FIRST ORDER THEORY OF THE s-DEGREES AND ARITHMETIC

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Abstract. We show that the first order theories of the s-degrees, and of the Q-degrees, are computably isomorphic to true second order arithmetic.

1. Introduction

When studying a computability theoretic reducibility, a natural goal is to investigate the first order theory of its degree structure: Typical questions concern the complexity of the theory, and the elementary differences with respect to other degree structures. A pioneering result along this line of research was Simpson's theorem, [19], stating that the first order theory of the poset of the Turing degrees is computably isomorphic to the set of true sentences of second order arithmetic. For an updated survey on first order theories of degree structures relative to models of computation based on Turing reducibility, see [18]. As regards degree structures arising from enumeration reducibility, or restricted versions of it, Slaman and Woodin [21] showed that the first order theory of the poset of the enumeration degrees is computably isomorphic to true second order arithmetic; more recently, by adapting Slaman and Woodin's proof, Marsibilio and Sorbi [12] have shown that this is true also of the bounded enumeration degrees. This paper is dedicated to an important restricted version of enumeration reducibility, known as s-reducibility. We show in Theorem 3.1 that the first order theory of the poset of s-degrees is as complicated as possible, namely it is computably isomorphic to true second order arithmetic. (This result was announced in [23].) Our proof is different from the proofs for the enumeration degrees and for the bounded enumeration degrees, for reasons that will be explained in the next section.

Via isomorphism of the s-degrees with the Q-degrees, this shows also that the first order theory of the Q-degrees is computably isomorphic to true second order arithmetic.

Our basic references for computability theory are the textbooks [6], [17], andv[22]. If $\leq_{\mathrm{r}}$ is a reducibility on sets of numbers, then by $\deg_{\mathrm{r}}(A)$ we denote the r-degree of A, i.e. the equivalence class of A under the equivalence relation $\equiv_{\mathrm{r}}$ generated by $\leq_{\mathrm{r}}$; the collection of r-degrees is a poset, denoted by $\langle \mathbf{D}_{\mathrm{r}}, \leq_{\mathrm{r}} \rangle$; its least element, if existing, is denoted by $\mathbf{0}_{\mathrm{r}}$. We recall that a set A is *enumeration reducible* (or, simply, e-*reducible*) to a set B (notation: $A \leq_{\mathrm{e}} B$), if there exists a computably enumerable (or simply, c.e.) set W such that

$$A = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in W \;\&\; D \subseteq B]\},$$

where finite sets are identified with their canonical indices: It is common to write in this case, $A = W(B)$, thus viewing the c.e. set W as an operator, called an

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enumeration operator (or, simply, an e-*operator*) mapping sets of numbers to sets of numbers. A particular but important restriction of e-reducibility is provided by s-reducibility: We say that A is s-*reducible* to B (notation: $A \leq_{\mathrm{s}} B$), if $A = W(B)$ for some e-operator W such that

$$(\forall x, D)[\langle x, D\rangle \in W \Rightarrow |D| \leq 1],$$

where $|D|$ denotes the cardinality of D: An e-operator with this property is called an s-*operator*. The s-degrees form an upper semilattice with least element $\mathbf{0}_{\mathrm{s}}$ consisting of the c.e. sets, and with supremum given by the usual operation $\oplus$ of disjoint union of sets. If $\{A_i : i \in I\}$, with $I \subseteq \omega$ is a family of set, then we let $\bigoplus_{i\in I} A_i = \bigcup_{i\in I}(\{i\} \times A_i)$. If, for every $i \in I$, $A_i \subseteq \omega^{[i]}(= \{x : (\exists y)[x = \langle i, y\rangle]\})$, then $\bigoplus_{i\in I} A_i \equiv_{\mathrm{s}} \bigcup_{i\in I} A_i$: This feature will be used in the proof of Theorem 3.1. If $\boldsymbol{a}_i = \deg_{\mathrm{s}}(A_i)$ then we let $\bigcup_{i\in I} \boldsymbol{a}_i = \deg_{\mathrm{s}}(\bigoplus_{i\in I}(A_i))$. Moreover, if I is finite, then $\bigcup_{i\in I} \boldsymbol{a}_i$ gives the supremum of the degrees.

We also recall that a set A is Q-*reducible* to a set B (denoted by $A \leq_{\mathrm{Q}} B$) if there exists a computable function f such that, for every x,

$$x \in A \Leftrightarrow W_{f(x)} \subseteq B.$$

It is easy to see, [7] that, if $B \neq \omega$ then $A \leq_{\mathrm{s}} B$ if and only if $\overline{A} \leq_{\mathrm{Q}} \overline{B}$: Thus that the poset of the Q-degrees is isomorphic to the poset of the s-degrees.

Among the so-called strong enumeration reducibilities (i.e. subreducibilities $\leq_{\mathrm{r}}$ of $\leq_{\mathrm{e}}$ such that $\mathbf{0}_{\mathrm{r}}$ consists of all c.e. sets: See [4] and [5] for exhaustive and well written surveys on strong enumeration reducibilities) s-reducibility is perhaps the most important and useful one. In most practical instances of $A \leq_{\mathrm{e}} B$, it often happens that one can in fact show that $A \leq_{\mathrm{s}} B$: as argued in [16] this is perhaps due to the fact that $\leq_{\mathrm{e}}$ naturally embeds into $\leq_{\mathrm{s}}$, via $A \leq_{\mathrm{e}} B$ if and only if $A^* \leq_{\mathrm{s}} B^*$, where for a given set X, X^* is the set of all finite strings of elements of X. (In fact, see e.g. [13], or [9], the s-degree of X^* is the greatest s-degree inside the e-degree of X.) Interest in s-reducibility (often through its isomorphic presentation as Q-reducibility, see [15]), relies also in its many applications to computability theory and general mathematics. For instance Q-reducibility plays a key role in Marchenkov's solution of Post's Problem using Post's methods ([11]); and has applications to word problems (for instance, see [2], [10]) and to abstract computational complexity (for instance, see [3], [7]). Throughout the rest of the paper, we assume to have fixed some effective listing $\{\Psi_e : e \in \omega\}$ (henceforth called the *standard listing*) of the s-operators.

Our approach to second order arithmetic follows closely [17, Section 16.2]. The language, with identity, consists of: first order variables $v_0, v_1, \ldots$; second order variables $U_0, U_1, \ldots$; the function symbols 0, s, $+$, and $\times$, interpreted in the standard model $\mathbb{N}$ of arithmetic with the number 0, successor , sum, and multiplication, respectively; a binary predicate symbol $<$, interpreted in $\mathbb{N}$ as the strict natural ordering on numbers; formulas are built up in the obvious way from first order atomic formulas, and second order atomic formulas (these latter ones having the form $t \in U$, where t is a first order term, and U is a second order variable), allowing also second order quantification, i.e. quantification over second order variables. Second order formulas and sentences will be interpreted in the structure $(\mathbb{N}, \mathcal{P}(\omega))$, via interpretation in $\mathbb{N}$ of the first order part of the language, and evaluation of second order variables with subsets of ω, with $\in$ interpreted as the usual set theoretic membership. If $\psi(\overrightarrow{x}, \overrightarrow{U})$ is a second order arithmetical formula, (with free first order variables occurring among $\overrightarrow{x}$, and second order variables occurring among $\overrightarrow{U}$), and $\overrightarrow{n}$, $\overrightarrow{X}$ are a vector of natural numbers, and a vector of sets of numbers, respectively (with $\overrightarrow{n}$ and $\overrightarrow{X}$ having the same lengths as $\overrightarrow{x}$, and $\overrightarrow{U}$, respectively) then $(\mathbb{N}, \mathcal{P}(\omega)) \models \psi(\overrightarrow{n}, \overrightarrow{X})$ denotes that evaluation of the variables in $\overrightarrow{x}$ with the corresponding elements of $\overrightarrow{n}$, and evaluation of the second order variables in $\overrightarrow{U}$ with the corresponding elements of $\overrightarrow{X}$, yield a property that holds in $(\mathbb{N}, \mathcal{P}(\omega))$ through the above described interpretation of ψ in the structure. Similarly, if $\psi(\overrightarrow{x})$ is a first order formula in the language with identity of posets (thus having a binary predicate symbol $<$ as the only non-logical constant, in addition to the symbol $=$ for identity), with free variables occurring among $\overrightarrow{x}$, and $\overrightarrow{\boldsymbol{a}}$ is a list of elements of $\mathbf{D}_{\mathrm{s}}$, having the same length as $\overrightarrow{x}$, then $\mathbf{D}_{\mathrm{s}} \models \psi(\overrightarrow{\boldsymbol{a}})$ denotes that evaluation of the variables in $\overrightarrow{x}$ with the corresponding elements of $\overrightarrow{\boldsymbol{a}}$ yields a statement that holds in $\mathbf{D}_{\mathrm{s}}$.

2. The first order theory of the s-degrees

Let $\mathrm{Th}(\mathbf{D}_{\mathrm{s}})$ be the set of first order sentences, in the language with identity of posets, that are true in $\mathbf{D}_{\mathrm{s}}$; and let $\mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$ be the set of second order arithmetical sentences that are true in $(\mathbb{N}, \mathcal{P}(\omega))$. We want to show that $\mathrm{Th}(\mathbf{D}_{\mathrm{s}}) \equiv \mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$, where the symbol $\equiv$ denotes computable isomorphism. By the Myhill Isomorphism Theorem [14], it suffices to show that $\mathrm{Th}(\mathbf{D}_{\mathrm{s}}) \equiv_1 \mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$, i.e. the two theories are 1-reducible to each other, and finally, since theories are cylinders (see [17]), it is enough to show that $\mathrm{Th}(\mathbf{D}_{\mathrm{s}}) \leq_{\mathrm{m}} \mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$ and $\mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega)) \leq_{\mathrm{m}} \mathrm{Th}(\mathbf{D}_{\mathrm{s}})$.

One direction is easy and standard:

Lemma 2.1. $\mathrm{Th}(\mathbf{D}_{\mathrm{s}}) \leq_{\mathrm{m}} \mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$.

Proof. A simple calculation shows that s-reducibility is a Σ^0_3 relation in the arithmetical hierarchy, hence there exists a Σ^0_3 formula of second order arithmetic $\psi(U, V)$, having U, V as free set variables , such that, for all pairs of sets X, Y,

$$X \leq_{\mathrm{s}} Y \Leftrightarrow (\mathbb{N}, \mathcal{P}(\omega)) \models \psi(X, Y).$$

This gives a way of effectively translating sentences in the language of posets, into second order arithmetical sentences, upon translation of $x \leq y$ with the Σ^0_3 definition of the reducibility, so that $\mathrm{Th}(\mathbf{D}_{\mathrm{s}}) \leq_{\mathrm{m}} \mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$. $\square$

The rest of the paper is devoted to showing that $\mathrm{Th}(\mathbb{N}, 2^{\mathbb{N}}) \leq_{\mathrm{m}} \mathrm{Th}(\mathbf{D}_{\mathrm{s}})$. Since s-reducibility is a subreducibility of enumeration reducibility, a first reasonable attempt towards this goal would be to try and adapt to the s-degrees, Slaman and Woodin's proof in [21] for the enumeration degrees. (This was indeed possible for bounded enumeration reducibility, [12].) We give some intuitive motivations as to why this approach presents intrinsic difficulties. Recall,

Definition 2.2. In a poset, a set is an *antichain* if its elements are pairwise incomparable.

Slaman and Woodin's machinery for the enumeration degrees, as well as its adaptation to the bounded enumeration degrees in [12], relies on the fact that in the given degree structure every countable antichain is uniformly definable from finitely many parameters. Namely, for every countable antichain $\mathcal{C}$, there exist three parameters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ such that $\mathcal{C}$ consists exactly of the degrees that are minimal solutions of the property (in $\boldsymbol{x}$)

$$\boldsymbol{x} \leq \boldsymbol{c} \;\&\; (\boldsymbol{x}] \neq (\boldsymbol{x} \cup \boldsymbol{a}] \cap (\boldsymbol{x} \cup \boldsymbol{b}].$$

(The symbol $(\boldsymbol{u}]$ denotes the principal ideal generated by $\boldsymbol{u}$.) For the proof of this basic result, given an antichain $\mathcal{C} = \{\boldsymbol{c}_n : n \in \omega\}$ (where, say $\boldsymbol{c}_n$ is the degree of the set C_n), one builds three sets A, B, C (with corresponding degrees $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$) with $C = \bigoplus_{n \in \omega} C_n$. As to show that $(\boldsymbol{c}_n] \neq (\boldsymbol{c}_n \cup \boldsymbol{a}] \cap (\boldsymbol{c}_n \cup \boldsymbol{b}]$, one defines a witness D_n by:

$$x \in D_n \;\Leftrightarrow\; (\exists \text{ finite } E)[\langle x, E \rangle \in A^{[n]} \;\&\; E \subseteq C_n],$$

where $A^{[n]} = A \cap \omega^{[n]}$ denotes the nth column of A. Clearly D_n is enumeration reducible to $C_n \oplus A$ (and also bounded enumeration reducible if we restrict to those finite sets E that are either singletons or the empty set). The construction guarantees also that D_n is reducible to $C_n \oplus B$ by the same reduction. Finally, by diagonalization one ensures that D_n is not reducible to C_n.

When trying to adapt this proof to the s-degrees, one encounters the following obstacle: The definition

$$D_n = \{x : (\exists u)[\langle x, E\rangle \in A^{[n]} \ \&\ E \subseteq C_n]\},$$

does not allow for an s-reduction of D_n to $C_n \oplus A$ and to $C_n \oplus B$, as the enumeration reduction of D_n to these sets, coming from the definition of D_n requires axioms of the form $\langle x, \{\langle n, \langle x, E\rangle\rangle\} \oplus E\rangle$ which may not be consistent with s-operators, in that the oracle set may have more than one element. This obstacle has shown up in all our other attempts at uniformly defining in $\mathbf{D}_{\mathrm{s}}$ the countable antichains and relations. Although not able to show that the countable antichains and relations are uniformly definable from parameters, we can however code the standard model of arithmetic through the construction of a particular copy within the degree structure

To carry out the reduction of second order arithmetic into the s-degrees, it is convenient to see arithmetic as presented through a language involving only predicate symbols: in other words it is convenient to express 0, successor, sum, and multiplication, by predicate symbols rather than by function symbols: The details of this translation of the language into a purely relational form are standard, and can be worked out for instance from [8]. In the following we freely use logical abbreviations and conventions that are standard and of common use: for instance, we write $x \neq y$ for $\neg x = y$, and $x \leq y$ for $x = y \vee x < y$.

Let us start with the following definitions:

Definition 2.3. Let $\mathcal{A} = \{\boldsymbol{a}_i\}_{i\in\omega}$ be a set of s-degrees.

