Journal of Logic and Computation Advance Access published October 14, 2010

Embedding distributive lattices in the Σ_2^0 enumeration degrees

HRISTO GANCHEV and MARIYA SOSKOVA, *Department of Mathematical Logic, Faculty of Mathematics and Computer Science, Sofia University, Bulgaria. E-mail: ganchev@fmi.uni-sofia.bg; msoskova@fmi.uni-sofia.bg*

Abstract

We prove that every countable distributive lattice is embeddable in the Σ_2^0 enumeration degrees via a 0–1 preserving monomorphism. Moreover, we prove that every countable distributive lattice is embeddable below arbitrary Δ_2^0 degree via a 0 preserving monomorphism.

Keywords: Enumeration degrees, local theory, distributive lattices, embeddings.

1 Introduction

The local structure of the enumeration degrees $\mathcal{G}_e$ is the partially ordered set of the enumeration degrees below the first jump $\mathbf{0}_e{}'$ of the least enumeration degree $\mathbf{0}_e$. Cooper [3] shows that $\mathcal{G}_e$ consists exactly of the Σ_2^0 enumeration degrees, degrees which contain Σ_2^0 sets, or equivalently consist entirely of Σ_2^0 sets. In investigating structural complexity of $\mathcal{G}_e$, the natural question of what other structures are embeddable in $\mathcal{G}_e$ arises. For example, if we view $\mathcal{G}_e$ as a countable partial ordering, we might ask what other partial orderings are embedded in $\mathcal{G}_e$. The complete answer to this question is provided by Bianchini [2], who proves that every countable partial ordering can be embedded densely in $\mathcal{G}_e$, i.e. in any non-empty interval of Σ_2^0 enumeration degrees; see also Sorbi [11] for a published proof of Bianchini's result.

As $\mathcal{G}_e$ is an interval of enumeration degrees, $\mathcal{G}_e$ is a countable upper semi-lattice with least and greatest elements. In this article, we investigate a further question of characterizing special types of partially ordered structures, lattices, that are embeddable in $\mathcal{G}_e$.

We start by outlining preliminary results on this topic. McEvoy and Cooper [8] prove that the standard embedding ι of the Turing degrees in the enumeration degrees preserves greatest lower bounds for low c.e. degrees, i.e. if $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathcal{R}$ and $\mathbf{a}'=\mathbf{b}'=\mathbf{c}'=\mathbf{0}_T{}'$, then

$$\mathbf{a}\wedge\mathbf{b}=\mathbf{c}\Longrightarrow\iota(\mathbf{a})\wedge\iota(\mathbf{b})=\iota(\mathbf{c}).$$

This allows us to transfer known embeddability results for the low c.e. Turing degrees into the substructure of the low Π_1^0 enumeration degrees. An unpublished result by Lachlan and independently by Lerman is that every countable distributive lattice can be embedded in the low c.e. degrees preserving the least element (See Soare [9] for a proof of this result.) This is also the best result that can be obtained in this way, as Lachlan's Nondiamond Theorem [6], yields the four elements lattice $\{0,\mathbf{a},\mathbf{b},1\}$ for which $\mathbf{a}\not\leq\mathbf{b}$ and $\mathbf{b}\not\leq\mathbf{a}$ (the diamond lattice) is not embeddable in the c.e. degrees preserving least and greatest elements.

This limitation of the c.e. Turing degrees, however, does not apply to the local enumeration degrees. Indeed, Ahmad [1] shows that the diamond lattice is embeddable in the Σ_2^0 enumeration

doi:10.1093/logcom/exq042

degrees preserving least and greatest elements, providing the first evidence for the fact that the local structures of the Turing degrees and the enumeration degrees are not elementarily equivalent. Furthermore, her proof embeds the intermediate degrees of the diamond in the low Σ^0_2 enumeration degrees. Lempp and Sorbi [7] extend this result and show that every finite lattice is embeddable in the low Σ^0_2 degrees preserving least and greatest elements. In this article, we extend the characterization of partially ordered structures embeddable in $\mathcal{G}_e$ to include countable distributive lattices. Our two main results are as follows.

