

ω -Degree Spectra

Alexandra A. Soskova ^{*}

Faculty of Mathematics and Computer Science,
Sofia University,
asoskova@fmi.uni-sofia.bg

Abstract. We present a notion of a degree spectrum of a structure with respect to countably many sets, based on the notion of ω -enumeration reducibility. We prove that some properties of the degree spectrum such as the minimal pair theorem and the existence of quasi-minimal degree are true for the ω -degree spectrum.

1 Introduction

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_s)$ be a structure, where \mathbb{N} is the set of all natural numbers, each R_i is a subset of \mathbb{N}^{r_i} and the equality $=$ and the inequality \neq are among R_1, \dots, R_s . An enumeration f of \mathfrak{A} is a total mapping from \mathbb{N} onto \mathbb{N} .

Given an enumeration f of \mathfrak{A} and a subset A of \mathbb{N}^a let $f^{-1}(A) = \{\langle x_1, \dots, x_a \rangle \mid (f(x_1), \dots, f(x_a)) \in A\}$. Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_s)$.

Given a set X of natural numbers denote by $d_e(X)$ the enumeration degree of X and by $d_T(X)$ the Turing degree of X .

The notion of *Turing degree spectrum* of \mathfrak{A} is introduced by Richter [6]: $DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is an injective enumeration of } \mathfrak{A}\}$. Soskov [8] initiated the study of the properties of the degree spectra as sets of enumeration degrees. *The enumeration degree spectrum of \mathfrak{A}* (called shortly degree spectrum of \mathfrak{A}) is the set: $DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}$.

The benefit of considering all enumerations of the structure instead of only the injective ones is that every degree spectrum is upwards closed with respect to total enumeration degrees [8]. Soskov considered the notion of *co-spectrum* $CS(\mathfrak{A})$ of \mathfrak{A} as the set of all lower bounds of the elements of the degree spectrum and proved several properties which show that the degree spectra behave with respect to their co-spectra very much like the cones of the enumeration degrees $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{a}\}$ behave with respect to the ideals $\{\mathbf{x} \mid \mathbf{x} \leq \mathbf{a}\}$. Further properties true of the degree spectra but not necessarily true of all upwards closed sets are: the minimal pair theorem for the degree spectrum and the existence of quasi-minimal degree for the degree spectrum.

In this paper we shall relativize Soskov's approach to degree spectra by considering multi-component spectra, i.e. a degree spectrum with respect to a given sequence of sets of natural numbers, considering the ω -enumeration reducibility introduced and studied in [10–12]. It is a uniform reducibility between sequences

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of sets of natural numbers. We shall prove that the so defined ω -degree spectrum preserves almost all properties of the degree spectrum and generalizes the notion of relative spectrum, i.e multi-component spectrum of a structure with respect to finitely many structures, introduced in [13].

2 ω -Enumeration Degrees

We assume the reader is familiar with enumeration reducibility and refer to [3] for further background. Denote by \mathcal{D}_e the set of all enumeration degrees. Recall that an enumeration degree \mathbf{a} is total if \mathbf{a} contains a total set, i.e. a set A such that $A \equiv_e A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. If X is a total set then $A \leq_e X \iff A$ is c.e. in X . Cooper [2] introduced the jump operation “ $'$ ” for enumeration degrees. By A' we denote the enumeration jump of the set A .

Denote by \mathcal{S} the set of all sequences of sets of natural numbers. For each element $\mathcal{B} = \{B_n\}_{n < \omega}$ of \mathcal{S} call *the jump class of \mathcal{B}* the set

$$J_{\mathcal{B}} = \{d_T(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} .$$

For every two sequences \mathcal{A} and \mathcal{B} let $\mathcal{A} \leq_{\omega} \mathcal{B}$ (\mathcal{A} is ω -enumeration reducible to \mathcal{B}) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$ and let $\mathcal{A} \equiv_{\omega} \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$. The relation \equiv_{ω} is an equivalence relation on \mathcal{S} . Let the ω -enumeration degree of \mathcal{B} be $d_{\omega}(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_{\omega} \mathcal{B}\}$ and $\mathcal{D}_{\omega} = \{d_{\omega}(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$. If $\mathbf{a} = d_{\omega}(\mathcal{A})$ and $\mathbf{b} = d_{\omega}(\mathcal{B})$ then $\mathbf{a} \leq_{\omega} \mathbf{b}$ if $\mathcal{A} \leq_{\omega} \mathcal{B}$. Denote by $\mathbf{0}_{\omega} = d_{\omega}(\emptyset_{\omega})$, where \emptyset_{ω} is the sequence with all members equal to \emptyset . There is a natural embedding of the enumeration degrees into the ω -enumeration degrees. Given a set A denote by $A \uparrow \omega$ the sequence $\{A_n\}_{n < \omega}$, where $A_0 = A$ and for all $n > 0$ $A_n = \emptyset$. For every $A, B \subseteq \mathbb{N}$ we have that $A \leq_e B \iff A \uparrow \omega \leq_{\omega} B \uparrow \omega$. So the mapping $\kappa(d_e(A)) = d_{\omega}(A \uparrow \omega)$ gives an isomorphic embedding of \mathcal{D}_e to \mathcal{D}_{ω} . We shall identify the enumeration degree $d_e(A)$ with its representation $d_{\omega}(A \uparrow \omega)$ in \mathcal{D}_{ω} . So when $\mathbf{a} = d_e(A)$ and $\mathbf{b} \in \mathcal{D}_{\omega}$ then by writing $\mathbf{a} \leq_{\omega} \mathbf{b}$ ($\mathbf{b} \leq_{\omega} \mathbf{a}$) we mean $d_{\omega}(A \uparrow \omega) \leq_{\omega} \mathbf{b}$ ($\mathbf{b} \leq_{\omega} d_{\omega}(A \uparrow \omega)$).