(i) $\mathcal{A}$ is *independent* if $\boldsymbol{a}_i \not\leq \bigcup_{j\in F} \boldsymbol{a}_j$, for every finite set F and $i \notin F$;
(ii) $\mathcal{A}$ is *computably independent* if $\boldsymbol{a}_i \not\leq \bigcup_{j\in I} \boldsymbol{a}_j$, for every computable set I and $i \notin I$.

Note that an independent set is in fact an antichain, and a computably independent set is an independent set too; moreover it is easy to see that $\mathcal{A} = \{\boldsymbol{a}_i\}_{i\in\omega}$ is computably independent if and only if $\boldsymbol{a}_i \not\leq \bigcup_{j\neq i} \boldsymbol{a}_j$, for every $i \in \omega$.

Definition 2.4. Let $\mathcal{A} = \{\boldsymbol{a}_i\}_{i\in\omega} \subseteq \mathbf{D}_{\mathrm{s}}$, and let $\boldsymbol{a} \in \mathbf{D}_{\mathrm{s}}$ be a degree. We say that:

(1) $\boldsymbol{a}$ *codes* a subset $\mathcal{X} \subseteq \mathcal{A}$, if

$$\mathcal{X} = \{\boldsymbol{x} \in \mathcal{A} : \boldsymbol{x} \leq \boldsymbol{a}\};$$

(2) $\boldsymbol{a}$ *codes* a set $X \subseteq \omega$ *in* $\mathcal{A}$ if $\boldsymbol{a}$ codes $\{\boldsymbol{a}_i : i \in X\}$.

It is clear that every $\boldsymbol{a} \in \mathbf{D}_{\mathrm{s}}$ codes a set of numbers X in $\mathcal{A}$. A more delicate problem is to show that for every $X \subseteq \omega$ there exists a degree $\boldsymbol{a}_X \in \mathbf{D}_{\mathrm{s}}$ such that $\boldsymbol{a}_X$ codes X in $\mathcal{A}$. As shown by the following theorem, this is true if $\mathcal{A}$ is independent.

Theorem 2.5. *For every independent set* $\mathcal{A} = \{\boldsymbol{a}_i\}_{i\in\omega}$ *in* $\mathbf{D}_{\mathrm{s}}$*, and for every* $X \subseteq \omega$*, there exists* $\boldsymbol{a}_X \in \mathbf{D}_{\mathrm{s}}$ *such that* $\boldsymbol{a}_X$ *codes* X *in* $\mathcal{A}$.

Proof. Let $\mathcal{A} = \{\boldsymbol{a}_i\}_{i\in\omega}$ be independent in $\mathbf{D}_{\mathrm{s}}$, with $\boldsymbol{a}_i = \deg_{\mathrm{s}}(A_i)$, for every $i \in \omega$. Given $X \subseteq \omega$, we build a set A_X such that

$$i \in X \Leftrightarrow A_i \leq_{\mathrm{s}} A_X.$$

Thus $\boldsymbol{a}_X = \deg_{\mathrm{s}}(A_X)$ is the desired degree coding the set X in $\mathcal{A}$.

Suitable requirements to be satisfied for a successful set A_X are, for every pair of numbers i, e:

$$R_{2i} : \ i \in X \Rightarrow A_X^{\{i\}} =^* A_i,$$
$$R_{2\langle i,e\rangle+1} : \ i \notin X \Rightarrow A_i \neq \Psi_e(A_X),$$

where $\{\Psi_e\}_{e\in\omega}$ is the standard enumeration of the s-operators, $=^*$ denotes equality modulo finite sets, and $A_X^{\{i\}} = \{x : \langle i, x\rangle \in A_X\}$ is the ith projection of A_X throgh the Cantor pairing function.

Construction. The construction is by stages. At stage s, we define an approximation $\alpha_{X,s}$ to the characteristic function of A_X. In the rest of the proof, A_i is often identified with its characteristic function.

Stage 0: $\alpha_{X,0} = \emptyset$.

Stage $s + 1$: We distinguish two cases:

- $s = 2i$. If $i \in X$ then *fill up* $A_X^{\{i\}}$ *with* A_i, i.e. define
$$\alpha_{X,s+1} = \alpha_{X,s} \cup \{\langle\langle i, x\rangle, A_i(x)\rangle : \langle i, x\rangle \notin \text{domain}(\alpha_{X,s})\}.$$
Otherwise, if $i \notin X$ then *fill up* $A_X^{\{i\}}$ *with zeros*, i.e. let
$$\alpha_{X,s+1} = \alpha_{X,s} \cup \{\langle\langle i, x\rangle, 0\rangle : \langle i, x\rangle \notin \text{domain}(\alpha_{X,s})\}.$$
- $s = 2\langle i, e\rangle + 1$. If $i \notin X$ then look for a finite set F (with $|F| \leq 1$) *compatible with* $\alpha_{X,s}$ (i.e. if $x \in F$ and $\alpha_{X,s}(x)\downarrow$, then $\alpha_{X,s}(x) = 1$) and an element y such that $y \in \Psi_e(F) \setminus A_i$. If F and y exist, then choose the least such F, and set
$$\alpha_{X,s+1} = \alpha_{X,s} \cup \{\langle x, 1\rangle : x \in F\}.$$
Otherwise, set $\alpha_{X,s+1} = \alpha_{X,s}$.
If $i \in X$ then $\alpha_{X,s+1} = \alpha_{X,s}$.

Verification. Each requirement R_{2i} is clearly satisfied by construction at stage $2i+1$, since at the beginning of this stage the "column" $A_X^{\{i\}}$ is finite. Now suppose that some requirement $R_{2\langle i,e\rangle+1}$ is not satisfied, i.e. $i \notin X$ and $A_i = \Psi_e(A_X)$. This means that at stage $s + 1$ with $s = 2\langle i, e\rangle + 1$ we can not diagonalize. Then, for every y, $y \in A_i$ if and only if there exists a finite set F (with $|F| \leq 1$) such that $\langle y, F\rangle \in \Psi_e$ and F compatible with $\alpha_{X,s}$. Now, at the beginning of stage $s + 1$ the "columns" completely filled up are $A_X^{\{0\}}, \ldots, A_X^{\{\langle i,e\rangle\}}$; on the other hand $\alpha_{X,s}$ is defined only on finitely many numbers $x \in \bigcup_{j>\langle i,e\rangle} \omega^{[j]}$ and on these numbers $\alpha_{X,s}(x) = 1$. Thus, F compatible with $\alpha_{X,s}$ is equivalent to saying that if $x \in F \cap \bigcup_{j\leq\langle i,e\rangle} \omega^{[j]}$ then $\alpha_X(x) = 1$. Hence $A_i \leq_{\rm s} \bigoplus_{j\leq\langle i,e\rangle} A_X^{\{j\}}$; on the other hand, $A_X^{\{j\}} =^* A_j$, for every $j \leq \langle i, e\rangle$, so $\bigoplus_{j\leq\langle i,e\rangle} A_X^{\{j\}} \leq_{\rm m} \bigoplus_{j\leq\langle i,e\rangle} A_j$, where $\leq_{\rm m}$ denotes many-one reducibility; since $\leq_{\rm m} \subseteq \leq_{\rm s}$, we conclude that $A_i \leq_{\rm s} A_0 \oplus \cdots \oplus A_{\langle i,e\rangle}$. But this is a contradiction, since $\{\deg_{\rm s}(A_i)\}_{i\in\omega}$ is an independent set. □

Definition 2.6. An n-ary relation $\mathcal{R}$ on $\mathbf{D}_{\rm s}$, with $n \geq 1$, is *definable in* $\mathbf{D}_{\rm s}$ *with parameters*, if there exists a first order formula of posets $\varphi(\overrightarrow{x}; \overrightarrow{w})$, (where $\overrightarrow{x} = \langle x_1, \ldots, x_n\rangle$ and $\overrightarrow{w} = \langle w_1, \ldots, w_m\rangle$, for some $m \geq 0$: Here and throughout the paper, we adopt the convention of using a semicolon to separate in a first order formula, variables used for parameters, from the other variables), and degrees $\overrightarrow{\boldsymbol{w}} = \langle \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \rangle$ such that, for every $\overrightarrow{\boldsymbol{x}} \in (\mathbf{D}_{\mathrm{s}})^n$,

$$\mathcal{R}(\overrightarrow{\boldsymbol{x}}) \Leftrightarrow \mathbf{D}_{\mathrm{s}} \models \varphi(\overrightarrow{\boldsymbol{x}}; \overrightarrow{\boldsymbol{w}}).$$

We say in this case that $\mathcal{R}$ is *defined by* φ *from the parameters* $\overrightarrow{\boldsymbol{w}}$. A class $\mathbb{X}$ of n-ary relations (with $n \geq 1$) is *uniformly definable with parameters in* $\mathbf{D}_{\mathrm{s}}$, if there exists a first order formula of posets $\varphi(\overrightarrow{x}; \overrightarrow{w})$, such that $\mathbb{X}$ coincides with the class of relations that are defined by φ from all possible choices of parameters $\overrightarrow{\boldsymbol{w}}$. We say in this case that φ *uniformly defines* $\mathbb{X}$ in $\mathbf{D}_{\mathrm{s}}$.

We now introduce a special class of uniformly definable countable antichains due to Slaman and Woodin [20].

Definition 2.7. A set $\mathcal{G} \subseteq \mathbf{D}_{\mathrm{s}}$ is called a *Slaman-Woodin set* (or, simply, *SW-set*) if for some degrees $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}$, we have that $\mathcal{G}$ is the set of degrees $\boldsymbol{x}$, that are minimal with respect to the property:

$$\boldsymbol{x} \leq \boldsymbol{g} \;\&\; \boldsymbol{a} \leq \boldsymbol{x} \cup \boldsymbol{b}.$$

Thus the class of SW-sets is uniformly defined in $\mathbf{D}_{\mathrm{s}}$ by the formula:

$$\varphi_{SW}(x; a, b, g) := x \leq g \;\&\; a \leq x \cup b \;\&\; \neg(\exists y)(y \leq g \;\&\; a \leq y \cup b \;\&\; y < x),$$

where $\cup$ is of course definable in the language with identity of posets.

2.1. **Plan of the proof.** We outline the plan to show that $\mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega)) \leq_{\mathrm{m}} \mathrm{Th}(\mathbf{D}_{\mathrm{s}})$.

(1) We exhibit first order formulas, in the language with identity of posets, $\varphi_N(x; \overrightarrow{w})$, $\varphi_s(x, y; \overrightarrow{w})$, $\varphi_+(x, y, z; \overrightarrow{w})$, $\varphi_\times(x, y, z; \overrightarrow{w})$, $\varphi_<(x, y; \overrightarrow{w})$. Starting from these formulas, for every list $\overrightarrow{\boldsymbol{w}}$ of parameters interpreting the variables in the list $\overrightarrow{w}$, one can consider the structure for arithmetic

$$\mathbb{N}_{\overrightarrow{\boldsymbol{w}}} = \langle \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}, s_{\overrightarrow{\boldsymbol{w}}}, +_{\overrightarrow{\boldsymbol{w}}}, \times_{\overrightarrow{\boldsymbol{w}}}, <_{\overrightarrow{\boldsymbol{w}}} \rangle,$$

where $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{x} : \mathbf{D}_{\mathrm{s}} \models \varphi_N(\boldsymbol{x}; \overrightarrow{\boldsymbol{w}})\}$ is the universe, and $s_{\overrightarrow{\boldsymbol{w}}}$, $+_{\overrightarrow{\boldsymbol{w}}}$, $\times_{\overrightarrow{\boldsymbol{w}}}$, $<_{\overrightarrow{\boldsymbol{w}}}$, are the relations (interpreting in order, $s, +, \times, <$) on $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, obtained by restricting to the universe the relations defined by the formulas φ_s, φ_+, $\varphi_\times$, $\varphi_<$, respectively, from the parameters $\overrightarrow{\boldsymbol{w}}$.

(2) We show that there is a first order formula $\alpha_{st}(\overrightarrow{w})$ (called a *correctness condition*), such that and for every list of degrees $\overrightarrow{\boldsymbol{w}}$ as above, if $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}})$ then $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}}$ is isomorphic to $\mathbb{N}$, the standard model of arithmetic. In other words, among the structures $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}}$, uniformly individuated as above, the correctness condition picks only ones that are copies of the standard model of arithmetic.

(3) Satisfaction of the correctness condition will guarantee that $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ is an infinite computably independent set of degrees, so if $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}})$ then we may use Theorem 2.5 (enabling us to interpret second order quantification on subsets of any countably infinite independent set of degrees with first order quantification in $\mathbf{D}_{\mathrm{s}}$), to show that there is a computable mapping $\sigma \mapsto \sigma^*(\overrightarrow{w})$ taking second order arithmetical sentences to first order formulas in the language of posets such that, if $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}})$ then

$$(\mathbb{N}, \mathcal{P}(\omega)) \models \sigma \Leftrightarrow \mathbf{D}_{\mathrm{s}} \models \sigma^*(\overrightarrow{\boldsymbol{w}}),$$

so that the mapping $\sigma \mapsto (\exists \overrightarrow{w})(\alpha_{st}(\overrightarrow{w}) \,\&\, \sigma^*(\overrightarrow{w}))$ gives the desired reduction of $\mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega))$ to $\mathrm{Th}(\mathbf{D}_{\mathrm{s}})$.

(4) Finally, we show the existence of suitable parameters $\overrightarrow{\boldsymbol{w}}$ which satisfy the correctness condition.

2.2. **How to build suitable parameters.** In Theorem 3.1, we build a list of s-degrees

$$\overrightarrow{\boldsymbol{v}} = \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle$$

obeying some simple properties. Before proving the theorem in next section, we summarize these properties here, and show how to use them to augment $\overrightarrow{\boldsymbol{v}}$ to a larger list of parameters

$$\overrightarrow{\boldsymbol{w}} = \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \rangle,$$

such that $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}}$ is isomorphic to $\mathbb{N}$. To do so, we contextually exhibit suitable order formulas $\varphi_N(x; \overrightarrow{w})$, $\varphi_s(x, y; \overrightarrow{w})$, $\varphi_+(x, y, z; \overrightarrow{w})$, $\varphi_\times(x, y, z; \overrightarrow{w})$, $\varphi_<(x, y, z; \overrightarrow{w})$; the desired effective translation $\sigma \mapsto \sigma^*(\overrightarrow{w})$ of second order arithmetical sentences into first order formulas in the language of posets; and the existence of a suitable correctness condition $\alpha_{st}(\overrightarrow{w})$.