THEOREM 1
Every countable distributive lattice is embeddable in $[\mathbf{0}_e, \mathbf{0}_e{}']$ preserving both least and greatest elements. Moreover, the range of the embedding contains only low quasiminimal enumeration degrees, except for the image of the least and greatest elements.

THEOREM 2
Every countable distributive lattice is embeddable preserving the least element in every non-trivial interval $[\mathbf{a},\mathbf{b}] \subseteq \mathcal{G}_e$, for which $\mathbf{a},\mathbf{b}$ are Δ^0_2 enumeration degrees and $\mathbf{a}$ is low. Moreover, the range of the embedding contains only enumeration degrees quasiminimal and low over $\mathbf{a}$, except for the image of the least and greatest elements.

A relativization of the proofs of Theorems 1 and 2 provides us with further insight to the global structure of the enumeration degrees. Theorem 2 can be as usual only relativized above any total enumeration degree. Theorem 1, however, provides an interesting example of a structural property of the interval $[\mathbf{0}_e, \mathbf{0}_e{}']$ which can be relativized to every interval $[\mathbf{u}, \mathbf{u}']$, where $\mathbf{u}$ is an arbitrary enumeration degree.

As a further corollary of the proof of Theorem 2, we shall obtain that if $\mathbf{v}$ is downwards properly Σ^0_2, i.e. a Σ^0_2 degree, which does not bound any non-trivial Δ^0_2 degrees, then every countable distributive lattice is embeddable in $[\mathbf{v}, \mathbf{0}_e{}']$ in such a way, that the range of the embedding consists only of degrees low over $\mathbf{v}$ degrees except for the image of the greatest element. Harris [4] has recently announced a result, that yields the existence of a downwards properly Σ^0_2 degree in every jump class of the high/low hierarchy of the Σ^0_2 enumeration degrees, except for L_1. Combing this with our result we get that every countable distributive lattice is embeddable in L_n for $n>1$, H_n for $n\geq 1$ and I.

We shall prove both theorems using the notion of Kalimullin pairs ($\mathcal{K}$-pairs). This notion is introduced and used by Kalimullin to prove the definability of the enumeration jump.

DEFINITION 1 [5]
A pair of sets $\{A,B\}$ is a $\mathcal{K}$-pair over U, if there is a set $W \leq_e U$, such that $A\times B \subseteq W$ and $\overline{A}\times\overline{B}\subseteq\overline{W}$. If $A,B \not\leq_e U$, we call this $\mathcal{K}$-pair non-trivial. If U is a c.e. set, then we refer to $\{A,B\}$ just as a $\mathcal{K}$-pair.

The enumeration degrees generated by $\mathcal{K}$-pairs exhibit some very interesting properties [5]. If $\mathbf{a}=\mathbf{d}_e(A)$, $\mathbf{b}=\mathbf{d}_e(B)$ and $\mathbf{u}=\mathbf{d}_e(U)$, then $\{A,B\}$ is a $\mathcal{K}$-pair over U if and only if

$$\forall \mathbf{x}\in\mathbf{D}_e[\mathbf{x}\vee\mathbf{u}=(\mathbf{x}\vee\mathbf{u}\vee\mathbf{a})\wedge(\mathbf{x}\vee\mathbf{u}\vee\mathbf{b})]. \tag{1.1}$$

Additionally, if $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair over U then the degrees $A\oplus U$ and $B\oplus U$ are quasiminimal over U. Furthermore, if A,B are e-reducible to the enumeration jump of U, then both $A\oplus U$ and $B\oplus U$ are low over U.

From now on we shall use the term $\mathcal{K}$-pairs both for sets, as in Definition 1, and for degrees that satisfy (1.1).