Given a sequence of sets of natural numbers $\mathcal{B} = \{B_n\}_{n < \omega}$ we define the respective *jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n < \omega}$ by induction on n :

- (1) $\mathcal{P}_0(\mathcal{B}) = B_0$;
- (2) $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$.

Note that if $X \subseteq \mathbb{N}$, then $\mathcal{P}_n(X \uparrow \omega) \equiv_e X^{(n)}$ uniformly in n .

The following theorem of Soskov and Kovachev [10] gives an explicit characterization of the uniform reducibility.

Theorem 1. *Let $\mathcal{A} = \{A_n\}_{n < \omega}$ and $\mathcal{B} = \{B_n\}_{n < \omega}$ be elements of \mathcal{S} . The following conditions are equivalent:*

- (1) $\mathcal{A} \leq_{\omega} \mathcal{B}$, i.e. for every total set X , if $B_n \leq_e X^{(n)}$ uniformly in n then $A_n \leq_e X^{(n)}$ uniformly in n .
- (2) $A_n \leq_e \mathcal{P}_n(\mathcal{B})$ uniformly in n , i.e. there is a computable function g such that $A_n = \Gamma_{g(n)}(\mathcal{P}_n(\mathcal{B}))$ for every n .

It follows that if $X \subseteq \mathbb{N}$ then for every sequence $\mathcal{A} = \{A_n\}_{n < \omega}$ we have: $A_n \leq_e X^{(n)}$ uniformly in n if and only if $\mathcal{A} \leq_\omega \{X^{(n)}\}_{n < \omega}$ if and only if $\mathcal{A} \leq_\omega X \uparrow \omega$. It is clear also that $\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A})$.

With a slight modification of the proof of Theorem 1 we have the following:

Corollary 2. *Let $\mathcal{A}_0, \dots, \mathcal{A}_r, \dots$ be sequences of sets such that for every r , $\mathcal{A}_r \not\leq_\omega \mathcal{B}$. There is a total set X such that $\mathcal{B} \leq_\omega \{X^{(n)}\}_{n < \omega}$ and $\mathcal{A}_r \not\leq_\omega \{X^{(n)}\}_{n < \omega}$ for each r .*

The jump operator on the ω -enumeration degrees is defined by Soskov [11]. For every $\mathcal{A} \in \mathcal{S}$ the ω -enumeration jump of \mathcal{A} is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n < \omega}$ and $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$. Furthermore $\mathcal{A}^{(k+1)} = (\mathcal{A}^{(k)})'$ and $d_\omega(\mathcal{A})^{(k+1)} = d_\omega(\mathcal{A}^{(k+1)})$. Then $\mathcal{A}^{(k)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{n < \omega}$ for each k .

3 The ω -Degree Spectra of Structures

In this section we shall generalize the notion of degree spectrum of \mathfrak{A} by considering a multi-component spectrum. The first step in this direction was the notion of relative spectrum of the structure \mathfrak{A} with respect to finitely many given structures $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ studied in [13]. The relative spectrum $\text{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ of the structure \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set $\{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A} \text{ s. t. } (\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_e f^{-1}(\mathfrak{A})^{(k)})\}$. It turns out that all properties of the degree spectra obtained by Soskov [8] remain true for the relative spectra.

We shall define here the notion of a spectrum of the structure \mathfrak{A} with respect to a given infinite sequence of sets using the ω -enumeration reducibility.

Let $\mathcal{B} = \{B_n\}_{n < \omega}$ be a sequence of sets of natural numbers. An enumeration f of \mathfrak{A} is called *acceptable with respect to the sequence \mathcal{B}* if for every n , $f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)}$ uniformly in n . Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ the class of all acceptable enumerations of \mathfrak{A} with respect to the sequence \mathcal{B} .

Definition 3. *The ω -degree spectrum of the structure \mathfrak{A} with respect to the sequence \mathcal{B} is the set $\text{DS}(\mathfrak{A}, \mathcal{B}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$.*

The notion of the ω -degree spectrum is a generalization of the relative spectrum since $\text{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \text{DS}(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k < \omega}$, $B_0 = \emptyset$, B_k is the positive diagram of the structure \mathfrak{A}_k for $0 < k \leq n$ and $B_k = \emptyset$ for all $k > n$.

Given an enumeration f of \mathfrak{A} denote by $\mathcal{P}^f = \{\mathcal{P}_n^f\}_{n < \omega}$ the respective jump sequence of the sequence $\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \dots, f^{-1}(B_n), \dots\}$ where $\mathcal{P}_n^f = \mathcal{P}_n(\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \dots, f^{-1}(B_n), \dots\})$. Note that if f is an acceptable enumeration of \mathfrak{A} with respect to \mathcal{B} then $\mathcal{P}^f \equiv_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}_{n < \omega} \equiv_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$. So $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{P}^f \leq_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$.

First we shall see that the ω -degree spectrum of the structure \mathfrak{A} with respect to \mathcal{B} is upwards closed with respect to total enumeration degrees.