The universe. We build a computably independent SW-set

$$\mathcal{G}_{\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{g}_{\boldsymbol{n}} : n \in \omega\},$$

defined by φ_{SW} from the triple $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g} \rangle$. For every n, let $\boldsymbol{a}_n = \boldsymbol{g}_{2n}$, $\boldsymbol{b}_n = \boldsymbol{g}_{2n+1}$, and $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{a}_n : n \in \omega\}$, $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{b}_n : n \in \omega\}$. The parameters $\boldsymbol{n}$ and $\boldsymbol{p}$ are such that

$$\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{g} \in \mathcal{G} : \boldsymbol{g} \leq \boldsymbol{n}\}$$
$$\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}} = \{\boldsymbol{g} \in \mathcal{G} : \boldsymbol{g} \leq \boldsymbol{p}\} :$$

Notice that given $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}$, the existence of $\boldsymbol{n}$ and $\boldsymbol{p}$ is guaranteed by Theorem 2.5. Clearly $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ and $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ are definable with parameters by first order formulas of posets $\varphi_N(x; a, b, g, n)$ (from the parameters $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{n} \rangle$), and $\varphi_P(x; a, b, g, n)$ (from the parameters $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{p} \rangle$), respectively. The set $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ will be the universe of our copy of the standard model.

Successor, 0, and $<$. To define the successor function, we use the two parameters $\boldsymbol{p}_0, \boldsymbol{p}_1$, guaranteeing that for every $\boldsymbol{a}_i \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, there exists a unique element $\boldsymbol{a}_j \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ (with $\boldsymbol{a}_i \neq \boldsymbol{a}_j$) such that $\boldsymbol{a}_j \leq \boldsymbol{a}_i \cup \boldsymbol{p}_0$ or $\boldsymbol{a}_j \leq \boldsymbol{a}_i \cup \boldsymbol{p}_1$. Given $\boldsymbol{a}_i$, this unique element is actually $\boldsymbol{a}_{i+1}$. Furthermore, we ensure that $\boldsymbol{a}_0$ is not the "successor" of any element in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$. As specified, we need two parameters: $\boldsymbol{p}_0$ is used to define the "successor" for the elements $\boldsymbol{a}_{2i}$, and $\boldsymbol{p}_1$ is used to define the "successor" for the elements $\boldsymbol{a}_{2i+1}$, with $i \in \omega$. Formally, the successor function in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ is then coded by the binary relation $s_{\overrightarrow{w}}$ on $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$,

$$s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} \neq \boldsymbol{y} \ \& \ [\boldsymbol{y} \leq \boldsymbol{x} \cup \boldsymbol{p}_0 \ \vee \ \boldsymbol{y} \leq \boldsymbol{x} \cup \boldsymbol{p}_1],$$

which is clearly definable by a first order formula $\varphi_s(x, y; a, b, g, n, p_0, p_1)$ from the parameters $\overrightarrow{\boldsymbol{w}}$.

It is interesting to see that it is already possible to define at this point the natural order on numbers.

Corollary 2.8. *The relation* $<_{\overrightarrow{\boldsymbol{w}}}$ *on* $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$*,*

$$\boldsymbol{x} <_{\overrightarrow{\boldsymbol{w}}} \boldsymbol{y} := (\exists m, n \in \omega)[m < n \ \& \ \boldsymbol{x} = \boldsymbol{a}_m \ \& \ \boldsymbol{y} = \boldsymbol{a}_n]$$

is definable from $\overrightarrow{\boldsymbol{w}}$ *by a first order formula* $\varphi_<(x, y; a, b, g, n, p_0, p_1)$*.*

Proof. It is easy to see, using Theorem 2.5, that for every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, we have that $\boldsymbol{x} <_{\overrightarrow{\boldsymbol{w}}} \boldsymbol{y}$ holds if and only if

$$(\exists \boldsymbol{g})(\forall \boldsymbol{c} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}})[(s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{c}) \leq \boldsymbol{g} \Rightarrow \boldsymbol{c} \leq \boldsymbol{g}] \ \& \ \boldsymbol{x} \leq \boldsymbol{g} \ \& \ \boldsymbol{y} \not\leq \boldsymbol{g}],$$

where for simplicity, we use the unary function symbol $s_{\overrightarrow{\boldsymbol{w}}}$ for successor, translating into functional form its relational definition given above: If $\boldsymbol{c} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, then $s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{c})$ is meant to be the unique $\boldsymbol{y} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ such that $s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{x}, \boldsymbol{y})$. □

Notice also that $\boldsymbol{a}_0$ is first order definable too, from the parameters $\overrightarrow{\boldsymbol{w}}$. In view of the previous results, it is appropriate to identify $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ with ω, via the mapping $i \mapsto \boldsymbol{a}_i$.

Second order quantification. Next, we show how to code second order quantification. The set $\mathcal{G}_{\overrightarrow{\boldsymbol{w}}}$ is computably independent in $\mathbf{D}_{\mathrm{s}}$ and so it is independent too. Thus, both $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ and $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ are independent. By Theorem 2.5, it follows that for every $X \subseteq \omega$ there exist elements $\boldsymbol{n}_X, \boldsymbol{p}_X \in \mathbf{D}_{\mathrm{s}}$ such that $\boldsymbol{n}_X$ codes X in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, and $\boldsymbol{p}_X$ codes X in $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$, i.e., $i \in X$ if and only if $\boldsymbol{a}_i \leq \boldsymbol{n}_X$; and, $i \in X$ if and only if $\boldsymbol{b}_i \leq \boldsymbol{p}_X$.

Ordered pairs. We now show how to talk in a first order way about ordered pairs of elements of $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$. To do this, we use the two parameters $\boldsymbol{p}_2, \boldsymbol{p}_3$. These parameters allow us to pick ordered pairs of elements of the universe. More precisely, $\boldsymbol{p}_2, \boldsymbol{p}_3$ are built so that for every $(\boldsymbol{a}_k, \boldsymbol{a}_n) \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} \times \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$, there exists a unique element $\boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ such that $\boldsymbol{a}_k$ is the unique solution in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ of the equation in $\boldsymbol{x}$,

$$\boldsymbol{x} \leq \boldsymbol{b} \cup \boldsymbol{p}_2,$$

and $\boldsymbol{a}_n$ is the unique solution in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ of the equation in $\boldsymbol{x}$

$$\boldsymbol{x} \leq \boldsymbol{b} \cup \boldsymbol{p}_3,$$

and viceversa every $\boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ bounds via $\boldsymbol{p}_2, \boldsymbol{p}_3$ a unique pair of elements in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} \times \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$. In our construction, the element $\boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ corresponding to $(\boldsymbol{a}_k, \boldsymbol{a}_n)$ will be $\boldsymbol{b}_{\langle k,n\rangle} = \boldsymbol{g}_{2\langle k,n\rangle+1}$. Thus, we can see ordered pairs of elements of $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ as elements of $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$, via identification of $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}} \times \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ with the unique $\boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ such that

$$\boldsymbol{x} \leq \boldsymbol{b} \cup \boldsymbol{p}_2 \ \& \ \boldsymbol{y} \leq \boldsymbol{b} \cup \boldsymbol{p}_3.$$

The needed properties of $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ (which work as projections for the pairing function $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{b}$) can be summarized by the first order condition:

$$\begin{aligned}
&(\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}})(\exists \boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}})[\boldsymbol{x} \leq \boldsymbol{b} \cup \boldsymbol{p_2} \ \& \ \boldsymbol{y} \leq \boldsymbol{b} \cup \boldsymbol{p_3}] \ \& \\
&\qquad (\forall \boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}})(\exists \boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}})[\boldsymbol{x} \leq \boldsymbol{b} \cup \boldsymbol{p_2} \ \& \ \boldsymbol{y} \leq \boldsymbol{b} \cup \boldsymbol{p_3}] \ \& \\
&\qquad\qquad (\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}})(\forall \boldsymbol{b} \in \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}) \\
&\qquad\qquad\qquad [\boldsymbol{x}, \boldsymbol{u} \leq \boldsymbol{b} \cup \boldsymbol{p_2} \ \& \ \boldsymbol{y}, \boldsymbol{v} \leq \boldsymbol{b} \cup \boldsymbol{p_3} \Rightarrow \boldsymbol{x} = \boldsymbol{u} \ \& \ \boldsymbol{y} = \boldsymbol{v}].
\end{aligned}$$

Sum. Let $\boldsymbol{y} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$: we say that a subset $P \subseteq \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ has the *$(+, \boldsymbol{y})$-closure property* if

(1) $(\boldsymbol{a}_0, \boldsymbol{y}) \in P$;
(2) $(\forall \boldsymbol{x}, \boldsymbol{z} \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}))[(\boldsymbol{x}, \boldsymbol{z}) \in P \Rightarrow (s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{x}), s_{\overrightarrow{\boldsymbol{w}}}(\boldsymbol{z})) \in P]$.

(Notice our identification of subsets of $\mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$, with sets of pairs of elements of $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$. We use also the already observed definability of $\boldsymbol{a}_0$.) The idea here is that P contains the pairs of s-degrees in $\mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}}$ corresponding to the pairs of natural numbers $(x, x+y)$, where y is the natural number corresponding to $\boldsymbol{y}$.

Lemma 2.9. *The property $P_+(\boldsymbol{q}, \boldsymbol{y})$ of* s*-degrees $\boldsymbol{q}$, $\boldsymbol{y}$,*

$\boldsymbol{q}$ codes a set $P \subseteq \mathcal{G}_{P,\overrightarrow{\boldsymbol{w}}}$ having the $(+, \boldsymbol{y})$-closure property

(where "$\boldsymbol{q}$ codes P" is as in Definition 2.4(1)) is definable from the parameters $\overrightarrow{\boldsymbol{w}}$.

Proof. The first order formula defining the property is obtained by combining φ_N, φ_P, φ_s; the parameters are those needed for these formulas, plus $\boldsymbol{p}_2$, $\boldsymbol{p}_3$. □

It follows that $+$ on $\mathcal{G}_{N,\vec{\boldsymbol{w}}}$ is coded by the relation $+_{\vec{\boldsymbol{w}}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$:

$$(\forall P \subseteq \mathcal{G}_{P,\vec{\boldsymbol{w}}})[P \text{ has the } (+,\boldsymbol{y})\text{-closure property} \Rightarrow (\boldsymbol{x},\boldsymbol{z}) \in P]$$

which by Lemma 2.9, is first order definable from the parameters $\vec{\boldsymbol{w}}$ by a formula $\varphi_+(x,y,z;\vec{w})$.

Multiplication. We say that a subset $P \subseteq \mathcal{G}_P$ has the $(\times,\boldsymbol{y})$-*closure property* if
(i) $(\boldsymbol{a}_0,\boldsymbol{a}_0) \in P$;
(ii) $(\forall \boldsymbol{x},\boldsymbol{z} \in \mathcal{G}_{N,\vec{\boldsymbol{w}}}))[(\boldsymbol{x},\boldsymbol{z}) \in P \Rightarrow (s_{\vec{\boldsymbol{w}}}(\boldsymbol{x}), \boldsymbol{z} +_{\vec{\boldsymbol{w}}} \boldsymbol{y})) \in P]$,
(where for simplicity we translate into functional form the relational definition $+_{\vec{\boldsymbol{w}}}$ given above). Then multiplication on $\mathcal{G}_{N,\vec{\boldsymbol{w}}}$ is coded by the relation $\times_{\vec{\boldsymbol{w}}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$:

$$(\forall P \subseteq \mathcal{G}_{P,\vec{\boldsymbol{w}}})[P \text{ has the } (\times,\boldsymbol{y})\text{-closure property} \Rightarrow (\boldsymbol{x},\boldsymbol{z}) \in P],$$

which is clearly first order definable from the parameters $\vec{\boldsymbol{w}}$ by, say, a formula $\varphi_\times(x,y,z;\vec{w})$.

Lemma 2.10. *If $\vec{\boldsymbol{w}}$ is as above, then the structure $\mathbb{N}_{\vec{\boldsymbol{w}}}$ is isomorphic to the standard model of arithmetic, and there is a computable mapping $\sigma \mapsto \sigma^*(\vec{w})$ taking second order arithmetical sentences to first order formulas (with free variables among $\vec{w}$) in the language of posets such that*

$$(\mathbb{N},\mathcal{P}(\omega)) \models \sigma \Leftrightarrow \mathbf{D}_{\mathrm{s}} \models \sigma^*(\vec{\boldsymbol{w}})$$

Proof. The isomorphism of $\mathbb{N}$ with $\mathbb{N}_{\vec{\boldsymbol{w}}}$ is given by the mapping $i \mapsto \boldsymbol{a}_i$. As to the mapping $\sigma \mapsto \sigma^*(\vec{w})$, define $\sigma^*(\vec{w})$, for a given unnested second order arithmetical sentence σ, by induction on the complexity of the subformulas α of σ as follows (assume that distinct occurrences of quantifiers are relative to distinct variables, and for every set variable U, choose a first order variable p_U which is uniquely targeted for U):

(1) α is first order atomic: obtain α^* by replacing $s,+,\times,<$ with φ_s, φ_+, $\varphi_\times$, $\varphi_<$, respectively;
(2) $(u \in U)^* := \varphi_N(u,\vec{w})$ & $u \leq p_U$;
(3) * commutes with the propositional connectives;
(4) $((\exists v)\beta)^* := (\exists v)(\varphi_N(v,\vec{w})$ & $\beta^*)$; $((\forall v)\beta)^* := (\forall v)(\varphi_N(v,\vec{w}) \to \beta^*)$;
(5) $((\exists U)\beta)^* := (\exists p_U)\beta^*$; similarly, $((\forall U)\beta)^* := (\forall p_U)\beta^*$.