Equality (1.1) makes $\mathcal{K}$-pairs a powerful tool for embedding distributive lattices in intervals of enumeration degree. In order to illustrate this, consider a finite $\mathcal{K}$-system $\{\mathbf{a}_i \mid 0\leq i\leq n-1\}$, i.e. for each $i \neq j$, $\{\mathbf{a}_i, \mathbf{a}_j\}$ is a non-trivial $\mathcal{K}$-pair. Using induction on $|X|+|Y|$, we shall prove that, whenever X and Y are disjoint non-empty subsets of $\{0,1,\ldots,n-1\}$, the pair

$$\left\{\bigvee_{i\in X}\mathbf{a}_i, \bigvee_{i\in Y}\mathbf{a}_i\right\} \text{ is a } \mathcal{K}\text{-pair.} \tag{1.2}$$

For $|X|+|Y|=2$ the statement follows from the definition of a $\mathcal{K}$-system. Suppose that $|X|+|Y|>2$ and let $|X|\geq 2$. Fix an arbitrary $\mathbf{x}\in\mathcal{G}_e$ and let

$$\mathbf{y}\leq \mathbf{x}\vee\bigvee_{i\in X}\mathbf{a}_i, \mathbf{x}\vee\bigvee_{i\in Y}\mathbf{a}_i. \tag{1.3}$$

Fix $i_0\in X$ and let $X_0 = X-\{i_0\}$. From (1.3), we obtain

$$\mathbf{y}\leq(\mathbf{x}\vee\mathbf{a}_{i_0})\vee\bigvee_{i\in X_0}\mathbf{a}_i,\ (\mathbf{x}\vee\mathbf{a}_{i_0})\bigvee_{i\in Y}\mathbf{a}_i.$$

As $|X_0|+|Y|<|X|+|Y|$ and $X_0\neq\emptyset$, we have that $\{\bigvee_{i\in X_0}\mathbf{a}_i, \bigvee_{i\in Y}\mathbf{a}_i\}$ is a $\mathcal{K}$-pair and hence $\mathbf{y}\leq\mathbf{x}\vee\mathbf{a}_{i_0}$. But $1+|Y|<|X|+|Y|$ and again by the induction hypothesis $\{\mathbf{a}_{i_0}, \bigvee_{i\in Y}\mathbf{a}_i\}$ is a $\mathcal{K}$-pair. From here $\mathbf{y}\leq\mathbf{x}$ and so (1.2) is satisfied.

Note that (1.1) implies that if $\mathbf{u}\leq_e\mathbf{v}$ and $\{\mathbf{a},\mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{u}$, then $\{\mathbf{a},\mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{v}$. Thus, (1.2) implies that if $\mathbf{v}$ bounds a $\mathcal{K}$-system of n degrees omitting $\mathbf{u}$, then the lattice $(2^n,\cap,\cup)$ is embeddable in the interval $[\mathbf{u},\mathbf{v}]$. By Birkhoff's Theorem, every finite distributive lattice is embeddable in $(2^n,\cap,\cup)$ for an appropriate n and so we may conclude that every finite distributive lattice is embeddable in $[\mathbf{u},\mathbf{v}]$, given that $\mathbf{v}$ bounds a sufficiently large $\mathcal{K}$-system avoiding $\mathbf{u}$. Our strategy to prove Theorems 1 and 2 is to generalize (1.2) for special countable $\mathcal{K}$-systems and to prove that such $\mathcal{K}$-systems exist.

2 Preliminaries

Throughout this article, we shall use standard notation. We refer the reader to Cooper [3] and Sorbi [10] for an extensive survey of results on both the global and local theory of the enumeration degrees. We outline the basic notions and facts used in the article.

By $W_0, W_1, \ldots,$ we denote the c.e. sets with their Gödel indices. For every natural number i and every set of natural numbers A, we denote by $W_i(A)$ the set

$$W_i(A)=\{x\mid\exists u[\langle x,u\rangle\in W_i\,\&\,D_u\subseteq A\},$$

where D_u is the finite set with canonical index u. Thus, every c.e. set can be viewed as an operator on sets, an enumeration operator. Its elements will be called axioms.

The relation enumeration reducibility is defined by $B\leq_e A$ if and only if $B=W_i(A)$ for some natural i. This relation defines a preorder on the sets of natural numbers and induces an equivalence relation $\equiv_e$. The equivalence class of a set A, denoted by $\mathbf{d}_e(A)$, is the enumeration degree of the set A. The enumeration degrees are ordered in the natural way by $\mathbf{d}_e(B)\leq\mathbf{d}_e(A)$ if and only if $B\leq_e A$.