Lemma 4. *Let f be an enumeration of \mathfrak{A} and F be a total set such that $f^{-1}(\mathfrak{A}) \leq_e F$ and $f^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n . Then there exists an acceptable enumeration g of \mathfrak{A} with respect to \mathcal{B} such that $g^{-1}(\mathfrak{A}) \equiv_e F$.*

Proof. The construction of g is the following. Let $s \neq t \in \mathbb{N}$.

$$g(x) \simeq \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z + 1 \text{ and } z \in F, \\ t & \text{if } x = 2z + 1 \text{ and } z \notin F. \end{cases}$$

It is easy to see that $F \oplus f^{-1}(\mathfrak{A}) \equiv_e g^{-1}(\mathfrak{A})$ and hence $F \equiv_e g^{-1}(\mathfrak{A})$.

Moreover for every set $B \subseteq \mathbb{N}$ we have that $g^{-1}(B) \leq_e F \oplus f^{-1}(B)$.

Then $g^{-1}(B_n) \leq_e F \oplus f^{-1}(B_n) \leq_e F \oplus F^{(n)} \equiv_e F^{(n)} \equiv_e g^{-1}(\mathfrak{A})^{(n)}$ uniformly in n . And thus g is an acceptable enumeration of \mathfrak{A} with respect to \mathcal{B} . \square

Using Theorem 1 and the previous lemma one can find an acceptable enumeration $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $f^{-1}(\mathfrak{A})$ is a total set.

Another corollary of Lemma 4 is the following:

Proposition 5. *The ω -degree spectrum is upwards closed with respect to total e -degrees, i.e. if \mathbf{b} is a total e -degree and for some $\mathbf{a} \in \text{DS}(\mathfrak{A}, \mathcal{B})$, $\mathbf{b} \geq_e \mathbf{a}$ then $\mathbf{b} \in \text{DS}(\mathfrak{A}, \mathcal{B})$.*

It is obvious that $\text{DS}(\mathfrak{A}) \supseteq \text{DS}(\mathfrak{A}, \mathcal{B})$ for every structure \mathfrak{A} and every $\mathcal{B} \in \mathcal{S}$. It is easy to find a structure \mathfrak{A} and a sequence of sets \mathcal{B} so that $\text{DS}(\mathfrak{A}) \neq \text{DS}(\mathfrak{A}, \mathcal{B})$. For example consider the structure $\mathfrak{A} = \{\mathbb{N}, S, =, \neq\}$, where $S \subseteq \mathbb{N}^2$ is defined as $S = \{(n, n+1) \mid n \in \mathbb{N}\}$. It is clear that the structure \mathfrak{A} admits an effective enumeration f , i.e. $f^{-1}(\mathfrak{A})$ is c.e. Thus $\mathbf{0}_e \in \text{DS}(\mathfrak{A})$. By Proposition 5 all total enumeration degrees are elements of $\text{DS}(\mathfrak{A})$. Consider now an arbitrary acceptable enumeration f of \mathfrak{A} with respect to $\mathcal{B} = \{B_n\}_{n < \omega}$. Fix a number x_0 such that $f(x_0) = 0$. Then $k \in B_n \iff (\exists x_1) \dots (\exists x_k)(f^{-1}(S)(x_0, x_1) \& \dots \& f^{-1}(S)(x_{k-1}, x_k) \& x_k \in f^{-1}(B_n))$. Then $B_n \leq_e f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)}$. Let $B_0 = \emptyset'$ and let $B_n = \emptyset$ for each $n \geq 1$. Then $\emptyset' \leq_e B_0 \leq_e f^{-1}(\mathfrak{A})$. Thus $\mathbf{0}_e \notin \text{DS}(\mathfrak{A}, \mathcal{B})$.

Let $k \in \mathbb{N}$. The k th ω -jump spectrum of \mathfrak{A} with respect to \mathcal{B} is the set $\text{DS}_k(\mathfrak{A}, \mathcal{B}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \text{DS}(\mathfrak{A}, \mathcal{B})\}$.

Proposition 6. *The k th ω -jump spectrum of \mathfrak{A} with respect to \mathcal{B} is upwards closed with respect to total e -degrees, i.e. if \mathbf{b} is a total e -degree, $\mathbf{b} \geq_e \mathbf{a}^{(k)}$ for some $\mathbf{a} \in \text{DS}(\mathfrak{A}, \mathcal{B})$ then $\mathbf{b} \in \text{DS}_k(\mathfrak{A}, \mathcal{B})$.*

Proof. Let G be a total set, $d_e(G) = \mathbf{b}$ and let $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$. Then $f^{-1}(\mathfrak{A})^{(k)} \leq_e G$ and $\mathcal{P}_k^f \leq_e G$ since $\mathcal{P}_k^f \leq_e f^{-1}(\mathfrak{A})^{(k)}$. By the jump inversion theorem from [7] there exists a total set F such that $G \equiv_e F^{(k)}$, $f^{-1}(\mathfrak{A}) \leq_e F$ and $f^{-1}(B_i) \leq_e F^{(i)}$ for $i \leq k$. Moreover $f^{-1}(B_{n+k}) \leq_e f^{-1}(\mathfrak{A})^{(n+k)} \leq_e G^{(n)} \equiv_e F^{(n+k)}$ uniformly in n . By Lemma 4 there is an acceptable enumeration g of \mathfrak{A} with respect to \mathcal{B} so that $g^{-1}(\mathfrak{A}) \equiv_e F$. Thus $d_e(g^{-1}(\mathfrak{A})) \in \text{DS}(\mathfrak{A}, \mathcal{B})$ and $g^{-1}(\mathfrak{A})^{(k)} \equiv_e G$. Hence $d_e(G) \in \text{DS}_k(\mathfrak{A}, \mathcal{B})$. \square

For every $\mathcal{D} \subseteq \mathcal{D}_e$ denote by $\text{co}(\mathcal{D}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \& (\forall \mathbf{a} \in \mathcal{D})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition 7. *The ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is the set $\text{CS}(\mathfrak{A}, \mathcal{B}) = \text{co}(\text{DS}(\mathfrak{A}, \mathcal{B}))$.*

For each $\mathcal{A} \in \mathcal{S}$ it holds that $d_\omega(\mathcal{A}) \in \text{CS}(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{A} \leq_\omega \mathcal{P}^f$ for every acceptable enumeration f of \mathfrak{A} with respect to \mathcal{B} . Actually the ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is a countable ideal of ω -enumeration degrees.