□

Finally, we exhibit the desired correctness condition $\alpha_{st}(\vec{w})$, such that every list of parameters $\vec{\boldsymbol{w}}' = \langle \boldsymbol{a}',\boldsymbol{b}',\boldsymbol{g}',\boldsymbol{n}',\boldsymbol{p}',\boldsymbol{p}'_0,\boldsymbol{p}'_1,\boldsymbol{p}'_2,\boldsymbol{p}'_3\rangle$ in $\mathbf{D}_{\mathrm{s}}$, satisfying the condition (i.e. $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\vec{\boldsymbol{w}}'))$ codes a copy of $\mathbb{N}$ in $\mathbf{D}_{\mathrm{s}}$. The condition $\alpha_{st}(\vec{w})$ states that $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\vec{\boldsymbol{w}}')$ if and only if:

(1) the triple $\langle \boldsymbol{a}',\boldsymbol{b}',\boldsymbol{g}'\rangle$ provides, via $\varphi_{SW}(x;a,b,g)$, a computably independent set $\mathcal{G}_{\vec{\boldsymbol{w}}'}$ of degrees: this is a first order condition on degrees, since a set $\mathcal{A}$ of degrees is computably independent if and only if
$$(\forall \boldsymbol{x} \in \mathcal{A})(\exists \boldsymbol{h})(\forall \boldsymbol{y} \in \mathcal{A})[\boldsymbol{x} \neq \boldsymbol{y} \Rightarrow \boldsymbol{y} \leq \boldsymbol{h} \;\&\; \boldsymbol{x} \not\leq \boldsymbol{h}];$$
(2) the additional parameters $\boldsymbol{n}',\boldsymbol{p}'$ partition $\mathcal{G}$ into two halves $\mathcal{G}_{N,\vec{\boldsymbol{w}}'}$ and $\mathcal{G}_{P,\vec{\boldsymbol{w}}'}$ which are still computably independent (hence quantification on subsets of these two sets correspond via Theorem 2.5 to quantification on degrees);
(3) $\boldsymbol{p}'_0,\boldsymbol{p}'_1$ code the successor function on $\mathcal{G}_{N,\vec{\boldsymbol{w}}'}$ via $s_{\vec{\boldsymbol{w}}'}$, in the same first order way as in our previous discussion for $\vec{\boldsymbol{w}}$;
(4) $\boldsymbol{p}'_2,\boldsymbol{p}'_3$ codes the set of ordered pairs of $\mathcal{G}_{N,\vec{\boldsymbol{w}}'}$ in the same first order way as in our previous discussion, satisfying the first order condition with parameters therein exhibited;

(5) the relations $s_{\overrightarrow{\boldsymbol{w}}'}$, $+_{\overrightarrow{\boldsymbol{w}}'}$, $\times_{\overrightarrow{\boldsymbol{w}}'}$, introduced by uniform definitions from the parameters $\overrightarrow{\boldsymbol{w}}'$, satisfy the (finitely many) axioms of Robinson's Arithmetic (in the appropriate relational language); notice that this allows to define "zero", $\mathbf{0}_{\overrightarrow{\boldsymbol{w}}'}$, in $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}'}$;

(6) the first order translation (via Theorem 2.5) of second order induction is satisfied, i.e. the following holds in $\mathbf{D}_{\mathrm{s}}$:

$$(\forall U \subseteq \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}'}) \left[\mathbf{0}_{\overrightarrow{\boldsymbol{w}}'} \in U \ \& \ (\forall u \in \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}'}) \left[u \in U \to s_{\overrightarrow{\boldsymbol{w}}'}(u) \in U\right] \to U = \mathcal{G}_{N,\overrightarrow{\boldsymbol{w}}'}\right].$$

Clearly, the correctness condition $\alpha_{st}(\overrightarrow{w})$ can be expressed in a first order way, and the following holds:

Lemma 2.11. *If* $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}}')$ *holds, then* $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}'}$ *is isomorphic to* $\mathbb{N}$, *and*

$$(\mathbb{N}, \mathcal{P}(\omega)) \models \sigma \Leftrightarrow \mathbf{D}_{\mathrm{s}} \models \sigma^*(\overrightarrow{\boldsymbol{w}}').$$

Proof. Obvious by the previous remarks. Clearly $\mathbb{N}_{\overrightarrow{\boldsymbol{w}}'}$ has no nonstandard elements, as follows from satisfaction of second order induction. □

Corollary 2.12. $\mathrm{Th}(\mathbb{N}, \mathcal{P}(\omega)) \leq_{\mathrm{m}} \mathrm{Th}(\mathbf{D}_{\mathrm{s}})$.

Proof. By Theorem 3.1 there exists $\overrightarrow{\boldsymbol{w}}$ such that $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}})$. Then, by the previous lemma, for every second order arithmetical sentence σ,

$$(\mathbb{N}, \mathcal{P}(\omega)) \models \sigma \Leftrightarrow \mathbf{D}_{\mathrm{s}} \models (\exists \overrightarrow{w})(\alpha_{st}(\overrightarrow{w}) \ \& \ \sigma^*(\overrightarrow{w})).$$

□

3. A copy of the standard model of arithmetic within $\mathbf{D}_{\mathrm{s}}$

It remains to show that there exists a list of parameters $\overrightarrow{\boldsymbol{w}}$ in $\mathbf{D}_{\mathrm{s}}$, such that $\mathbf{D}_{\mathrm{s}} \models \alpha_{st}(\overrightarrow{\boldsymbol{w}})$. The following theorem is a strengthening of a result in [1], where it is shown that there exists a computable independent SW-set consisting of Σ^0_2 s-degrees: This result was used to show that the first order theory of the Σ^0_2 s-degrees is undecidable. The proof of Theorem3.1 is based on [1]: One of the reasons to reproduce here in full detail even the parts of the proof inherited from [1] is to make up for a few typos and inaccuracies therein appearing.

Theorem 3.1. *In* $\mathbf{D}_{\mathrm{s}}$ *there exist degrees* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{g}$, $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ *and a computably independent SW-set* $\mathcal{G} = \{\boldsymbol{g}_i\}_{i\in\omega}$ *such that letting* $\boldsymbol{a}_i = \boldsymbol{g}_{2i}$, $\boldsymbol{b}_i = \boldsymbol{g}_{2i+1}$, $\mathcal{G}_N = \{\boldsymbol{a}_i : i \in \omega\}$, *we have:*

- $\mathcal{G}$ *is defined by* $\varphi_{SW}(x; a, b, g)$ *from the parameters* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{g}$;
- *for every* i, $\boldsymbol{a}_{2i+1}$ *is the unique element* $\boldsymbol{a}$ *of* $\mathcal{G}_N$ *such that* $\boldsymbol{a} \neq \boldsymbol{a}_{2i}$ *and* $\boldsymbol{a} \leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0$; *for every* i, $\boldsymbol{a}_{2i+2}$ *is the unique element* $\boldsymbol{a}$ *of* $\mathcal{G}_N$ *such that* $\boldsymbol{a} \neq \boldsymbol{a}_{2i+1}$ *and* $\boldsymbol{a} \leq \boldsymbol{a}_{2i+1} \cup \boldsymbol{p}_1$;
- *for every* k, n, *the pair* $(\boldsymbol{a}_k, \boldsymbol{a}_n)$ *is the unique pair* $(\boldsymbol{a}, \boldsymbol{a}') \in \mathcal{G}_N^2$ *such that* $\boldsymbol{a} \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_2$, *and* $\boldsymbol{a}' \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_3$.

Proof. We build a set $G = \bigcup_{i\in\omega} G_i$ (with $G_i \subseteq \omega^{[i]}$, for every i, so, up to $\equiv_{\mathrm{s}}$, we may identify $\bigoplus_{i\in I} G_i$ with $\bigcup_{i\in I} G_i$, for every $I \subseteq \omega$) so that the requirements displayed below are satisfied, for every i, k, n and for every pair Φ, Ψ of s-operators, where we let $A_i = G_{2i}$, $B_i = G_{2i+1}$, and Δ_i, Ξ_i, Ω_i, $\Lambda_{k,n}$, $\Pi_{k,n}$ and $\Gamma_{\Phi,\Psi,i}$ are s-operators built by us; $x \dot{-} 1$ denotes $x - 1$ if $x \geq 0$, and $x \dot{-} 1 = 0$ otherwise.

$$\mathcal{D}_i : A = \Delta_i(G_i \oplus B)$$
$$\mathcal{M}_{\Phi,\Psi} : A = \Phi(\Psi(G) \oplus B) \Rightarrow (\exists i, \Gamma_{\Phi,\Psi,i})[G_i = \Gamma_{\Phi,\Psi,i}(\Psi(G))]$$
$$\mathcal{C}_i^{od} : A_{2i+1} = \Xi_i(A_{2i} \oplus P_0)$$
$$\mathcal{C}_i^{ev} : A_{2i+2} = \Omega_i(A_{2i+1} \oplus P_1)$$
$$\mathcal{C}_{k,n,2} : A_k = \Lambda_{k,n}(B_{\langle k,n\rangle} \oplus P_2)$$
$$\mathcal{C}_{k,n,3} : A_n = \Pi_{k,n}(B_{\langle k,n\rangle} \oplus P_3)$$
$$\mathcal{I}_{i,\Psi} : G_i \neq \Psi(\bigoplus_{j\neq i} G_j)$$
$$\mathcal{I}_{i,0,\Psi}^{od} : A_{2i+1} \neq \Psi(\bigoplus_{j\neq i} A_{2j} \oplus P_0)$$
$$\mathcal{I}_{i,1,\Psi}^{od} : A_{2i+1} \neq \Psi(\bigoplus_{j\neq i} A_{2j+1} \oplus P_1)$$
$$\mathcal{I}_{i,0,\Psi}^{ev} : A_{2i} \neq \Psi(\bigoplus_{j\neq i} A_{2j} \oplus P_0)$$
$$\mathcal{I}_{i,1,\Psi}^{ev} : A_{2i} \neq \Psi(\bigoplus_{2j+1\neq 2i\dot{-}1} A_{2j+1} \oplus P_1)$$
$$\mathcal{I}_{k,2,\Psi} : A_k \neq \Psi(\bigoplus_{\substack{n\\ i\neq k}} B_{\langle i,n\rangle} \oplus P_2)$$
$$\mathcal{I}_{n,3,\Psi} : A_n \neq \Psi(\bigoplus_{\substack{k\\ j\neq n}} B_{\langle k,j\rangle} \oplus P_3).$$

We distinguish the following groups of requirements (where $i, k, n \in \omega$, and Φ, Ψ are s-operators): the *$\mathcal{D}$-requirements*, of the form $\mathcal{D}_i$, for some i; the *$\mathcal{M}$-requirements*, of the form $\mathcal{M}_{\Phi,\Psi}$; the *$\mathcal{C}^{od}$-requirements*, of the form $\mathcal{C}_i^{od}$; the *$\mathcal{C}^{ev}$-requirements*, of the form $\mathcal{C}_i^{ev}$; the *$\mathcal{C}_2$-requirements*, of the form $\mathcal{C}_{k,n,2}$; the *$\mathcal{C}_3$-requirements*, of the form $\mathcal{C}_{k,n,3}$; the *$\mathcal{I}$-requirements*, of the form $\mathcal{I}_{i,\Psi}$; the *$\mathcal{I}_0^{od}$-requirements*, of the form $\mathcal{I}_{i,0,\Psi}^{od}$; the *$\mathcal{I}_1^{od}$-requirements*, of the form $\mathcal{I}_{i,1,\Psi}^{od}$; the *$\mathcal{I}_0^{ev}$-requirements*, of the form $\mathcal{I}_{i,0,\Psi}^{ev}$; the *$\mathcal{I}_1^{ev}$-requirements*, of the form $\mathcal{I}_{i,1,\Psi}^{ev}$; the *$\mathcal{I}_2$-requirements*, of the form $\mathcal{I}_{k,2,\Psi}$; the *$\mathcal{I}_3$-requirements*, of the form $\mathcal{I}_{k,3,\Psi}$.

We then talk about the *minimality requirements*, i.e., the $\mathcal{M}$-requirements; the *comparability requirements*, which include the $\mathcal{C}^{od}$-requirements, the $\mathcal{C}^{ev}$-requirements, the $\mathcal{C}_2$-requirements, and the $\mathcal{C}_3$-requirements; and finally, the *incomparability requirements*, which include the $\mathcal{I}$-requirements, the $\mathcal{I}_0^{od}$-requirements, the $\mathcal{I}_1^{od}$-requirements, the $\mathcal{I}_0^{ev}$-requirements, the $\mathcal{I}_1^{ev}$-requirements, the $\mathcal{I}_2$-requirements, and lastly the $\mathcal{I}_3$-requirements.

If the sets $A, B, G, \{G_i\}_{i\in\omega}, P_0, P_1, P_2, P_3$ satisfy all the above requirements, then the corresponding s-degrees $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{g}, \{\boldsymbol{g}\}_{i\in\omega}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ have the desired properties. Indeed:

(1) satisfaction of the $\mathcal{D}$-requirements and the fact that $\boldsymbol{g} = \bigcup_{i\in\omega} \boldsymbol{g}_i$ guarantee that each $\boldsymbol{g}_i$'s is a solution of
$$\boldsymbol{a} \leq \boldsymbol{x} \cup \boldsymbol{b} \;\&\; \boldsymbol{x} \leq \boldsymbol{g};$$
(2) satisfaction of all $\mathcal{M}$-requirements guarantees that the $\boldsymbol{g}_i$ are minimal solutions of the above system of two inequalities: If $\boldsymbol{y} = \deg_s(Y)$ is a solution, then $Y = \Psi(G)$ for some s-operator Ψ, and there exists an s-operator Φ such

that $A = \Phi(\Psi(G) \oplus B)$. But then there exists an i such that $G_i \leq_{\mathrm{s}} Y$ via the s-operator $\Gamma_{\Phi,\Psi,i}$;

(3) satisfaction of the $\mathcal{I}$-requirements guarantees that $\{\boldsymbol{g}_i\}_{i\in\omega}$ is a computably independent set of s-degrees;

(4) requirement $\mathcal{C}_i^{od}$ implies that $\boldsymbol{a}_{2i+1} \leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0$. On the other hand $\boldsymbol{a}_{2i+1}$ is the unique element $\boldsymbol{x}$ of $\mathcal{G}_N$ that satisfies

$$\boldsymbol{x} \neq \boldsymbol{a}_{2i} \ \&\ \boldsymbol{x} \leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0.$$

In fact, if $i' \neq i$ then by satisfaction of every $\mathcal{I}^{od}_{i',0,\Psi}$ we have that $\boldsymbol{a}_{2i'+1} \not\leq \bigcup_{j\neq i'} \boldsymbol{a}_{2j} \cup \boldsymbol{p}_0$. But then $\boldsymbol{a}_{2i'+1} \not\leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0$, as $\boldsymbol{a}_{2i}$ is one of the addenda in $\bigcup_{j\neq i'} \boldsymbol{a}_{2j}$. Furthermore, if $i' \neq i$ then by satisfaction of every $\mathcal{I}^{ev}_{i',0,\Psi}$ we have that $\boldsymbol{a}_{2i'} \not\leq \bigcup_{j\neq i'} \boldsymbol{a}_{2j} \cup \boldsymbol{p}_0$; hence $\boldsymbol{a}_{2i'} \not\leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0$, as $\boldsymbol{a}_{2i}$ is one of the addenda in $\bigcup_{j\neq i'} \boldsymbol{a}_{2j}$;

(5) a similar argument, involving the requirements of the form $\mathcal{I}^{od}_{i',1,\Psi}$ and $\mathcal{I}^{ev}_{i',1,\Psi}$, shows that $\boldsymbol{a}_{2i+2}$ is the unique element $\boldsymbol{x}$ of $\mathcal{G}_N$ that satisfies system of inequalities

$$\boldsymbol{x} \neq \boldsymbol{a}_{2i+1} \ \&\ \boldsymbol{x} \leq \boldsymbol{a}_{2i+1} \cup \boldsymbol{p}_1;$$

on the other hand, for no $\boldsymbol{a}_i$, $i \neq 0$, can we have $\boldsymbol{a}_0 \leq \boldsymbol{a}_{2i} \cup \boldsymbol{p}_0$, and for no i can we have $\boldsymbol{a}_0 \leq \boldsymbol{a}_{2i+1} \cup \boldsymbol{p}_1$;

(6) for fixed k, satisfaction of all requirements $\mathcal{C}_{k,n,2}$ gives that $\boldsymbol{a}_k \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_2$; on the other hand, satisfaction of all requirements of the form $\mathcal{I}_{k,2,\Psi}$ gives that $\boldsymbol{a}_k$ is the unique degree $\boldsymbol{a} \in \mathcal{G}_N$ such that $\boldsymbol{a} \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_2$, for some n;

(7) finally, for fixed n, satisfaction of all requirements $\mathcal{C}_{k,n,3}$ gives that $\boldsymbol{a}_n \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_3$; on the other hand, satisfaction of all requirements of the form $\mathcal{I}_{k,3,\Psi}$ gives that $\boldsymbol{a}_n$ is the unique degree $\boldsymbol{a} \in \mathcal{G}_N$ such that $\boldsymbol{a} \leq \boldsymbol{b}_{\langle k,n\rangle} \cup \boldsymbol{p}_3$, for some n;

It follows from the previous two items that for fixed k, n, the pair $(\boldsymbol{a}_k, \boldsymbol{a}_n)$ is the unique ordered pair of elements of $\mathcal{G}_N$ which is "coded" by $\boldsymbol{b}_{\langle k,n\rangle}$ through the "projections" $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$.

