbb{N}$. Set $\mathcal{F}^X = \{\{n\} \oplus F | F \text{ is a finite set and } F \neq W_n^X\}$.

8.9. Lemma. *(*[12]*) There does not exist a c.e. in X enumeration U of $\mathcal{F}^X$.*

8.10. Lemma. *Suppose that $B \not\leq_T X$. Then one can compute uniformly in B and X an enumeration U of $\mathcal{F}^X$.*

Proof. Let $B^+ = B \oplus \bar{B}$. Clearly B^+ is not c.e. in X. Fix a computable in B one to one enumeration $x_0, \ldots, x_n, \ldots$ of B^+ and set for $s \in \omega$, $\nu_s = \langle x_0, \ldots, x_s \rangle$.

Given $n \in \mathbb{N}$, finite set F and $i \in \mathbb{N}$, let $U^0_{\langle n,F,i\rangle} = \{n\} \oplus F$. At step s set $V^s_{\langle n,F,i\rangle} = \{x | 2x+1 \in U^s_{\langle n,F,i\rangle}\}$ and check for each triple $a = \langle n, F, i\rangle < s$ whether $V^s_a = W^X_{n,s}$. If the answer is negative do nothing, otherwise add $2\nu_s + 1$ to U^s_a. Let $U^{s+1}_a = U^s_a$ if $s \le a$.

Set $U_a = \bigcup_s U^s_a$. Denote by V_a the set $\{x|2x+1 \in U_a\}$. Notice that if $a = \langle n, F, i\rangle$ then $U_a = \{n\} \oplus V_a$.

Fix $a = \langle n, F, i\rangle$. Then $V_a = W^X_n$ if and only if there exist arbitrary large s such that $V^s_a = W^X_{n,s}$. Indeed, since both sequences V^s_a and $W^X_{n,s}$ are monotone, from the existence of arbitrary large s such that $V^s_a = W^X_{n,s}$ it follows that $V_a = W^X_n$. Assume that $V_a = W^X_n$ but there exists a t such that for all $s > t$, $V^s_a \neq W^X_{n,s}$. Then by the construction the set V_a is finite and hence W^X_n is also finite. Then there exists a s_0 such that for all $s > s_0$, $V^s_a = V_a = W^X_n = W^X_{n,s}$. A contradiction. Assume now that $V_a = W^X_n$. Then W^X_n contains the elements of F and arbitrary large segments of B^+. Hence B^+ is c.e. in X which is not possible. So, $V_a \neq W^X_n$ and hence it is finite. Thus $U_a \in \mathcal{F}^X$.

Suppose that F is a finite set and $F \neq W^X_n$. Let t be a stage such that for all $s > t$, $F \neq W_{n,s}$. Fix an i such that $t < a = \langle n, F, i\rangle$. Then for all s, if $a < s$ then $F = V^s_a \neq W^X_{n,s}$. So $V_a = F$ and hence $U_a = \{n\} \oplus F$.

Thus the set $U = \{(a, x)|x \in U_a\}$ is an enumeration of $\mathcal{F}^X$.

It remains to see that U is computable in $B \oplus X$. This follows from the fact that

$$\begin{aligned}(\langle n, F, i\rangle, x) \in U \iff & ((x = 2n) \vee (x = 2y + 1 \wedge y \in F) \vee \\ & (x = 2\nu_s + 1 \wedge \langle n, F, i\rangle < s \wedge V^s_{\langle n,F,i\rangle} = W^X_{n,s})).\end{aligned}$$

□

Let $\mathfrak{A}_n$ be the Wehner's structure corresponding to the family $\mathcal{F}^{\emptyset^{(n)}}$, $n \ge 0$. Set $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$. By Theorem 7.10 there exists a structure $\mathfrak{M}$ such that

$$Sp(\mathfrak{M}) = \{d_T(B)|(\exists\{Y_n\} \in Jsp(\vec{\mathfrak{A}}))((\forall n)(Y_n \le_T J^n_T(B)) \text{ uniformly in } n)\}.$$

Suppose that $d_T(B) \in Sp(\mathfrak{M})$. Let $Y_n = f^{-1}_n(\mathfrak{A}_n)$ be a sequence of $Jsp(\vec{\mathfrak{A}})$ such that for all n, $Y_n \le_T J^n_T(B)$ uniformly in n. Since $J^n_T(B)$ computes the diagram Y_n of an isomorphic copy of $\mathfrak{A}_n$, $J^n_T(B) \not\le_T \emptyset^{(n)}$. Hence $(\forall n)(\mathbf{0}^{(n)} <_T d_T(B)^{(n)})$.

Let $d_T(B)$ be a Turing degree such that $(\forall n)(\mathbf{0}^{(n)} < d_T(B)^{(n)})$. Since $\emptyset^{(n)} \le_T J^n_T(B)$ uniformly in n we can compute uniformly in $J^n_T(B)$ an enumeration U_n of $\mathcal{F}^{\emptyset^{(n)}}$. Hence we can compute in $J^n_T(B)$ the diagram Y_n of an isomorphic copy of $\mathfrak{A}_n$ uniformly in n. Thus $d_T(B) \in Sp(\mathfrak{M})$.

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