The k th ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is the set $\text{CS}_k(\mathfrak{A}, \mathcal{B}) = \text{co}(\text{DS}_k(\mathfrak{A}, \mathcal{B}))$. It is clear that $d_\omega(\mathcal{A}) \in \text{CS}_k(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{A} \leq_\omega \{\mathcal{P}_{n+k}^f\}_{n < \omega}$ for every acceptable enumeration f of \mathfrak{A} with respect to \mathcal{B} . As we shall see in section 4. Corollary 22, the k th ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is the least ideal containing all k th ω -enumeration jumps of the elements of $\text{CS}(\mathfrak{A}, \mathcal{B})$.

In order to obtain a forcing normal form of the sequences with ω -enumeration degrees in $\text{CS}(\mathfrak{A}, \mathcal{B})$ we shall define the notions of a forcing relation $\tau \Vdash_n F_e(x)$ and a relation $f \Vdash_n F_e(x)$.

Let f be an enumeration of \mathfrak{A} . For every n and $e, x \in \mathbb{N}$, define the relations $f \Vdash_n F_e(x)$ and $f \Vdash_n \neg F_e(x)$ by induction on n :

1. $f \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x \rangle \in W_e \ \& \ D_v \subseteq f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0));$
2. $f \Vdash_{n+1} F_e(x) \iff (\exists v)(\langle v, x \rangle \in W_e \ \& \ (\forall u \in D_v)($
 $(u = \langle 0, e_u, x_u \rangle \ \& \ f \Vdash_n F_{e_u}(x_u)) \vee$
 $(u = \langle 1, e_u, x_u \rangle \ \& \ f \Vdash_n \neg F_{e_u}(x_u)) \vee$
 $(u = \langle 2, x_u \rangle \ \& \ x_u \in f^{-1}(B_{n+1})));$
3. $f \Vdash_n \neg F_e(x) \iff f \not\Vdash_n F_e(x) .$

Lemma 8. (a) Let $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$. Then $A \leq_e \mathcal{P}_n^f$ if and only if $A = \{x \mid f \Vdash_n F_e(x)\}$ for some $e \in \mathbb{N}$.

(b) Let $\mathcal{A} = \{A_n\}_{n < \omega}$. Then $\mathcal{A} \leq_\omega \mathcal{P}^f$ if and only if there exists a computable function g such that $A_n = \{x \mid f \Vdash_n F_{g(n)}(x)\}$ for every n .

The forcing conditions, called *finite parts*, are finite mappings τ of \mathbb{N} to \mathbb{N} . We will denote the finite parts by letters δ, τ, ρ .

For each n and $e, x \in \mathbb{N}$ and for every finite part τ , define the forcing relations $\tau \Vdash_n F_e(x)$ and $\tau \Vdash_n \neg F_e(x)$ following the definition of the relation " \Vdash_n ".

1. $\tau \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x \rangle \in W_e \ \& \ D_v \subseteq \tau^{-1}(\mathfrak{A}) \oplus \tau^{-1}(B_0));$
2. $\tau \Vdash_{n+1} F_e(x) \iff (\exists v)(\langle v, x \rangle \in W_e \ \& \ (\forall u \in D_v)($
 $(u = \langle 0, e_u, x_u \rangle \ \& \ \tau \Vdash_n F_{e_u}(x_u)) \vee$
 $(u = \langle 1, e_u, x_u \rangle \ \& \ \tau \Vdash_n \neg F_{e_u}(x_u)) \vee$
 $(u = \langle 2, x_u \rangle \ \& \ x_u \in \tau^{-1}(B_{n+1})));$
3. $\tau \Vdash_n \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \not\Vdash_n F_e(x)) .$

An enumeration f of \mathfrak{A} is k -generic with respect to \mathcal{B} if for every $j < k$ and $e, x \in \mathbb{N}$ it holds that $(\exists \tau \subseteq f)(\tau \Vdash_j F_e(x) \vee \tau \Vdash_j \neg F_e(x))$.

Lemma 9. (1) If $\tau \subseteq \rho$ then $\tau \Vdash_k (\neg)F_e(x) \Rightarrow \rho \Vdash_k (\neg)F_e(x)$.

(2) For every $(k+1)$ -generic enumeration f of \mathfrak{A} with respect to \mathcal{B}
 $f \Vdash_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x))$.

Definition 10. Let $\mathcal{A} = \{A_n\}_{n < \omega}$. The sequence \mathcal{A} is *forcing definable* on \mathfrak{A} with respect to \mathcal{B} if there exist a finite part δ and a computable function g such that for every $n \in \mathbb{N}$ [$x \in A_n \iff (\exists \tau \supseteq \delta)(\tau \Vdash_n F_{g(n)}(x))$].