3.1. **Strategies to meet the requirements.** We now describe the strategies that will be used to meet each requirement in isolation. Strategies will be usually classified following the classification of the requirements that they are intended to meet: Thus for instance we talk in general of *incomparability strategies*, or of *comparability strategies*, if the strategies refer to incomparability requirements, or comparability requirements, respectively; similarly, according to the particular requirements addressed by the strategies, we may talk about *$\mathcal{D}$-strategies*, *$\mathcal{M}$-strategies*, *$\mathcal{C}^{od}$-strategies*, *$\mathcal{C}^{ev}$-strategies*, *$\mathcal{C}_2$-strategies*, *$\mathcal{C}_3$-strategies*, *$\mathcal{I}$-strategies*, *$\mathcal{I}_0^{od}$-strategies*, *$\mathcal{I}_1^{od}$-strategies*, *$\mathcal{I}_0^{ev}$-strategies*, *$\mathcal{I}_1^{ev}$-strategies*, *$\mathcal{I}_2$-strategies*, or *$\mathcal{I}_3$-strategies*.

For the sake of notational simplicity we use lower case Greek letters to denote strategies, and, for a given strategy α, we will often use the same symbol α also to index (some of) the relevant ingredients of the addressed requirement: For instance, if α addresses the requirement $\mathcal{C}_i^{od} : A_{2i+1} = \Xi_i(A_{2i} \oplus P_0)$ then, with obvious suggestion, the requirement will be often written as $A_\alpha = \Theta_\alpha(A_{\alpha-1} \oplus P_\alpha)$, where $A_\alpha = A_{2i+1}$, $\Theta_\alpha = \Xi_i$, $A_{\alpha-1} = A_{2i}$, $P_\alpha = P_0$; similarly, if α is an $\mathcal{I}$-strategy, then the corresponding requirement may be written as $G_\beta \neq \Psi_\beta(\bigoplus_{j\neq\beta} G_j)$, and so on.

3.1.1. ***Incomparability strategies.*** To address incomparability requirements, we use strategies that are appropriate variations of the classical Friedberg-Muchnick strategy.

We observe that an incomparability requirement has the form $X \neq \Psi(Y)$, where Ψ is an s-operator, and X and Y are sets (built by us): We have that $X = G_i$, for some i, and Y has two possible forms: There is set T of indices such that either $Y = \bigoplus_{j\in T} G_j$ if we are dealing with an $\mathcal{I}$-requirement; or $Y = \bigoplus_{j\in T} G_j \oplus P$ (with P also built by us), if we are dealing with an incomparability requirement that is not an $\mathcal{I}$-requirement. In any case, $X \neq G_j$ for every $j \in T$, and $X \neq P$.

To achieve $X \neq \Psi(Y)$:

(1) appoint a new witness $g \in X$;
(2) await $g \in \Psi(Y)$;
(3) extract g from X, and restrain $g \in \Psi(Y)$; (this can be done since X is none of the addenda appearing in $\bigoplus_{j\in T} G_j$, or in $\bigoplus_{j\in T} G_j \oplus P$.)

3.1.2. ***Comparability strategies.*** Next we treat the comparability strategies. A comparability strategy has the form $X = \Theta(C \oplus P)$, where X, C (both lying in $\{G_i : i \in \omega\}$), and P are sets built by us, and Θ is an s-operator built by us.

To achieve $X = \Theta(C \oplus P)$, the strategy aims to maintain a correct s-operator Θ: Whenever we have $g \in X$ (due to the action of some lower priority incomparability strategy), we add (if no such axiom is active, i.e. already defined and valid at the current stage) an axiom of the form $\langle g, \{g'\} \oplus \emptyset\rangle \in \Theta$, with $g' \in C$, or of the form $\langle g, \emptyset \oplus \{p\}\rangle \in \Theta$, with $p \in P$. We choose either form depending on which lower priority $\mathcal{I}$-strategy demands $g \in X$ as a witness. We will come back later to this point, when discussing in more detail the interactions between strategies. Of course, if later g is extracted from X (by the same incomparability strategy), then we must accordingly correct $\Theta(C \oplus P)$ by extracting g' from X, or p from P, as appropriate.

Although we are discussing here comparability strategies, it is clear by the above remarks that for a given comparability strategy a great deal of the task of maintaining a correct Θ is performed by the lower priority incomparability strategies that are enumerating numbers into X, rather than by the comparability strategy itself, which limits itself to some routine updating: In particular, if the strategy sees that some $g \in X$ will never be used again by the construction, then it permanently restrains $g \in X$, and in $\Theta(C \oplus P)$ by adding the axiom $\langle g, \emptyset \oplus \emptyset\rangle \in \Theta$.

3.1.3. ***Minimality strategies.*** The strategy in isolation to achieve $\mathcal{M}_{\Phi,\Psi}$ is the following:

(1) appoint a new witness $x \in A$;
(2) await $x \in \Phi(\Psi(G) \oplus B)$;
(3) extract x from A and restrain $x \in \Phi(\Psi(G) \oplus B)$. (In isolation, this is not a problem, since A is neither $\Psi(G)$, nor B.)

As we will see later, implementation of this simple strategy may be prevented by its interactions with higher priority $\mathcal{D}$-strategies. In this case, we switch to a backup strategy, which consists in constructing an s-operator Γ_i, for some i, such that, $G_i = \Gamma(\Psi(G))$: Details about this will be given later.

3.1.4. ***$\mathcal{D}$-strategies.*** These are comparability strategies of a different type. A $\mathcal{D}$-strategy α, addressing the requirement $A = \Delta_\alpha(G_\alpha \oplus B)$ (recall that A, B, and G_α are sets built by us, and Δ_α is an s-operator built by us) consists in maintaining correctness of the s-operator Δ_α. When we see some $x \in A$ (due to the action of some lower priority $\mathcal{M}$-strategy, which shares therefore the responsibility of maintaining a correct Δ_α), we add (if no such axiom is already active) an axiom of the form $\langle x, \{g\} \oplus \emptyset\rangle \in \Delta_\alpha$, with $g \in G_\alpha$, or of the form $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\alpha$, with $b \in B$. We choose either form depending on the particular action taken by the $\mathcal{M}$-strategy demanding $x \in A$ as a witness: We will come back later to this when discussing in more detail the interactions between strategies. If, later, x is extracted from A (by the same $\mathcal{M}$-strategy, that has appointed it as a witness), then we must accordingly correct $\Delta_\alpha(G_\alpha \oplus B)$ by extracting g from G_α, or b from B, as appropriate. When α

sees that some number x will never be used again by the construction, it then takes care of making $A(x) = \Delta_\alpha(G_\alpha \oplus B)(x)$: In particular, if this happens when $x \in A$, then it permanently restrains $x \in A$, and also $x \in \Delta_\alpha(G_\alpha \oplus B)$ by adding the axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_\alpha$.

3.2. **Interactions between strategies.** We begin by analyzing how the incomparability strategies interact with higher priority comparability strategies. Throughout this section we write $\alpha \subset \beta$ to mean that α has higher priority than β. (On the other hand, this reflects what happens in the later actual construction, where we place strategies on a tree of strategies (thus strategies correspond to strings), and the only delicate interactions between strategies α, β with α having higher priority than β will happen when α is an initial segment of β: The case of α "to the left of" β will be taken care of by initialization.)

3.2.1. ***β is an $\mathcal{I}$-strategy, and $\alpha \subset \beta$ is a comparability strategy.*** We need only consider the case when β is a strategy addressing a requirement $G_\beta \neq \Psi_\beta(\bigoplus_{j \neq \beta} G_j)$, and α is relative to $A_\alpha = \Theta_\alpha(A_{\alpha-1} \oplus P_\alpha)$, or to $A_\alpha = \Theta_\alpha(B_\alpha \oplus P_\alpha)$, and the two strategies do *interact*, i.e. $G_\beta = A_\alpha$:

When β appoints a witness $g \in G_\beta$, then it also adds an axiom $\langle g, \emptyset \oplus \{p\} \rangle \in \Theta_\alpha$, with $p \in P_\alpha$. Notice that if later β needs to extract g from G_β, then this can be done without injuring α by extracting p from P_α.

3.2.2. ***β is an incomparability strategy other than an $\mathcal{I}$-strategy, and $\alpha, \gamma \subset \beta$ are comparability strategies.*** We begin with some definitions.

We say that β is *P_r-related* ($r = 0, 1, 2, 3$) if β addresses an incomparability requirement of the form

$$A_\beta \neq \Psi_\beta(C_\beta \oplus P_r),$$

(i.e. $P_\beta = P_r$) where C_β is of the form $C_\beta = \bigoplus_{j \in T_\beta} C_j$. (Hence $\mathcal{I}_0^{od}$- and $\mathcal{I}_0^{ev}$-strategies are P_0-related; $\mathcal{I}_1^{od}$- and $\mathcal{I}_1^{ev}$-strategies are P_1-related; $\mathcal{I}_2$-strategies are P_2-related; and $\mathcal{I}_3$-strategies are P_3-related.)

Similarly, a comparability strategy α is *P_r-related* ($r = 0, 1, 2, 3$) if α addresses a comparability requirement of the form $A_\alpha = \Theta_\alpha(A_{\alpha-1} \oplus P_r)$ (in this case $r \in \{0, 1\}$: Namely, $r = 0$ if $\alpha - 1$ is an even index, and $r = 1$ if $\alpha - 1$ is odd), or of the form $A_\alpha = \Theta_\alpha(B_\alpha \oplus P_r)$ (in this case $r \in \{2, 3\}$: Namely, $r = 2$ if α is a $\mathcal{C}_2$-strategy, and $r = 3$ if α is a $\mathcal{C}_3$-strategy).

Next, we give the definition of interacting strategies:

Definition 3.2. Let β be an incomparability strategy, and let $\alpha, \gamma \subset \beta$ be comparability strategies:

(1) We say that β *interacts with* α if $A_\beta = A_\alpha$;
(2) We say that β *indirectly interact with* γ *via* α, if β and α interact, β and α are P_r-related with the same $r \in \{0, 1\}$, and $A_\gamma = A_{\alpha-1}$. (The intuition here is that if β needs to correct Θ_α by extracting some element from $A_{\alpha-1}$, then it also needs to correct Θ_γ.)

In order to describe the relevant interactions, we distinguish the following cases: In the first two cases, the action taken by β towards Θ_α does not instigate any γ to indirectly interact with β.

β P_r-related, α P_s-related, $r \neq s$. When β appoints a witness $g \in A_\beta$, then it also adds an axiom $\langle g, \emptyset \oplus \{p\} \rangle \in \Theta_\alpha$, with $p \in P_\alpha$. Notice that if later β needs to extract g from A_β, while restraining $g \in \Psi_\beta(C_\beta \oplus P_\beta)$, then this can be done without injuring α, by extracting p from P_α, which is not an addendum in C_β, nor does $P_\alpha = P_\beta$.

β **and** α P_r**-related, and** $r \in \{2,3\}$. When β appoints a witness $g \in A_\beta$, then it also adds an axiom $\langle g, \{g'\} \oplus \emptyset\rangle \in \Theta_\alpha$, with $g' \in B_\alpha$. Notice that if later β needs to extract g from A_β, while restraining $g \in \Psi_\beta(C_\beta \oplus P_\beta)$, then this can be done without injuring α by extracting g' from B_α, which is not an addendum in C_β (since $C_\beta \subseteq \{A_i : i \in \omega\}$), nor does $B_\alpha = P_\beta$. Notice also that extraction of g' from B_α does not require correction of any other s-operator, since B_α does not appear in the left-hand side of any comparability requirement.

β **and** α P_r**-related, and** $r \in \{0,1\}$. When β appoints a witness $g \in A_\beta$, then it also adds an axiom $\langle g, \{g'\} \oplus \emptyset\rangle \in \Theta_\alpha$, with $g' \in A_{\alpha-1}$. We must noe consider the case of $\gamma \subset \beta$ such that β indirectly interacts with γ via α, hence $A_\gamma = A_{\alpha-1}$. Since β and γ indirectly interact, we have that β is responsible for keeping Θ_γ correct, too. But notice that γ is P_s-related, with $r \neq s$. (Consider for instance the case $r = 0$: If β addresses the requirement $\mathcal{I}^{od}_{i,0,\Psi}$, then $g' \in A_{2i}$, and γ addresses either a $\mathcal{C}^{ev}$-, or a $\mathcal{C}_2$-, or a $\mathcal{C}_3$-requirement, but in any case γ is not P_0-related. The other cases are similar.) But if γ is not P_r-related, then β can comply with the task of maintaining Θ_γ correct, exactly as in the first item, since β is P_r-related and γ is P_s-related, with $r \neq s$: Thus β adds an axiom $\langle g', \emptyset \oplus \{p\}\rangle \in \Theta_\gamma$, with $p \in P_\gamma$. Notice that if later β needs to extract g from A_β, while restraining $g \in \Psi_\beta(C_\beta \oplus P_\beta)$, then this can be done without injuring either α or γ by extracting g' from $A_{\alpha-1}$ (this can be done since $A_{\alpha-1}$ not an addendum in C_β, nor does $A_{\alpha-1} = P_\beta$); and by extracting p from P_γ (this can be done since P_γ is not an addendumnin C_β, nor does $P_\gamma = P_\beta$).