The least upper bound of the enumeration degrees $\mathbf{d}_e(A)$ and $\mathbf{d}_e(B)$ is the degree of the *join* $A\oplus B=\{2a\mid a\in A\}\cup\{2b+1\mid b\in B\}$ of A and B. The *uniform join* of the indexed system of sets $\{A_i\mid i\in I\}, I\subseteq\mathbb{N}$, is given by $\bigoplus_{i\in I}A_i=\{\langle x,k\rangle\mid k\in I\,\&\,x\in A_k\}$. The uniform join is the least uniform upper bound for the system $\{A_i \mid i \in I\}$, i.e. if $A_i \leq_e B$ uniformly in $i \in I$ and $I \leq_e B$, then $\bigoplus_{i\in I} A_i \leq_e B$. Furthermore, for arbitrary computable sets R_1 and R_2

$$\bigoplus_{k\in R_1\cup R_2} A_k \equiv_e \bigoplus_{k\in R_1} A_k \oplus \bigoplus_{k\in R_2} A_k. \tag{2.1}$$

The enumeration jump of the set A is defined by $A' = L_A \oplus \overline{L}_A$, where $L_A = \{\langle x,i\rangle \mid x \in W_i(A)\}$. We say that a set B is low over A, if $A \leq_e B$ and $A' \equiv_e B'$.

We say that a set A is total if $\overline{A} \leq_e A$. A degree $\mathbf{a}$ is total if it contains a total set. The total degrees are the images of the Turing degrees under the standard embedding $\iota : \mathbf{D}_T \to \mathbf{D}_e$. The degrees containing no total set are called partial. Thus, the partial degrees are exactly the degrees in $\mathbf{D}_e \setminus \iota(\mathbf{D}_T)$. A degree $\mathbf{b}$ is said to be quasi-minimal over $\mathbf{a}$ if every degree $\mathbf{a} \lneq \mathbf{x} \leq \mathbf{b}$ is partial (in particular $\mathbf{b}$ is partial).

Equality (1.1), characterizing the $\mathcal{K}$-pairs with a lattice-theoretic property, follows from the following theorem.

THEOREM 3 [5]
Let A, B and U be sets of natural numbers.

(1) If $\{A,B\}$ is a $\mathcal{K}$-pair over U, then

$$\forall \mathbf{x} \in \mathbf{D}_e[\mathbf{x} \vee \mathbf{d}_e(U) = (\mathbf{x} \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(A)) \wedge (\mathbf{x} \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(B))].$$

(2) If $\{A,B\}$ is not a $\mathcal{K}$-pair over U, then there is a set $X \leq_e U' \oplus (A \oplus \overline{A}) \oplus (B \oplus \overline{B})$, for which

$$\mathbf{d}_e(X) \vee \mathbf{d}_e(U) \neq (\mathbf{d}_e(X) \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(A)) \wedge (\mathbf{d}_e(X) \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(B)).$$

From claim (1) of the theorem it follows, that if $\mathbf{a}$ and $\mathbf{b}$ are the degrees of a $\mathcal{K}$-pair of Σ^0_2 sets then

$$\forall \mathbf{x} \in \mathcal{G}_e[\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})]. \tag{2.2}$$

It is still an open question whether two Σ^0_2 degrees satisfying (2.2) have representatives forming a $\mathcal{K}$-pair. However, claim (2) settles the questions for Δ^0_2 degrees. Namely, two Δ^0_2 degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy (2.2) if and only if $\{A,B\}$ is a $\mathcal{K}$-pair for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

As we have mentioned in the Section 1 if $\{A,B\}$ is a $\mathcal{K}$-pair over U, then $A \oplus U$ and $B \oplus U$ are quasi-minimal over U and furthermore, if $A, B \leq_e U'$, then A and B are low over U. This statement follows from Theorem 3 and following lemma.

LEMMA 1 [5]
Let A, B and M be sets, such that $A \times B \subseteq M$, $\overline{A} \times \overline{B} \subseteq \overline{M}$ and $A \not\leq_e M$. Then

$$B \leq_e \overline{A} \oplus M \;\&\; \overline{B} \leq_e A \oplus \overline{M}.$$

The following properties of $\mathcal{K}$-pairs are only listed in [5]. As we will be using them in this article, for completeness, we restate them and provide a formal proof.