Proposition 11. *Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be a sequence not forcing definable on \mathfrak{A} with respect to \mathcal{B} . Then there exists an enumeration f of \mathfrak{A} such that $A \not\leq_{\omega} \mathcal{P}^f$.*

Proof. The enumeration f is constructed on stages. On each stage q we find a finite part δ_q so that $\delta_q \subseteq \delta_{q+1}$ and ultimately we define $f = \bigcup_q \delta_q$. We consider three kinds of stages. On stages $q = 3r$ we ensure that the mapping f is total and surjective. On stages $q = 3r + 1$ we ensure that f is k -generic for each $k > 0$ and on stages $q = 3r + 2$ we ensure that f satisfies the omitting condition: $A \not\leq_{\omega} \mathcal{P}^f$.

Let g_0, g_1, \dots be an enumeration of all computable functions. For each n, e , $x \in \mathbb{N}$ denote by $Y_{\langle e, x \rangle}^n$ the set of all finite parts ρ such that $\rho \Vdash_n F_e(x)$.

Let $\delta_0 = \emptyset$. Suppose that we have already defined δ_q .

(a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\text{dom}(\delta_q)$ and let t_0 be the least natural number which does not belong to the range of δ_q . Set $\delta_{q+1}(x_0) \simeq t_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

(b) Case $q = 3\langle e, n, x \rangle + 1$. Check whether there exists a finite part $\rho \in Y_{\langle e, x \rangle}^n$ that extends δ_q . If there is such a finite part then let δ_{q+1} be the least extension of δ_q that belongs to $Y_{\langle e, x \rangle}^n$. Otherwise let $\delta_{q+1} = \delta_q$.

(c) Case $q = 3r + 2$. Consider the function g_r . For each n denote by

$$C_n = \{x \mid (\exists \tau \supseteq \delta_q)(\tau \Vdash_n F_{g_r(n)}(x))\} .$$

Clearly $\mathcal{C} = \{C_n\}_{n < \omega}$ is forcing definable on \mathfrak{A} with respect to \mathcal{B} and hence $\mathcal{C} \neq \mathcal{A}$. Then $C_n \neq A_n$ for some n . Let $\langle x, n \rangle$ be the least pair such that

$$x \in C_n \ \& \ x \notin A_n \ \vee \ x \notin C_n \ \& \ x \in A_n .$$

(i) Suppose that $x \in C_n$. Then there exists a finite part τ such that

$$\delta_q \subseteq \tau \ \& \ \tau \Vdash_n F_{g_r(n)}(x) . \tag{1}$$

Let δ_{q+1} be the least τ satisfying (1).

(ii) If $x \notin C_n$ then set $\delta_{q+1}(x) \simeq \delta_q(x)$. Note that in this case we have that $\delta_{q+1} \Vdash_n \neg F_{g_r(n)}(x)$.

Let $f = \bigcup_q \delta_q$. The enumeration f is total and surjective. Let $k \in \mathbb{N}$. In order to prove that f is $(k + 1)$ -generic, suppose that $j \leq k$ and consider the stage $q = 3\langle e, j, x \rangle + 1$. If there is a finite part $\rho \supseteq \delta_q$ such that $\rho \Vdash_j F_e(x)$ then from the construction we have that $\delta_{q+1} \Vdash_j F_e(x)$. Otherwise $\delta_{q+1} \Vdash_j \neg F_e(x)$.

To prove that f satisfies the omitting condition suppose for a contradiction that $A \leq_{\omega} \mathcal{P}^f$. Then there exists a computable function g_s such that for each n we have that $A_n = \{x \mid f \Vdash_n F_{g_s(n)}(x)\}$. Since the enumeration f is $(n + 1)$ -generic, by Lemma 9 we have for each number x :

$$f \Vdash_n (\neg) F_{g_s(n)}(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_n (\neg) F_{g_s(n)}(x)) . \tag{2}$$

Consider the stage $q = 3s + 2$. From the construction there are numbers n and x such that one of the following two cases holds:

(i) $x \notin A_n$ & $\delta_{q+1} \Vdash_n F_{g_s(n)}(x)$. By (2) $f \Vdash_n F_{g_s(n)}(x)$ and hence $x \in A_n$.

A contradiction.

(ii) $x \in A_n$ and $(\forall \rho \supseteq \delta_q)(\rho \not\Vdash_n F_{g_s(n)}(x))$. Then $\delta_q \Vdash_n \neg F_{g_s(n)}(x)$. So by (2), $f \not\Vdash_n F_{g_s(n)}(x)$ and hence $x \notin A_n$. A contradiction. \square

Corollary 12. *Let $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_i \dots$ be a sequence of elements of \mathcal{S} such that each \mathcal{A}_i is not forcing definable on \mathfrak{A} with respect to \mathcal{B} . Then there exists an enumeration f of \mathfrak{A} such that $\mathcal{A}_i \not\leq_\omega \mathcal{P}^f$ for each i .*

The construction of the enumeration f is very similar to that in Proposition 11. On stages of the form $q = 3\langle r, i \rangle + 2$ we consider the computable function g_r and ensure that $\mathcal{A}_i \neq \mathcal{C}$, where the sequence \mathcal{C} is defined by the same way.