3.2.3. ***β is a minimality strategy and $\alpha \subset \beta$ is a $\mathcal{D}$-strategy.*** Let β be a minimality strategy, and α a $\mathcal{D}$-strategy, with $\alpha \subset \beta$. Suppose that β wants to extract x from A, and restrain $x \in \Phi_\beta(\Psi_\beta(G) \oplus B)$, but currently $x \in \Phi_\beta(\Psi_\beta(G) \oplus B)$ via an axiom $\langle x, \{y\} \oplus \emptyset\rangle \in \Phi_\beta$, and $y \in \Psi_\beta(G)$ via an axiom $\langle y, \{g\}\rangle \in \Psi_\beta$, with $g \in G$; on the other hand, β, when appointing x as a witness, has already defined an axiom $\langle x, \{g\} \oplus \emptyset\rangle \in \Delta_\alpha$, with $g \in G_\alpha$, so it is not possible to extract x from A, and restrain $x \in \Phi_\beta(\Psi_\beta(G) \oplus B)$, without injuring α. This conflict may be resolved as in [1] (which we closely follows in our presentation), using the following device:

We wait for axioms of a different form to appear for y in $\Psi_\beta(G)$: For instance $\langle y, \emptyset\rangle \in \Psi_\beta$ or $\langle y, \{g'\}\rangle \in \Psi_\beta$, with $g' \in G$, $g' \neq g$, such that g' can be restrained without preventing β from extracting x from A, and rectifying Δ_α; or for axioms of the form $\langle x, \emptyset \oplus \emptyset\rangle \in \Phi_\beta$, or $\langle x, \emptyset \oplus \{b\}\rangle \in \Phi_\beta$, $b \in B$, such that we can restrain $b \in B$, without preventing β from extracting x from A and rectifying Δ_α. If and when such an axiom appears, then we go ahead with the extraction of x from A, and at the same time we are able to keep $x \in \Phi_\beta(\Psi_\beta(G) \oplus B)$.

While waiting, we continue building an s-operator Γ_α (in fact, this operator is built by β, so it would be more appropriate to call it $\Gamma_{\beta,\alpha}$) , by enumerating the axiom $\langle g, \{y\}\rangle \in \Gamma_\alpha$; we extract from G all those numbers $\hat{g}$ such that there are axioms $\langle x, \{\hat{g}\} \oplus \emptyset\rangle \in \Delta_\gamma$, for all $\mathcal{D}$-strategies $\gamma \neq \alpha$ and $\gamma \subset \beta$. If a new axiom $\langle y, \{g'\}\rangle \in \Psi_\beta$, with $g' \in G$, as in the previous bullet, appears, then g' is different from g and all these $\hat{g}$'s. Therefore we are free to diagonalize A against $\Phi_\beta(\Psi_\beta(G) \oplus B)$, by restraining $g' \in G$, extracting x from A, and maintaining each Δ_γ correct, for each $\mathcal{D}$-strategy $\gamma \subset \beta$. On the other hand, if no new such axiom appears, then we have that $g \in G_\alpha$ if and only if $y \in \Psi_\beta(G)$, and thus $g \in G_\alpha$ if and only if $g \in \Gamma_\alpha(\Psi_\beta(G))$. The idea is then to "pass on" g to lower priority strategies for their own use, as they can freely use g without destroying the correctness of Γ_α at g. Notice, however, that our extraction of the relevant $\hat{g}$ from G_γ has made $x \notin \Delta_\gamma(G_\gamma \oplus B)$, even if $x \in A$. To set back $A(x) = \Delta_\gamma(G_\gamma \oplus B)(x)$, we select a new element b, with $b \in B$, and define the axiom $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\gamma$. If later we are able to diagonalize by extracting x from

A, then we must extract b from B (in addition to the already extracted $\hat{g}$) to preserve $A(x) = \Delta_\gamma(G_\gamma \oplus B)(x)$.

The above strategy makes us lose x as a diagonalization witness for β, so we must appoint a new witness x' in a new attempt at diagonalization. If all our attempts at diagonalization fail, then since there are only finitely many strategies having higher priority than β, the conclusion must be that there is a least a strategy α such that we define infinitely many axioms of the form $\langle g, \{y\} \rangle \in \Gamma_\alpha$, and the elements of the infinite set (called *stream*) of these g's can be used as witnesses by lower priority strategies. Thus Γ_α is correct at least on the numbers belonging to the stream. To make Γ_α correct on numbers used by other lower priority strategies, whenever we define Γ_α, we permanently restrain in G_α all numbers g' (not in the stream) currently in G_α, used by these lower priority strategies, and we consequently permanently restrain each such g' into $\Gamma_\alpha(\Psi_\beta(G))$, by defining the axiom $\langle g', \emptyset \rangle \in \Gamma_\alpha$.

3.2.4. ***Interaction of an $\mathcal{M}$-strategy β with a higher priority comparability $\mathcal{C}$-strategy γ.*** Here the interaction consists in the fact that when β defines an axiom $\langle x, \{g\} \oplus \emptyset \rangle \in \Delta_\alpha$ with $g \in G_\alpha$, on behalf of some higher priority $\mathcal{D}$-strategy α, then it has also to update the s-operators Θ_γ relative to the comparability requirements $A_\gamma = \Theta_\gamma(A_{\gamma-1} \oplus P_\gamma)$ or $A_\gamma = \Theta_\gamma(B_\alpha \oplus P_\gamma)$, with $\gamma \subset \beta$, such that $G_\alpha = A_\gamma$ (we say in this case that β *interacts* with γ): This can be done by adding an axiom $\langle g, \emptyset \oplus \{p\} \rangle \in \Theta_\gamma$ with $p \in P_\gamma$, so that extraction of p from P_γ can later correct extraction of g from $G_\alpha = A_\gamma$.

3.3. **The tree of strategies.** We place strategies on a *tree of strategies*, defined as follows. On the set $\omega \cup \{w, d\}$ we define the strict linear order $<$,

$$d < 0 < 1 < \cdots < w,$$

and we use $<$ to order lexicographically the set of strings $(\omega \cup \{w, d\})^{<\omega}$. We use standard terminology and notations for strings: In particular, λ denotes the empty string, if α and β are strings then $|\alpha|$ denotes the length of α, and $\alpha \subseteq \beta$ means that α is an initial segment of β; $\alpha \subset \beta$ means that $\alpha \subseteq \beta$ and $\alpha \neq \beta$; we write $\alpha \leq \beta$ to denote that $\alpha \subseteq \beta$ or α lexicographically precedes β; finally we write $\alpha < \beta$ to denote that $\alpha \leq \beta$ but $\alpha \neq \beta$. We write $\alpha <_L \beta$ if $\alpha \leq \beta$ but $\alpha \not\subseteq \beta$.

In the following, we refer to some computable linear ordering of all requirements, in which $\mathcal{D}_i$ precedes $\mathcal{D}_j$ if and only if $i < j$. We define the *tree* T, a subtree of $(\omega \cup \{w, d\})^{<\omega}$, and a *requirement assignment* function R to nodes of T, recursively as follows. (In the following, if R has been already defined on α, we write $R(\alpha) = R_\alpha$.)

- $\lambda \in T$; R_λ is the least requirement;
- Suppose that we have defined $T_n = T \cap \{\alpha : \alpha \in (\omega \cup \{w, d\})^{<\omega} \,\&,\, |\alpha| = n\}$, and R on all $\alpha \in T_n$. Define the strings of length $n+1$ that belong to T, and the function R on these strings as follows: For every $\alpha \in T_n$:
  - if R_α is a $\mathcal{D}$-requirement, then $\alpha^\frown\langle 0 \rangle \in T$;
  - if R_α is a comparability requirement, then $\alpha^\frown\langle 0 \rangle \in T$;
  - if R_α is an incomparability strategy, then $\beta \in T$ where $\beta \in \{\alpha^\frown\langle o \rangle : o \in \{w, d\}\}$;
  - If R_α is an $\mathcal{M}$-strategy then $\beta \in T$ where $\beta \in \{\alpha^\frown o : o \in \omega \cup \{w, d\}\}$;
  - if β is any of the strings of length $n+1$ that belong to T and $\beta = \alpha^\frown\langle o \rangle$, for some $\alpha \in T$, then R_β is the least requirement such that $R_\beta \notin \{R_\gamma : \gamma \subseteq \beta\}$.

In view of the above requirement assignment, strings (or, nodes) in T will be also called *strategies*: moreover we say that α is an R_α*-strategy* as α can be viewed as the strategy to meet R_α. Notice that for every infinite branch of T, there is a bijective correspondence between the set of requirements and the set of nodes along the branch.

3.3.1. ***The parameters.*** During the construction we define approximations A_s, B_s, $G_{i,s}$, $P_{0,s}$, $P_{1,s}$, $P_{2,s}$ $P_{3,s}$ to the sets A, B, G_i, P_0, P_1, P_2, P_3: The final values of these sets will be, for $X \in \{A, B, G_i, P_0, P_1, P_2, P_3\}$, respectively:

$$X = \{y : (\exists t)(\forall s \geq t)[y \in X_s]\}.$$

Moreover, we will guarantee that at each stage s, $G_{i,s} \subseteq \omega^{[i]}$, so that, for every $I \subseteq \omega$, we may take $\bigoplus_{i \in I} G_{i,s} = \bigcup_{i \in I} G_{i,s}$, and eventually $\bigoplus_{i \in I} G_i = \bigcup_{i \in I} G_i$.

The main parameters defined in the construction are:

- For every $\mathcal{D}$-node α we define an s-operator Δ_α; for every $\mathcal{C}^{od}$-strategy α we define an s-operator Ξ_α; for every $\mathcal{C}^{ev}$-strategy α we define an s-operator Ω_α; for every $\mathcal{C}_2$-strategy α we define an s-operator Λ_α; for every $\mathcal{C}_3$-strategy α we define an s-operator Π_α;
- If β is an $\mathcal{M}$-strategy, we define: witnesses $x_\beta(0), x_\beta(1), \ldots$; for each $\mathcal{D}$-strategy $\alpha \subset \beta$, traces $g_{\beta,\alpha}(0), g_{\beta,\alpha}(1), \ldots$; and traces $b_\beta(0), b_\beta(1), \ldots$; (the traces $g_{\beta,\alpha}(t)$ and $b_\beta(t)$ will be used to define suitable Δ_α-axioms for $x_\beta(t)$); finally, for every $\mathcal{D}$-strategy $\alpha \subset \beta$, with $\mathcal{R}_\alpha = \mathcal{D}_i$, an s-operator $\Gamma_{\beta^\frown\langle i\rangle}$ (which has been called $\Gamma_{\beta,\alpha}$ in the previous informal discussion);
- For every incomparability strategy β we define a witness g_β, and traces $g_{\beta,\alpha}$ $p_{\beta,\alpha}$ that will be used to axiomatize higher priority comparability strategies $\alpha \subset \beta$ (as described in the upcoming paragraph on the axiomatization procedure), with which β interacts or indirectly interacts. Of course we may choose $p_{\beta,\alpha} = p_{\beta,\alpha'}$, and similarly $g_{\beta,\alpha} = g_{\beta,\alpha'}$, when the axiomatization procedure requires the traces to be enumerated into the same set, although $\alpha \neq \alpha'$;
- for every α we define a set (called *stream*) S_α, which is given, stage by stage, by specifying its elements. The approximation $S_{\alpha,s}$ of this stream at stage s will be defined by specifying each column $S^{[i]}_{\alpha,s} = S_{\alpha,s} \cap \omega^{[i]}$, for every j: Strategies $\beta \supseteq \alpha$ may use only elements chosen from $S^{[i]}_\alpha$ in order to define G_i.

3.3.2. ***The initialization procedure.*** The *initialization procedure* for strategy α at stage s consists in the following: We set $\Delta_{\alpha,s} = \Xi_{\alpha,s} = \Omega_{\alpha,s} = \Lambda_{\alpha,s} = \Pi_{\alpha,s} = \Gamma_{\alpha,s} = S_{\alpha,s} = \emptyset$; $g_{\alpha,s} =\uparrow$ (undefined), $g_{\alpha,\beta,s} =\uparrow$, $p_{\alpha,\beta,s} =\uparrow$, $x_{\alpha,s}(t) = b_{\alpha,s}(t) = g_{\alpha,\beta}(t) =\uparrow$ for any t.

Dumping. Upon discarding the value of a parameter, which is a witness or a trace, the construction will not change its current membership state, so if the discarded value, say, y_α is currently in the corresponding set X_α, then y_α is *dumped*, i.e. permanently restrained, in the set, and thus if we want to make $X_\alpha = \Theta_\alpha(Y_\alpha)$ (where Θ_α is built by the construction), we also *dump* y_α *into* Θ_α, i.e., we permanently restrain y_α in $\Theta_\alpha(Y_\alpha)$ by adding at the given stage the axiom $\langle y_\alpha, \emptyset\rangle \in \Theta_\alpha$.

3.3.3. ***Choosing new numbers, and about the stream.*** Only numbers $g \in \omega^{[i]}$ are chosen to go into G_i.

At stage $s+1$ a number y is *new* for strategy α if either

(1) y needs to be chosen to be enumerated into one of the sets A, B, P_0, P_1, P_2, P_3, and y is bigger than any number that has been used so far by any strategy; or
(2) y needs to be chosen for enumeration into G_i, for some i, and $y \in S^{[i]}_{\alpha,s+1} \setminus S^{[i]}_{\alpha,s}$. It will follow from the construction that $S^{[i]}_{\alpha,s+1} \setminus S^{[i]}_{\alpha,s}$ has at most one element, so when a strategy β picks some new g for enumeration into G_i, then no other new g' will be available at that stage to lower strategies for enumeration in G_i, so when β acts in this way we end the stage at $s+1$.

At stage $s+1$, if o is the current outcome of α, any new element entering $S^{[i]}_{\alpha,s+1}$ will also be enumerated into $S^{[i]}_{\alpha^\frown\langle o\rangle,s+1}$ unless otherwise specified.

3.3.4. ***The axiomatization procedure.*** Some strategies are requested to define axioms for certain s-operators: This action will be called *axiomatization procedure.* It is prescribed only for incomparability and minimality strategies, and it is performed by these strategies vs. comparability strategies as described in the section on interactions between strategies. Notice that the $\mathcal{M}$-strategies are also requested to define axioms for the operators appearing in the $\mathcal{D}$-strategies: We do not include this action in the axiomatization procedure described here, since we prefer to explain it in full detail in the course of the construction.