LEMMA 2 [5]
If $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair over U, then $A \oplus U$ and $B \oplus U$ are quasi-minimal over U. Furthermore, if $A, B \leq_e U'$ then $A \oplus U$ and $B \oplus U$ are low over U, i.e. $(A \oplus U)' \equiv_e (B \oplus U)' \equiv_e U'$.

PROOF. Towards a contradiction assume that $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair over U and $A\oplus U$ is not quasi-minimal over U. Fix a total C such that $U\le_e C\le_e A\oplus U$. According to claim (1) of Theorem 3 for all $\mathbf{x}\ge\mathbf{d}_e(U)$ we have,

$$\mathbf{x}=(\mathbf{x}\vee\mathbf{d}_e(A\oplus U))\wedge(\mathbf{x}\vee\mathbf{d}_e(B\oplus U)).$$

From $C\le_e A\oplus U$, we obtain

$$\mathbf{x}=(\mathbf{x}\vee\mathbf{d}_e(C))\wedge(\mathbf{x}\vee\mathbf{d}_e(B)). \tag{2.3}$$

for every $\mathbf{x}\ge\mathbf{d}_e(U)$. Now claim (2) of Theorem 3 implies that $\{C,B\}$ is a $\mathcal{K}$-pair over U. Let $W\le_e U$ be such that $C\times B\subseteq W$ and $\overline{C}\times\overline{B}\subseteq\overline{W}$. Applying Lemma 1, we obtain $B\le_e\overline{C}\oplus W\le_e\overline{C}\le_e C$. But then (2.3) is possible only if $B\equiv_e U$, which contradicts the assumption that $\{A,B\}$ is a non-trivial of the $\mathcal{K}$-pair .

Now suppose that $A,B\le_e U'$. Since $A\equiv_e L_A$ and $B\equiv_e L_B$, applying consecutively (1) and (2) from Theorem 3, we obtain that $\{L_A,L_B\}$ is a $\mathcal{K}$-pair over U. Let $W\le_e U$, be such that $L_A\times L_B\subseteq W$ and $\overline{L}_A\times\overline{L}_B\subseteq\overline{W}$. Since $L_A,L_B\not\le_e U$ Lemma 1 yields $\overline{L}_A\le_e L_B\oplus\overline{W}$ and $\overline{L}_B\le_e L_A\oplus\overline{W}$. But L_A, L_B and $\overline{W}$ are enumeration reducible to U' and hence $\overline{L}_A,\overline{L}_B\le_e U'$. ■

Finally, we shall need some lattice-theoretic results about embeddability of distributive lattices. Birkhoff proves that every finite distributive lattice can be embedded in the Boolean algebra $(2^n,\cup,\cap)$ preserving least and greatest elements. From here using a compactness argument one can prove that every countable distributive lattice is embeddable in the countable atomless Boolean algebra preserving least and greatest elements. The countable atomless Boolean algebra is unique up to isomorphism. Take as an instance of it the algebra of finite unions of left semi-closed intervals of rational numbers. Since $(\mathbb{Q},\le)$ is a computable linear ordering, we thus obtain that the countable atomless Boolean algebra is embeddable in the Boolean algebra $\mathcal{R}$ of computable sets. Thus, in order to prove that every countable distributive lattice is embeddable in an interval of enumeration degrees $[\mathbf{u},\mathbf{v}]$, it is enough to prove that $\mathcal{R}$ is embeddable in it.

3 Uniform $\mathcal{K}$-systems

As we have seen in the Section 1, we need finite $\mathcal{K}$-systems in order to be able to embed finite distributive lattices in $\mathcal{G}_e$. For arbitrary countable distributive lattice, we shall need the notion of uniform $\mathcal{K}$-systems.

DEFINITION 2
We say that the system of sets $\{A_i\}_{i<\omega}$ is a uniform $\mathcal{K}$-system, if and only if for every natural i, $A_i\not\le_e\emptyset$ and there is a computable function r, such that whenever $i\neq j$

$$A_i\times A_j\subseteq W_{r(i,j)}\,\&\,\overline{A}_i\times\overline{A}_j\subseteq\overline{W}_{r(i,j)}.$$

For uniform $\mathcal{K}$-systems, we are able to prove an analogue of (1.2) as follows.