Proposition 13. *Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be a sequence not forcing definable on \mathfrak{A} with respect to \mathcal{B} . Then there exists an enumeration $g \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$ and the enumeration degree of $g^{-1}(\mathfrak{A})$ is total.*

Proof. By Proposition 11 there is an enumeration f of \mathfrak{A} such that $\mathcal{A} \not\leq_\omega \mathcal{P}^f$. Then by Theorem 1 there exists a total set F such that $\mathcal{P}^f \leq_\omega \{F^{(n)}\}_{n < \omega}$ and $\mathcal{A} \not\leq_\omega \{F^{(n)}\}_{n < \omega}$. By Lemma 4 there exists an acceptable enumeration g of \mathcal{A} with respect to \mathcal{B} such that $g^{-1}(\mathfrak{A}) \equiv_e F$ and hence $g^{-1}(\mathfrak{A})^{(n)} \equiv_e F^{(n)}$ uniformly in n . It is clear that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$. \square

Corollary 14. *For every sequence \mathcal{A} if $d_\omega(\mathcal{A}) \in \text{CS}(\mathcal{A}, \mathcal{B})$ then \mathcal{A} is forcing definable on \mathfrak{A} with respect to \mathcal{B} .*

Proof. If a sequence \mathcal{A} is not forcing definable on \mathfrak{A} with respect to \mathcal{B} then by Proposition 13 there exists an acceptable enumeration g of \mathfrak{A} with respect to \mathcal{B} such that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$. Hence $d_\omega(\mathcal{A}) \notin \text{CS}(\mathfrak{A}, \mathcal{B})$. \square

Definition 15. Let $k \in \mathbb{N}$, $\mathcal{A} \in \mathcal{S}$ and let $\mathcal{A} = \{A_n\}_{n < \omega}$. The sequence \mathcal{A} is *forcing k -definable* on \mathfrak{A} with respect to \mathcal{B} if there exist a finite part δ and a computable function g such that for every $n \in \mathbb{N}$ [$x \in A_n \iff (\exists \tau \supseteq \delta)(\tau \Vdash_{n+k} F_{g^{(n)}}(x))$].

Corollary 16. *For every sequence \mathcal{A} if $d_\omega(\mathcal{A}) \in \text{CS}_k(\mathcal{A}, \mathcal{B})$ then \mathcal{A} is forcing k -definable on \mathfrak{A} with respect to \mathcal{B} .*

We shall give an explicit form of all sequences which are forcing k -definable on \mathfrak{A} with respect to \mathcal{B} by means of recursive Σ_k^+ formulae. These formulae can be considered as a modification of Ash's formulae [1] appropriate for their use on abstract structures presented by Soskov and Baleva [9].

Let $\mathcal{L} = \{T_1, \dots, T_s\}$ be the first order language of the structure \mathfrak{A} . For each n let P_n be a new unary predicate representing the set B_n .

(1) An elementary Σ_0^+ formula with free variables among W_1, \dots, W_r is an existential formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atomic formulae in $\mathcal{L} \cup \{P_0\}$;

(2) A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulae;

(3) An elementary Σ_{n+1}^+ formula is a formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ and Σ_n^+ formulae or negations of Σ_n^+ formulae in $\mathcal{L} \cup \{P_0\} \cup \dots \cup \{P_n\}$.

Definition 17. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ and $k \in \mathbb{N}$. The sequence \mathcal{A} is *formally k -definable* on \mathfrak{A} with respect to \mathfrak{B} if there exists a recursive sequence $\{\Phi^{\gamma(n,x)}\}_{n,x < \omega}$ of formulae such that for every n , $\Phi^{\gamma(n,x)}$ is a Σ_{n+k}^+ formula with free variables among W_1, \dots, W_r and elements t_1, \dots, t_r of \mathbb{N} such that for every $x \in \mathbb{N}$, the following equivalence holds:
 $x \in A_n \iff (\mathfrak{A}, \mathfrak{B}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \dots, W_r/t_r)$.

This means that $x \in A_n$ if and only if the formula $\Phi^{\gamma(n,x)}$ is true in \mathfrak{A} with all sets B_n added as new predicates, under the variable assignment v such that $v(W_1) = t_1, \dots, v(W_r) = t_r$. With a uniform variant of the proof given in [9] we obtain the following:

Theorem 18. *If a sequence \mathcal{A} is forcing k -definable on \mathfrak{A} with respect to \mathfrak{B} then \mathcal{A} is formally k -definable on \mathfrak{A} with respect to \mathfrak{B} .*

Corollary 19. *Let $\mathcal{A} \in \mathcal{S}$ and let $k \in \mathbb{N}$. Then the following are equivalent:*

- (1) $d_\omega(\mathcal{A}) \in \text{CS}_k(\mathfrak{A}, \mathfrak{B})$;
- (2) \mathcal{A} is forcing k -definable on \mathfrak{A} with respect to \mathfrak{B} .
- (3) \mathcal{A} is formally k -definable on \mathfrak{A} with respect to \mathfrak{B} .

4 Properties of the ω -Degree Spectra

We prove that some properties of degree spectra from [8] remain true for ω -degree spectra. The first property follows directly from Theorem 1 and Lemma 4.

Proposition 20. $\text{CS}(\mathfrak{A}, \mathfrak{B}) = \text{co}(\{\mathbf{a} \mid \mathbf{a} \in \text{DS}(\mathfrak{A}, \mathfrak{B}) \text{ \& } \mathbf{a} \text{ is total } e\text{-degree}\})$.

The next property is an analogue of the minimal pair theorem for the degree spectrum of a structure \mathfrak{A} by Soskov [8]: There exist \mathbf{f} and \mathbf{g} in $\text{DS}(\mathfrak{A})$ such that $\mathbf{a} \leq_e \mathbf{f}^{(k)} \text{ \& } \mathbf{a} \leq_e \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \text{CS}_k(\mathfrak{A})$ for every $\mathbf{a} \in \mathcal{D}_e$ and each $k \in \mathbb{N}$.