Let β be an incomparability strategy: In the following α, γ are comparability strategies, with $\alpha, \gamma \subset \beta$: We suppose that α addresses the requirement $A_\alpha = \Theta_\alpha(A_{\alpha-1} \oplus P_\alpha)$, or the requirement $A_\alpha = \Theta_\alpha(B_\alpha \oplus P_\alpha)$, and γ addresses the requirement $A_\gamma = \Theta_\gamma(A_{\gamma-1} \oplus P_\gamma)$, or requirement $A_\gamma = \Theta_\gamma(B_\gamma \oplus P_\gamma)$. The following actions are taken by β for every α and γ such that β and α interact, and β and γ indirectly interact.

β **is an $\mathcal{I}$-strategy**. Action: β chooses a new $p = p_{\beta,\alpha}$, defines $p \in P_\alpha$, and adds the axiom $\langle g, \emptyset \oplus \{p\}\rangle \in \Theta_\alpha$.

β **P_r-related, α P_s-related, $r \neq s$.** Action: β chooses a new $p = p_{\beta,\alpha}$, defines $p \in P_\alpha$, and adds the axiom $\langle g, \emptyset \oplus \{p\}\rangle \in \Theta_\alpha$.

β **and α P_r-related, and $r \in \{2,3\}$.** Action: β chooses a new $g' = g_{\beta,\alpha}$, defines $g' \in B_\alpha$, and adds the axiom $\langle g, \{g'\} \oplus \emptyset\rangle \in \Theta_\alpha$.

β **and α P_r-related, and $r \in \{0,1\}$.** Action: β chooses a new $g' = g_{\beta,\alpha}$, defines $g' \in A_{\alpha-1}$, and adds the axiom $\langle g, \{g'\} \oplus \emptyset\rangle \in \Theta_\alpha$, with $g' \in A_{\alpha-1}$. In this case we may have that the interaction of β with α instigates indirect interaction of β with some γ: If β and γ indirectly interact via α, then β chooses a new $p = p_{\beta,\gamma}$, defines $p \in P_\gamma$, and adds an axiom $\langle g', \emptyset \oplus \{p\}\rangle \in \Theta_\gamma$.

β **is an $\mathcal{M}$-strategy**. Action: β chooses a new $p = p_{\beta,\alpha}$, defines $p \in P_\alpha$, adds the axiom $\langle g, \emptyset \oplus \{p\}\rangle \in \Theta_\alpha$.

How to choose the witness g_β and the trace $g_{\beta,\alpha}$. Notice that there is no problem to choose the traces $p_{\beta,\alpha}$ since almost all numbers p are new, and thus available, for this purpose. There are on the other hand restrictions on how to choose the witness g_β and the traces $g_{\beta,\alpha}$ since, for every i, there is at most one new element which is a candidate for a new element to be enumerated in G_i: See the remark made on choosing new numbers and about the stream.

On the other hand, we have observed that at each stage we choose at most one $g \in G_i$, for every i. Now, our definition of S_α and the construction will guarantee that if β is allowed to act and needs to choose witness and corresponding traces, then it will be able to pick up all needed numbers at once.

In all above cases we say that β *axiomatizes α (on behalf of a number g)*, or β *indirectly axiomatizes γ via α* .

3.3.5. ***The rectification procedure.*** Together with the axiomatization procedure, one has to cope with the rectification procedure, aiming at rectifying the various s-operators, following extractions of elements from the sets built in the construction. Once again, the description below is short of the rectification action performed by the $\mathcal{M}$-strategies on behalf of the $\mathcal{D}$-strategies: The details of this action will be fully given in the construction.

If β, α, and γ are related to each other as in the previous section regarding the axiomatization procedure, then we say that β *rectifies* α, and β *indirectly rectifies* γ *via* α, if β performs the following actions: It extracts all traces $g_{\beta,\alpha}$, $p_{\beta,\alpha}$ and $p_{\beta,\gamma}$ enumerated by β to interact with α or directly interact with γ, when appointing the a witness g_β.

3.4. **Construction.** The construction is by stages. At stage s we define a string δ_s such that $|\delta_s| \leq s$. We follow standard notations and terminology about tree constructions as can be found in most textbooks on computability theory, see e.g. [22]. In particular at stage $s+1$, when dealing with a parameter p, or an expression $\mathcal{A}$ (for instance $\mathcal{A}$ is of the form "$X = \Theta(Y)$", where X, Y are sets, and Θ is an s-operator, of which we consider stage by stage approximations X_t, Y_t, Θ_t), we work with the *current* value of the parameter, or of the expression, meaning that we work with their values $p(s)$, or $\mathcal{A}(s)$ (given by the values of their ingredients), respectively, as coming from the previous stage s; or we work with their new values $p(s+1)$, or $\mathcal{A}(s+1)$, respectively, if already redefined in the course of stage $s+1$. To simplify notation (hopefully things will always be clear from the context), we usually drop any mention of the stage, thus simply writing only p, or $\mathcal{A}$.

Given a string α, let us say that a stage s is *α-true*, or s is an *α-stage,* if $\alpha \subseteq \delta_s$.

Stage 0. Define $\delta_0 = \lambda$; initialize all strategies.

Stage $s+1$. Suppose we have defined $\alpha = \delta_{s+1} \restriction n$ and $S^{[j]}_\alpha$, for every j. Assume also that we have defined $S_{\lambda,s+1} = \{\langle x, y\rangle : y \leq s, x \in \omega\}$ (note that at stage $s+1$ a new element is added to each column of $S_{\lambda,s}$, i.e. for every j, $S^{[j]}_{\lambda,s+1} \setminus S^{[j]}_{\lambda,s} = \{s\}$). If $n = s+1$, or we end the stage, then go to stage $s+2$, and initialize all strategies $\alpha \geq \delta_{s+1}$. Otherwise, proceed as follows (we distinguish the various possibilities for R_α, and we act accordingly):

$R_\alpha = \mathcal{D}_i$. Consider all $x \in A$ such that either x has been enumerated into A by some strategy $\beta \not\supseteq \alpha$, or x has been enumerated into A by some strategy $\beta \supseteq \alpha$ which has been later initialized.

For every such number, dump x into A and into $\Delta_\alpha(G_i \oplus B)$. Let $\alpha^\frown\langle 0\rangle \subseteq \delta_{s+1}$.

$R_\alpha = \mathcal{M}_{\Phi,\Psi}$. In the following we say that a witness $x = x_\alpha(t)$ is *eligible to act* if $x \in A \cap \Phi(\Psi(G) \oplus B)$, and we can still restrain $x \in \Phi(\Psi(G) \oplus B)$, and extract x from $\Delta_\beta(G_\beta \oplus B)$ for every $\mathcal{D}$-strategy $\beta \subset \alpha$ (by extracting g from G_β, or b from B, if $\langle x, \{g\} \oplus \emptyset\rangle \in \Delta_\beta$ or $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\beta$ are currently active axioms) . (The idea is: x is eligible to act, if we can use x to have $x \in \Phi(\Psi(G) \oplus B) \setminus A$ by simply extracting x from A, maintaining $A(x) = \Delta_\beta(G_\beta \oplus B)(x)$ for every $\mathcal{D}$-strategy $\beta \subset \alpha$.)

Let s^- be the previous α-stage after last initialization of α, with the understanding that s^- is undefined if there is no such stage. We distinguish the following cases:

(1) s^- is undefined, or we ended s^- at α, or s^- was an $\alpha^\frown\langle i\rangle$-stage, for some $i \in \omega$: Then (assuming that n is the least number such that $x_\alpha(n)$ is still undefined), choose a new $x = x_\alpha(n)$; define $x \in A$; for every $\mathcal{D}$-strategy $\beta \subset \alpha$ (with, say, $\mathcal{D}_\beta = \mathcal{D}_j$), appoint a new number $g' = g_{\alpha,\beta}(n) \in S^{[j]}_\alpha$, define $g' \in G_j$, add the axiom $\langle x, \{g'\} \oplus \emptyset\rangle \in \Delta_\beta$ and let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$ (since

x is new, we have $x \notin \Phi(\Psi(G) \oplus B)$); end the stage. (We have exhausted the only available new numbers in the relevant streams, so we wait for new numbers to become available for lower priority strategies.)

(2) s^- was an $\alpha^\frown\langle w\rangle$-stage, at which we have defined $x_\alpha(n)$ for the largest n (notice that $b_\alpha(n)$ is still undefined): We further distinguish the following cases:
   (a) $x_\alpha(n) \notin \Phi(\Psi(G) \oplus B)$: Then let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$.
   (b) Some $x = x_\alpha(t)$, $t \leq n$, is eligible to act: Then extract x from A (i.e. define $x \notin A$) and *rectify* every $\mathcal{D}$-strategy $\beta \subset \alpha$, i.e. extract $g = g_{\alpha,\beta}(t)$ from G_β and $b = b_\alpha(t)$ from B (where $\langle x, \{g\} \oplus \emptyset\rangle \in \Delta_\beta$ or $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\beta$ are the axioms previously enumerated by α to put $x \in \Delta_\beta(G_\beta \oplus B)$); let $\alpha^\frown\langle d\rangle \subseteq \delta_{s+1}$ and end the stage.
   (c) Otherwise: Since $x = x_\alpha(n)$ is not eligible, then membership $x \in \Phi(\Psi^G \oplus B)$ is achieved only by Φ-axioms for x of the form $\langle x, \{y\} \oplus \emptyset\rangle \in \Phi$, with corresponding Ψ-axioms for y of the form $\langle y, \{g\}\rangle \in \Psi$, where $g = g_{\alpha,\beta}(n)$ for some $\mathcal{D}$-strategy $\beta \subset \alpha$ (thus, $\langle x, \{g\} \oplus \emptyset\rangle \in \Delta_\beta$), and $g \in G_\beta$, so that we can not restrain $g \in G_\beta$ and extract x from A without making it impossible to achieve $A(x) = \Delta_\beta(G_\beta \oplus B)(x)$. Pick the least such β, and suppose, say, that $R_\beta = \mathcal{D}_i$. (Notice that i is the least such number, due to the way requirements are ordered.) For every $\mathcal{D}$-strategies $\gamma \neq \beta$ such that $\gamma \subset \alpha$, extract $g_{\alpha,\gamma}(n) \notin G_\gamma$. Pick a new $b = b_\alpha(n)$, define $b \in B$ and add the axiom $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\gamma$, for any $\mathcal{D}$-node $\gamma \subset \alpha$. Add the axiom $\langle g, \{y\}\rangle \in \Gamma_{\alpha^\frown\langle i\rangle}$. For each $\hat{g}$ such that $\hat{g} \in G_i$ and $\hat{g}$ has been enumerated by some strategy $\beta \not\supseteq \alpha^\frown\langle i\rangle$, dump $\hat{g}$ into G_i and into $\Gamma_{\alpha^\frown\langle i\rangle}(\Psi(G))$ by defining the axiom $\langle \hat{g}, \emptyset\rangle \in \Gamma_{\alpha^\frown\langle i\rangle}$. Define $S^{[i]}_{\alpha^\frown\langle i\rangle,s+1} = S^{[i]}_{\alpha^\frown\langle i\rangle,s} \cup \{g\}$, and let $\alpha^\frown\langle i\rangle \subseteq \delta_{s+1}$.

(3) s^- was an $\alpha^\frown\langle d\rangle$-stage: Then let $\alpha^\frown\langle d\rangle \subseteq \delta_{s+1}$.

$R_\alpha = \mathcal{D}_i$. Say α addresses the requirement $R_\alpha : A = \Delta_\alpha(G_i \oplus B)$: Then consider all $x \in A$ such that either

(1) x has been enumerated into A by some strategy $\beta \not\supseteq \alpha$, or
(2) x has been enumerated into A by some strategy $\beta \supset \alpha$ which has been later initialized.

Dump every such x into A and into Δ_α. Let $\alpha^\frown\langle 0\rangle \subseteq \delta_{s+1}$.

$R_\alpha \in \{\mathcal{C}_i^{od}, \mathcal{C}_i^{ev}, \mathcal{C}_{k,2,n}, \mathcal{C}_{k,3,n}\}$. Say α addresses the requirement $R_\alpha : A_\alpha = \Theta_\alpha(C_\alpha \oplus P_\alpha)$: Then consider all $g \in A_\alpha$ such that either

(1) g has been enumerated into A_α by some strategy $\beta \not\supseteq \alpha$ (meaning, here and below, that β has put the number in the set either by appointing it as a witness, or a trace), or
(2) g has been enumerated into A_α by some strategy $\beta \supset \alpha$ which has been later initialized.

Dump every such g into A_α and into Θ_α. Let $\alpha^\frown\langle 0\rangle \subseteq \delta_{s+1}$.

$R_\alpha = \mathcal{I}_{i,\Psi}$. We distinguish the following possible cases, indicating the corresponding actions:

(1) there is no appointed witness (i.e. g_α is undefined): appoint a new witness g, i.e. let $g_\alpha = g$, define $g \in G_i$; axiomatize higher priority strategies $\beta \subset \alpha$ which interact with α; let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$; end the stage. (Here and below in similar cases, we have exhausted the only available new numbers in the

respective streams, so we wait for new numbers to become available for lower priority strategies.)

(2) $g = g_\alpha$ is defined and $g \in G_i \setminus \Psi(\bigoplus_{j\neq i} G_j)$: Let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$.

(3) $g \in \Psi(\bigoplus_{j\neq i} G_j)$: Extract g from G_i (i.e. let $g \notin G_i$); rectify higher priority strategies $\beta \subset \alpha$, which interact with α; let $\alpha^\frown\langle d\rangle \subseteq \delta_{s+1}$; if this is the first time we have taken this case since the last initialization of α, end the stage.

$R_\alpha \in \{\mathcal{I}^{od}_{i,0,\Psi}, \mathcal{I}^{od}_{i,1,\Psi}, \mathcal{I}^{ev}_{i,0,\Psi}, \mathcal{I}^{ev}_{i,1,\Psi}, \mathcal{I}_{k,2,\Psi}, \mathcal{I}_{n,3,\Psi}\}$. Say that α addresses the requirement $R_\alpha : A_\alpha \neq \Psi_\alpha(C_\alpha \oplus P_\alpha)$. We distinguish the following possible cases:

(1) there is no appointed witness (i.e. g_α is undefined): appoint a new witness g, i.e. let $g_\alpha = g$, define $g \in A_\alpha$, axiomatize, on behalf of g, higher priority strategies $\beta \subset \alpha$ interacting with α, and higher priority $\gamma \subset \alpha$, indirectly interacting with α; let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$; end the stage.