PROPOSITION 1
Let $\{A_i\}_{i<\omega}$ be a uniform $\mathcal{K}$-system and let R_1 and R_2 be disjoint computable sets. Then $\{\bigoplus_{i\in R_1}A_i,\bigoplus_{i\in R_2}A_i\}$ is a $\mathcal{K}$-pair.

PROOF. Let $\{A_i\}_{i<\omega}$ be a uniform $\mathcal{K}$-system and let R_1 and R_2 be disjoint computable sets. Consider the set

$$W=\{\langle\langle x,k\rangle,\langle y,j\rangle\rangle\mid k\in R_1,j\in R_2,\langle x,y\rangle\in W_{r(k,j)}\}. \tag{3.1}$$

It is clear, that W is c.e. First, we shall prove, that $\bigoplus_{i\in R_1} A_i \times \bigoplus_{i\in R_2} A_i \subseteq W$. Fix $\langle x,k\rangle \in \bigoplus_{i\in R_1} A_i$ and $\langle y,j\rangle \in \bigoplus_{i\in R_1} A_i$. We have $x\in A_k$, $y\in A_j$, $k\in R_1$ and $j\in R_2$. From $R_1\cap R_2=\emptyset$ we conclude $k\neq j$ and hence by the uniformity condition, we obtain $\langle x,y\rangle \in W_{r(k,j)}$. Therefore, $\langle\langle x,k\rangle,\langle y,j\rangle\rangle \in W$.

In order to prove $\overline{\bigoplus_{i\in R_1} A_i} \times \overline{\bigoplus_{i\in R_2} A_i} \subseteq \overline{W}$ fix $\langle x,k\rangle \notin \bigoplus_{i\in R_1} A_i$ and $\langle y,j\rangle \notin \bigoplus_{i\in R_1} A_i$. We shall consider two cases. First suppose that either $k\notin R_1$ or $j\notin R_2$. Then according to (3.1), $\langle\langle x,k\rangle,\langle y,j\rangle\rangle \notin W$. Now suppose, that $k\in R_1$ and $j\in R_2$. Then it should be the case $x\notin A_k$ and $y\notin A_j$. But R_1 and R_2 are disjoint and hence by the uniformity of the $\mathcal{K}$-system, we obtain $\langle x,y\rangle \notin W_{r(k,j)}$. Thus, in this case we also have $\langle\langle x,k\rangle,\langle y,j\rangle\rangle \notin W$. ■

LEMMA 3
Let $\{A_i\}_{i<\omega}$ be a uniform $\mathcal{K}$-system and let U be such that for all i, $A_i \not\leq_e U$. Then every countable distributive lattice is embeddable in the interval of enumeration degrees $[\mathbf{d}_e(U),\mathbf{d}_e(U\oplus\bigoplus_{i<\omega} A_i)]$ preserving least and greatest elements. Moreover, the range of the embedding, except for the image of the least and greatest elements, contains only degrees quasi-minimal over $\mathbf{d}_e(U)$. Furthermore, if $\bigoplus_{i<\omega} A_i \leq_e U'$ then all the images except for the image of the greatest element are low over $\mathbf{d}_e(U)$.

PROOF. Since every distributive lattice is embeddable preserving least and greatest elements in the lattice $\mathcal{R}$ of the computable sets, it is enough to prove the lemma for $\mathcal{R}$. Consider the mapping $\varphi:\mathcal{R}\to[\mathbf{d}_e(U),\mathbf{d}_e(U\oplus\bigoplus_{i<\omega} A_i)]$, acting by the rule

$$\varphi(R)=\mathbf{d}_e\left(U\oplus\bigoplus_{k\in R} A_k\right).$$

It is clear that $\varphi(\emptyset)=\mathbf{d}_e(U)$ and $\varphi(\mathbb{N})=\mathbf{d}_e(U\oplus\bigoplus_{i<\omega} A_i)$. From (2.1), we immediately obtain that φ preserves least upper-bounds. Thus, to show that φ is an embedding, it remains to show that φ preserves greatest lower-bounds. Fix two computable sets R_1 and R_2, and let $\widetilde{R}_1=R_1-(R_1\cap R_2)$ and $\widetilde{R}_2=R_2-(R_1\cap R_2)$. From (2.1), we obtain

$$U\oplus\bigoplus_{k\in R_1} A_k=\left(U\oplus\bigoplus_{k\in R_1\cap R_2} A_k\right)\oplus\bigoplus_{k\in\widetilde{R}_1} A_k$$

$$U\oplus\bigoplus_{k\in R_2} A_k=\left(U\oplus\bigoplus_{k\in R_1\cap R_2} A_k\right)\oplus\bigoplus_{k\in\widetilde{R}_2} A_k.$$