Theorem 21. *For every structure \mathfrak{A} and every sequence $\mathfrak{B} \in \mathcal{S}$ there exist total enumeration degrees \mathbf{f} and \mathbf{g} in $\text{DS}(\mathfrak{A}, \mathfrak{B})$ such that for every ω -enumeration degree \mathbf{a} and $k \in \mathbb{N}$:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \text{ \& } \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \text{CS}_k(\mathfrak{A}, \mathfrak{B}) . \quad (3)$$

Proof. First we shall construct total enumeration degrees \mathbf{f} and \mathbf{g} in $\text{DS}(\mathfrak{A}, \mathfrak{B})$ satisfying (3) for $k = 0$. Then we will show that \mathbf{f} and \mathbf{g} satisfy (3) for every k .

Let f be an acceptable enumeration of \mathfrak{A} with respect to \mathfrak{B} , such that $f^{-1}(\mathfrak{A})$ is a total set. Then $\mathcal{P}^f \equiv_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}_{n < \omega}$. Denote by $F = f^{-1}(\mathfrak{A})$. It is clear that $d_e(F) \in \text{DS}(\mathfrak{A}, \mathfrak{B})$.

Denote by $\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_r \dots$ all sequences ω -enumeration reducible to \mathcal{P}^f .

Consider the sequence $\mathfrak{C}_0, \mathfrak{C}_1, \dots, \mathfrak{C}_r \dots$ of these elements of $\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_r \dots$ which are not forcing definable on \mathfrak{A} with respect to \mathfrak{B} . By Proposition 12 there is an enumeration h such that $\mathfrak{C}_r \not\leq_\omega \mathcal{P}^h$ for all r . Then by Corollary 2 there is a total set G such that $\mathcal{P}^h \leq_\omega \{G^{(n)}\}_{n < \omega}$ and $\mathfrak{C}_r \not\leq_\omega \{G^{(n)}\}_{n < \omega}$ for all r . By

Lemma 4 there is an acceptable enumeration g of \mathfrak{A} with respect to \mathfrak{B} such that $g^{-1}(\mathfrak{A}) \equiv_e G$. Thus $d_e(G) \in \text{DS}(\mathfrak{A}, \mathfrak{B})$.

Suppose now that \mathcal{A} is a sequence such that $\mathcal{A} \leq_\omega \{F^{(n)}\}_{n < \omega}$ and $\mathcal{A} \leq_\omega \{G^{(n)}\}_{n < \omega}$. Then $\mathcal{A} = \mathfrak{X}_r$ for some r . If we assume that \mathcal{A} is not forcing definable on \mathfrak{A} with respect to \mathfrak{B} then $\mathcal{A} = \mathfrak{C}_l$ for some l and hence $\mathcal{A} \not\leq_\omega \{G^{(n)}\}_{n < \omega}$, which is a contradiction. Thus \mathcal{A} is forcing definable on \mathfrak{A} with respect to \mathfrak{B} and $d_\omega(\mathcal{A}) \in \text{CS}(\mathfrak{A}, \mathfrak{B})$ by Proposition 19. Then by setting $\mathbf{f} = d_e(F)$ and $\mathbf{g} = d_e(G)$ we obtain the desired minimal pair.

For each $\mathbf{a} \in \mathcal{D}_e$ denote by $I(\mathbf{a}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ \mathbf{b} \leq_\omega \mathbf{a}\} = \text{co}(\{\mathbf{a}\})$ the principal ideal generated by \mathbf{a} . We have that $\text{CS}(\mathfrak{A}, \mathfrak{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$, since $\mathbf{f}, \mathbf{g} \in \text{DS}(\mathfrak{A}, \mathfrak{B})$. We shall prove now that $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \text{CS}_k(\mathfrak{A}, \mathfrak{B})$ for every k . Since $\mathbf{f}^{(k)}, \mathbf{g}^{(k)} \in \text{DS}_k(\mathfrak{A}, \mathfrak{B})$ it follows that $\text{CS}_k(\mathfrak{A}, \mathfrak{B}) \subseteq I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$. Suppose that $\mathcal{A} = \{A_n\}_{n < \omega}$, $\mathcal{A} \leq_\omega F^{(k)} \uparrow \omega$ and $\mathcal{A} \leq_\omega G^{(k)} \uparrow \omega$. Denote by $\mathcal{C} = \{C_n\}_{n < \omega}$ the sequence such that $C_n = \emptyset$ for $n < k$, and $C_{n+k} = A_n$ for each n . Clearly $\mathcal{A} \leq_\omega \mathcal{C}^{(k)}$ and $\mathcal{C} \leq_\omega \{F^{(n)}\}_{n < \omega}$, $\mathcal{C} \leq_\omega \{G^{(n)}\}_{n < \omega}$. So $d_\omega(\mathcal{C}) \in \text{CS}(\mathfrak{A}, \mathfrak{B})$. Consider an arbitrary acceptable enumeration h of \mathfrak{A} with respect to \mathfrak{B} . Then $\mathcal{C} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}_{n < \omega}$ and thus $\mathcal{C}^{(k)} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}_{n < \omega}^{(k)}$. It follows that $\mathcal{A} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}_{n < \omega}^{(k)}$ for every $h \in \mathcal{E}(\mathfrak{A}, \mathfrak{B})$. Hence $d_\omega(\mathcal{A}) \in \text{CS}_k(\mathfrak{A}, \mathfrak{B})$. \square

Corollary 22. $\text{CS}_k(\mathfrak{A}, \mathfrak{B})$ is the least ideal containing all k th ω -jumps of the elements of $\text{CS}(\mathfrak{A}, \mathfrak{B})$.