(2) $g = g_\alpha$ is defined and $g \in A_\alpha \setminus \Psi_\alpha(C_\alpha \oplus P_\alpha)$: Let $\alpha^\frown\langle w\rangle \subseteq \delta_{s+1}$.

(3) $g \in \Psi_\alpha(C_\alpha \oplus P_\alpha)$: let $g \notin A_\alpha$; rectify higher priority strategies $\beta \subset \alpha$ which interact with α, and higher priority $\gamma \subset \alpha$ indirectly interacting with α; let $\alpha^\frown\langle d\rangle \subseteq \delta_{s+1}$; if this is the first time we have taken this case since the last initialization of α, then end the stage.

3.5. **Verification.** The verification breaks down into the following lemmas.

Lemma 3.3. *All the sets built in the construction are* 2*-c.e.*

Proof. This is clear form the construction, since for every number, we make at most two moves: We may enumerate it a first time, if needed; and after enumeration we may later extract it, if needed. ☐

Lemma 3.4. *For every n the following hold:*

(1) *$\alpha_n = \liminf_s \delta_s \upharpoonright n$ exists;*

(2) *α_n is eventually never initialized;*

(3) *after the last initialization of α_n there are infinitely many α_n-true stages s and at each such stage $S^{[j]}_{\alpha_n,s}$ contains a new element for every j;*

(4) *the witnesses g_{α_n}, the traces $g_{\alpha_n,\beta}$, $p_{\alpha_n,\beta}$ (used in the axiomatization procedures), and the parameters $x_{\alpha_n}(t)$, $b_{\alpha_n}(t)$, $g_{\alpha_n,\beta}(t)$ (used by minimality strategies) reach a limit.*

Proof. The proof is by induction on n. For $n = 0$, $\alpha_n = \lambda$ and the claims are immediate.

Suppose now that $\alpha_n = \liminf_s \delta_s \upharpoonright n$ exists, and claims (1)–(4) are true of n. For simplicity, let $\alpha = \alpha_n$. By (1), let u be the last stage at which α is initialized. It follows by inductive assumption that after this stage if α needs to appoint a witness g_α, traces $g_{\alpha,\beta}$, $p_{\alpha,\beta}$, $x_\alpha(t)$, $b_\alpha(t)$, then it is allowed to do so, and the values of these parameters will never change again: In particular, we use the inductive assumption (3), to conclude that α is allowed, when needed, to appoint a new witness or a new trace $g \in S^{[j]}_\alpha$, for enumeration of $g \in G_j$. Moreover, at all α-true stages $s > u$, the action taken by α, lets some $\alpha^\frown\langle o\rangle \subseteq \delta_s$. If we can show that α_{n+1} exists, then it follows that after t, α_{n+1} is initialized only if it ends the stage.

It is now trivial to show that if α is a $\mathcal{D}$-strategy, or a comparability strategy, or an incomparability strategy, then α_{n+1} exists, since at all α-true stages $s > u$ we define $\alpha^\frown\langle o\rangle \subseteq \delta_s$, and o lies in a finite set. Moreover, if α is a $\mathcal{D}$-strategy, or a comparability strategy, then α_{n+1} never ends the stage after u; if α is an incomparability strategy, then α_{n+1} ends the stage only at the first α_{n+1}-true stage after u, namely through Case 1, if $\alpha_{n+1} = \alpha^\frown\langle w\rangle$, or through Case 3, if $\alpha_{n+1} = \alpha_n^\frown\langle d\rangle$. Finally, by our conventions on the stream, $S^{[j]}_{\alpha_{n+1}} = S^{[j]}_\alpha$: Therefore (3) is true of $n+1$ as well.

Finally, let us consider the case $R_\alpha = \mathcal{M}_{\Phi,\Psi}$. Clearly there exists a greatest $m \in \omega \cup \{\omega\}$ such that for every $t < m$, $x_\alpha(t)$ is eventually appointed. If $m \in \omega$ then we eventually have outcome w or d, which are both finitary, and the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ carries through. Thus assume that $m = \omega$ and i is the least such that there are infinitely many $\alpha^\frown\langle i\rangle$-stages: Such an outcome i exists since there are only finitely many $\mathcal{D}$-strategy $\beta \subset \alpha$. Notice that whenever we visit $\alpha^\frown\langle i\rangle$ we add a new element g to the set $S^{[i]}_{\alpha^\frown\langle i\rangle}$. Thus the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ holds. Finally notice that after t, we end the stage at most once at α_{n+1}-true stages, namely at the first α_{n+1}-stage if $\alpha_{n+1} = \alpha^\frown\langle o\rangle$ if $o \in \{d, w\}$, after which α_{n+1} is never initialized.

□

Let $f = \bigcup_n \alpha_n$ be the *true path*, defined by

$$\alpha_n = \liminf_s \delta_s \restriction n.$$

Lemma 3.5. *For every n, R_{α_n} is satisfied.*

Proof. Let n be given. Let $\alpha = \alpha_n$, and suppose by the previous lemma that t_n is the least stage such that at no $t \geq t_n$ is α initialized, and we do not stop the stage after α acts.

$R_\alpha = \mathcal{D}_i$. Let x be given. We must check that, for every x, $A(x) = \Delta_\alpha(G_i \oplus B)(x)$. Since a number x may enter A (and consequently $\Delta_\alpha(G_i \oplus B)$) only due to the action of some $\mathcal{M}$-strategy β, clearly we need only to check this for those numbers x such that there are β and t with $x = x_\beta(t)$, as appointed at some stage. Only strategy β is responsible for keeping x in or out of A. Without loss of generality, we may assume that x has been appointed at a stage $u \geq t_n$. There are two possible cases:

Assume first that $\beta \not\supseteq \alpha$: At the first α-true stage $s > u$ if $x \in A$ then we dump x into A and into Δ_α, which makes $x \in \Delta_\alpha(G_i \oplus B)$. Otherwise, if $x \notin A$ at s, then at no α-stage after last initialization of α do we have $x \in A$, hence we do not define any Δ_α-axiom for x.

The other case to consider is $\beta \supseteq \alpha$: If β appoints x and β is initialized before ever extracting x, then $x \in A$, but on the other hand at the first α-stage after initialization of β, we dump x into $\Delta_\alpha(G_i \oplus B)$. Otherwise, at stage u, when β appoints x, β enumerates also an axiom $\langle x, \{g\} \oplus \emptyset\rangle\rangle \in \Delta_\alpha$ (where $g = g_{\beta,\alpha}(t)$) and defines $g \in G_i$, which makes $x \in \Delta_\alpha(G_i \oplus B)$ as long as β takes outcome w, waiting for $x \in \Phi_\beta(\Psi_\beta(G) \oplus B)$. Then either β jumps immediately from outcome w to outcome d because x is the least eligible witness in $\Phi_\beta(\Psi_\beta(G) \oplus B)$, and extracts x from A and g from G_i, implying $A(x) = \Delta_\alpha(G_i \oplus B)(x)$; or β takes some outcome j, keeps $x \in A$, adds an axiom $\langle x, \emptyset \oplus \{b\}\rangle \in \Delta_\alpha$ ($b = b_\beta(t)$), letting $b \in B$, which keeps $x \in \Delta_\alpha(G_i \oplus B)$, even if some lower priority strategy extracts g from G_i; until β takes, if this is ever the case, outcome d, due again to x becoming the least eligible witness, and extracts x from A, g from G_i, and b from B, making $x \notin \Delta_\alpha(G_i \oplus B)$. In all cases $A(x) = \Delta_\alpha(G_i \oplus B)(x)$.

$R_\alpha = \mathcal{M}_{\Phi,\Psi}$. If $\alpha_{n+1} = \alpha^\frown\langle w\rangle$ then there exists t such that $x = x_\alpha(t)$ is defined, no $x_\alpha(m)$ is ever defined for $m > t$, and $x \in A \setminus \Phi(\Psi(G) \oplus B)$. If $\alpha_{n+1} = \alpha^\frown\langle d\rangle$ then there is some $x = x_\alpha(t)$ (among finitely many witnesses $x_\alpha(0), \ldots, x_\alpha(m)$) which α has defined after last initialization) such that $x \in \Phi(\Psi(G) \oplus B) \setminus A$.

It remains to consider the case when $\alpha_{n+1} = \alpha^\frown\langle i\rangle$ for some $i \in \omega$. We claim in this case that $G_i = \Gamma_{\alpha^\frown\langle i\rangle}(\Psi(G))$, where $\Gamma_{\alpha^\frown\langle i\rangle}$ is the s-operator, as enumerated by α after the last initialization of α.

If g is eventually used by a strategy $\beta \leq \alpha$, then either $g \notin G_i$, and in this case there is no $\Gamma_{\alpha^\frown\langle i\rangle}$-axiom for g, or $g \in G_i$, in which case by construction we add an axiom

$\langle g, \emptyset \rangle \in \Gamma_{\alpha^\frown\langle i \rangle}$. Next, for every g which is ever used by any strategy $\beta >_L \alpha^\frown\langle i \rangle$, we have (at the moment when we discard g by initialization) either $g \notin G_i$, in which case we have $G_i(g) = \Gamma_{\alpha^\frown\langle i \rangle}(\Psi(G))(g)$ since we never define any axiom in $\Gamma_{\alpha^\frown\langle i \rangle}$ for g, or we have $g \in G_i$, in which case we add an axiom $\langle g, \emptyset \rangle \in \Gamma_{\alpha^\frown\langle i \rangle}$.

So we need only to show that for every g such that g is enumerated into $S^{[i]}_{\alpha^\frown\langle i \rangle}$ at some $\alpha^\frown\langle i \rangle$-stage,

$$g \in G_i \Leftrightarrow g \in \Gamma_{\alpha^\frown\langle i \rangle}(\Psi(G)).$$

The reason we have enumerated g into $S^{[i]}_{\alpha^\frown\langle i \rangle}$ at some stage $t' \geq t_n$ is that we have found an axiom $\langle y, \{g\} \rangle \in \Psi$, with $g \in G_i$, in correspondence with some witness x, for which there is an axiom $\langle x, \{g\} \oplus \emptyset \rangle \in \Delta_\beta$ (where $\beta \subset \alpha$ is such that $R_\beta = \mathcal{D}_i$). Moreover there is no other axiom $\langle y, \{g'\} \rangle \in \Psi$ with $g' \in G$: Indeed, such an axiom can not appear after t' since in this case we would be able to diagonalize (since x is now eligible) and give outcome d. If it is present at stage t', then since we give outcome i at t' there must be an axiom $\langle x, \{g'\} \oplus \emptyset \rangle \in \Delta_j$ with $j > i$ (here $\mathcal{D}_j = \mathcal{D}_\gamma$, for some $\gamma \subset \alpha$), but in this case we extract g' from G by construction.

We are therefore able to conclude:

$$g \in G_i \Leftrightarrow y \in \Psi(G) \Leftrightarrow g \in \Gamma_{\alpha^\frown\langle i \rangle}(\Psi(G)),$$

as desired.

$R_\alpha \in \{\mathcal{C}^{od}_i, \mathcal{C}^{ev}_i, \mathcal{C}_{k,n,2}, \mathcal{C}_{k,n,3}\}$. We must check that $A_\alpha = \Theta_\alpha(C_\alpha \oplus P_\alpha)$. Fix g: As for the case of $\mathcal{D}$-strategies, we must consider the incomparability strategy β which is responsible for dealing at some stage u with g, and enumerates g into A_α. As for the case of a $\mathcal{D}$-strategy, if $\beta \not\supseteq \alpha$ then at the first α-true stage $s > u$ if $g \in A_\alpha$ then we dump g into A_α and Θ_α; otherwise, if $g \notin A_\alpha$ then at no α-stage s after last initialization of α do we have $g \in A$, hence we do not define any Δ_α-axiom for g. It is left to consider the case when $\alpha \subseteq \beta$. If there is an α-true stage $s \geq t_n$ at which $g \in A_\alpha$, and β has been initialized, then again by dumping, as before, we achieve $A_\alpha(g) = \Theta_\alpha(C_\alpha \oplus P_\alpha)(g) = 1$. Otherwise, notice that g can be enumerated by β into A_α, either because β and α interact, or because β and α indirectly interact, and the axiomatization procedure performed by β towards α makes $A_\alpha(g) = \Theta_\alpha(C_\alpha \oplus P_\alpha)(g) = 1$ as long as we do not need to extract g from A_α; but, if and when we need to extract the witness $g \notin A_\alpha$, then the rectification procedure guarantees that $A_\alpha(g) = \Theta_\alpha(C_\alpha \oplus P_\alpha)(g) = 0$.

$R_\alpha = \mathcal{I}_{i,\Psi}$. We must check that $G_i \neq \Psi(\bigoplus_{j\neq i} G_j)$. Consider a stage after which α does not change g_α anymore. By Lemma 3.4 such a stage exists. If at no future α-stage do we have $g_\alpha \in \Psi(\bigoplus_{j\neq i} G_j)$ then $\alpha_{n+1} = \alpha^\frown\langle w \rangle$ and the requirement is satisfied. Otherwise at some future α-stage we have that $g_\alpha \in \Psi(\bigoplus_{j\neq i} G_j)$. As explained in the construction, at the first such stage, we restrain (by initialization) $g_\alpha \in \Psi(\bigoplus_{j\neq i} G_j)$, and we extract g_α from G_i, thus letting $g_\alpha \in \Psi(\bigoplus_{j\neq i} G_j) \setminus G_i$.

$R_\alpha \in \{\mathcal{I}^{od}_{i,0,\Psi}, \mathcal{I}^{od}_{i,1,\Psi}, \mathcal{I}^{ev}_{i,0,\Psi}, \mathcal{I}^{ev}_{i,1,\Psi}, \mathcal{I}_{k,2,\Psi}, \mathcal{I}_{k,3,\Psi}\}$. We must check that the requirement $A_\alpha \neq \Psi_\alpha(C_\alpha \oplus P_\alpha)$, is satisfied. Consider a stage after which α does not change g_α anymore. By Lemma 3.4 such a stage exists. If at no future α-stage do we have $g_\alpha \in \Psi_\alpha(C_\alpha \oplus P_\alpha)$ then $\alpha_{n+1} = \alpha^\frown\langle w \rangle$ and the requirement is satisfied. Otherwise at some future α-stage we have that $g_\alpha \in \Psi_\alpha(C_\alpha \oplus P_\alpha)$. As explained in the construction, at the first such stage, we restrain $g_\alpha \in \Psi_\alpha(C_\alpha \oplus P_\alpha)$, and we extract g_α from A_α, thus letting $g_\alpha \in \Psi_\alpha(C_\alpha \oplus P_\alpha) \setminus A_\alpha$. □

This concludes the proof of Theorem 3.1. □

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