$\widetilde{R}_1$ and $\widetilde{R}_2$ are disjoint, so that Proposition 1 yields that $\{\bigoplus_{k\in\widetilde{R}_1} A_k,\bigoplus_{k\in\widetilde{R}_2} A_k\}$ is a $\mathcal{K}$-pair. Now from Theorem 3 we obtain

$$\varphi(R_1)\wedge\varphi(R_2)=$$
$$\mathbf{d}_e\left(\left(U\oplus\bigoplus_{k\in R_1\cap R_2} A_k\right)\oplus\bigoplus_{k\in\widetilde{R}_1} A_k\right)\wedge\mathbf{d}_e\left(\left(U\oplus\bigoplus_{k\in R_1\cap R_2} A_k\right)\oplus\bigoplus_{k\in\widetilde{R}_1} A_k\right)=$$
$$\mathbf{d}_e\left(U\oplus\bigoplus_{k\in R_1\cap R_2} A_k\right)=\varphi(R_1\cap R_2).$$

It remains to prove that $\varphi(R)$ is quasi-minimal and low over $\mathbf{d}_e(U)$ whenever R is non-trivial. Fix a computable R and consider $\overline{R}$. We have that R and $\overline{R}$ are disjoint computable sets, and hence by

Proposition 1, $\{\bigoplus_{k\in R}A_k, \bigoplus_{k\in \overline{R}}A_k\}$ is a non-trivial $\mathcal{K}$-pair. But $\bigoplus_{k\in R}A_k, \bigoplus_{k\in \overline{R}}A_k \not\leq_e U$ and hence $\{\bigoplus_{k\in R}A_k, \bigoplus_{k\in \overline{R}}A_k\}$ is a non-trivial $\mathcal{K}$-pair over U. Applying Lemma 1, we obtain that both $\varphi(R)$ and $\varphi(\overline{R})$ are quasiminimal and low over $\mathbf{d}_e(U)$. ■

4 Existence of uniform $\mathcal{K}$-systems

In this section, we prove the two main theorems announced in Section 1. By Lemma 3 both proofs will follow from the existence of certain uniform $\mathcal{K}$ systems. We start by proving that there is a uniform $\mathcal{K}$-system, whose uniform join is equivalent to $\emptyset'$, and thus concluding the proof of Theorem 1.

THEOREM 4
There is a uniform $\mathcal{K}$-system $\{A_i\}_{i<\omega}$, such that $\bigoplus_{i<\omega}A_i \equiv_e \emptyset'$.

PROOF. We assume that an effective coding of all finite strings of 0 and 1 is fixed. As usual, we shall identify a string with its code. We denote by T the collection of all strings. If $\sigma,\tau\in T$, denote the concatenation of σ and τ by $\sigma*\tau$. If τ is an initial segment of σ, we write $\tau\subseteq\sigma$. By $\tau<_L\sigma$, we mean that there is a $\rho\in T$, such that $\rho*0\subseteq\tau$ and $\rho*1\subseteq\sigma$. We denote the length of the string σ by $|\sigma|$. Furthermore, we denote by $\lambda x.(x)_0$ and $\lambda x.(x)_1$ the computable functions for which $x=\langle(x)_0,(x)_1\rangle$ for arbitrary x.