Proof. Ganchev [5] proved that if the enumeration degrees \mathbf{f} and \mathbf{g} form an exact pair for a countable ideal I of ω -enumeration degrees, i.e. $I = I(\mathbf{f}) \cap I(\mathbf{g})$ then for every k the pair $\mathbf{f}^{(k)}, \mathbf{g}^{(k)}$ form an exact pair for the least ideal $I^{(k)}$ containing all k th ω -jumps of the elements of I , i.e. $I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$. Let \mathbf{f} and \mathbf{g} be the minimal pair from Theorem 21. Since $I = \text{CS}(\mathfrak{A}, \mathfrak{B})$ is a countable ideal, $I \subseteq \mathcal{D}_\omega$ and $\text{CS}(\mathfrak{A}, \mathfrak{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$ then $I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for each k . On the other hand $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \text{CS}_k(\mathfrak{A}, \mathfrak{B})$ for each k . \square

Soskov [8] showed that for any structure \mathfrak{A} , there is a quasi-minimal e-degree \mathbf{q} with respect to $\text{DS}(\mathfrak{A})$, i.e. $\mathbf{q} \notin \text{CS}(\mathfrak{A})$ and if \mathbf{a} is a total e-degree and $\mathbf{a} \geq_e \mathbf{q}$ then $\mathbf{a} \in \text{DS}(\mathfrak{A})$ and if \mathbf{a} is a total e-degree and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in \text{CS}(\mathfrak{A})$. We can give an analogue of this theorem.

Theorem 23. For every structure \mathfrak{A} and $\mathfrak{B} \in \mathcal{S}$, there exists a set $F \subseteq \mathbb{N}$ such that for $\mathbf{q} = d_\omega(F \uparrow \omega)$ it holds:

- (1) $\mathbf{q} \notin \text{CS}(\mathfrak{A}, \mathfrak{B})$;
- (2) If \mathbf{a} is a total e-degree and $\mathbf{a} \geq_\omega \mathbf{q}$ then $\mathbf{a} \in \text{DS}(\mathfrak{A}, \mathfrak{B})$;
- (3) If \mathbf{a} is a total e-degree and $\mathbf{a} \leq_\omega \mathbf{q}$ then $\mathbf{a} \in \text{CS}(\mathfrak{A}, \mathfrak{B})$.

Proof. Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_s)$ and $\mathfrak{B} = \{B_n\}_{n < \omega}$. Consider the structure $\mathfrak{A}_0 = (\mathbb{N}; R_1, \dots, R_s, B_0)$.

Soskov [8] proved that there is a partial generic enumeration f of \mathfrak{A}_0 such that $d_e(f^{-1}(\mathfrak{A}_0))$ is quasi-minimal with respect to $\text{DS}(\mathfrak{A}_0)$. Moreover if $i = \lambda x.x$

then $f^{-1}(\mathfrak{A}_0) \not\leq_e i^{-1}(\mathfrak{A}_0)$. Ganchev [4] showed that there is a set F such that $f^{-1}(\mathfrak{A}_0) <_e F$, $f^{-1}(B_n) \leq_e F^{(n)}$ uniformly in n and for any total set X , if $X \leq_e F$ then $X \leq_e f^{-1}(\mathfrak{A}_0)$. We call the set F quasi-minimal over $f^{-1}(\mathfrak{A}_0)$ with respect to $\{f^{-1}(B_n)\}_{n < \omega}$. The set F is constructed as a partial regular enumeration. Set $\mathbf{q} = d_\omega(F \uparrow \omega)$. We will prove that \mathbf{q} has the desired properties.

Suppose for a contradiction that $\mathbf{q} \in \text{CS}(\mathfrak{A}, \mathfrak{B})$. Then $d_\omega(f^{-1}(\mathfrak{A}_0) \uparrow \omega) \in \text{CS}(\mathfrak{A}, \mathfrak{B})$ since $f^{-1}(\mathfrak{A}_0) <_e F$. It follows that $f^{-1}(\mathfrak{A}_0) \uparrow \omega$ is forcing definable on \mathfrak{A} with respect to \mathfrak{B} . Then $f^{-1}(\mathfrak{A}_0) \leq_e i^{-1}(\mathfrak{A}) \oplus B_0 \equiv_e i^{-1}(\mathfrak{A}_0)$. A contradiction.

If X is a total set and $X \leq_e F$ then $X \leq_e f^{-1}(\mathfrak{A}_0)$ as F is quasi-minimal over $f^{-1}(\mathfrak{A}_0)$. Thus $d_e(X) \in \text{CS}(\mathfrak{A}_0)$ by the choice of $f^{-1}(\mathfrak{A}_0)$. But $\text{DS}(\mathfrak{A}, \mathfrak{B}) \subseteq \text{DS}(\mathfrak{A}_0)$. So $d_\omega(X \uparrow \omega) \in \text{CS}(\mathfrak{A}, \mathfrak{B})$.

If X is a total set and $X \geq_e F$ then $X \geq_e f^{-1}(\mathfrak{A}_0)$. Since “=” is among the predicates of \mathfrak{A} , $\text{dom}(f) \leq_e X$ and since X is a total set, $\text{dom}(f)$ is c.e. in X . Let ρ be a recursive in X enumeration of $\text{dom}(f)$. Set $h = \lambda n.f(\rho(n))$. Thus $h^{-1}(\mathfrak{A}) \leq_e X$ and $h^{-1}(B_n) \leq_e X^{(n)}$ uniformly in n . By Lemma 4 there is an acceptable enumeration g of \mathfrak{A} such that $g^{-1}(\mathfrak{A}) \equiv_e X$. And hence $d_e(X) \in \text{DS}(\mathfrak{A}, \mathfrak{B})$. \square

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