We start the proof by constructing a sequence of finite binary strings $\delta(0)\subseteq\delta(1)\subseteq\cdots\subseteq\delta(n)\subseteq\cdots$, such that $|\delta(n)|=n$. We set $\delta(0)=\emptyset$ and

$$\delta(n+1)=\begin{cases}\delta(n)*0, & \delta(n)\in W_{(n)_0}(\emptyset),\\ \delta(n)*1, & \delta(n)\notin W_{(n)_0}(\emptyset).\end{cases}$$

Consider the following sets

$$\begin{aligned}
R &= \{\ \sigma\in T\ \ |\exists n[\delta(n)<_L\sigma]\},\\
S &= \{\ \sigma\in T\ \ |\exists n[\delta(n)<_L\sigma\vee\delta(n)=\sigma]\},\\
A &= \{\ \ \delta(n)\ \ |\delta(n+1)=\delta(n)*1\}\cup R,\\
W &= \{\langle\sigma_0,\sigma_1\rangle|\sigma_0\in R\vee\sigma_1\in R\vee(\sigma_0\in S\,\&\,\sigma_0*1\subseteq\sigma_1)\vee\\
&\qquad(\sigma_1\in S\,\&\,\sigma_1*1\subseteq\sigma_0)\}.
\end{aligned}$$

To provide some visual intuition about the above defined sets we observe the following. The sequence $\{\delta(n)\}_{n<\omega}$ defines an infinite path δ in the tree T. The set R is the collection of all finite binary strings that are strictly to the right of the path δ. The set S is the set of strings to the right of or on the path δ. The set A is specially chosen representative of $\mathbf{0}_e{}'$.

We prove that $R\leq_e\emptyset$, $S\leq_e\emptyset$, $\mathrm{Graph}(\delta)\leq_e A$, $W\leq_e\emptyset$ and $A\equiv_e\emptyset'$.

- $R\leq_e\emptyset$ follows from

$$\sigma\in R\iff\exists\tau\subsetneq\sigma[\tau*1\subseteq\sigma\,\&\,\tau\in W_{(|\tau|)_0}(\emptyset)\,\&\,\forall\rho\subsetneq\tau[\rho*0\subseteq\tau\Rightarrow\rho\in W_{(|\rho|)_0}(\emptyset)]].$$

- $S\leq_e\emptyset$ follows from

$$\sigma\in S\iff\sigma\in R\vee\forall\rho\subsetneq\sigma[\rho*0\subseteq\sigma\Rightarrow\rho\in W_{(|\rho|)_0}(\emptyset)].$$

- $\mathrm{Graph}(\delta)\leq_e A$ follows from

$$\delta(n)\in A\Rightarrow\delta(n+1)=\delta(n)*1,$$
$$\delta(n)\in W_{(n)_0}(\emptyset)\Rightarrow\delta(n+1)=\delta(n)*0.$$

- $W \leq_e \emptyset$ follows directly from $R, S \leq_e \emptyset$.
- Finally, to see that $A \equiv_e \emptyset'$ we need only to prove that $\emptyset' \leq_e A$ (the converse is obvious). For, fix a computable function g, such that $x \in L_\emptyset \Rightarrow W_{g(x)}(\emptyset) = \omega$ and $x \notin L_\emptyset \Rightarrow W_{g(x)}(\emptyset) = \emptyset$. Then

$$x \notin L_\emptyset \iff \delta(\langle g(x), 0\rangle) \notin W_{g(x)}(\emptyset) \iff \delta(\langle g(x), 0\rangle) \in A.$$

From here $\overline{L}_\emptyset \leq_e \text{Graph}(\delta) \oplus A \leq_e A$ and so $\emptyset' \equiv_e A$.

Next we shall see that

$$(A \times A) \setminus \{\langle \sigma, \sigma \rangle \mid \sigma \in A\} \subseteq W \text{ and } \overline{A} \times \overline{A} \subseteq \overline{W}. \tag{4.1}$$

First, let $\sigma_0, \sigma_1 \in A$ and $\sigma_0 \neq \sigma_1$. If either $\sigma_0 \in R$ or $\sigma_1 \in R$, then $\langle \sigma_0, \sigma_1 \rangle \in W$. Now suppose that $\sigma_0, \sigma_1 \notin R$. Then $\sigma_0 = \delta(n)$ and $\sigma_1 = \delta(m)$ for some n and m, such that $\delta(n+1) = \delta(n) * 1$ and $\delta(m+1) = \delta(m) * 1$. Without loss of generality, let $n < m$. Then $\sigma_0 * 1 = \delta(n+1) \subseteq \delta(m) = \sigma_1$. But $\sigma_0 = \delta(n)$ implies $\sigma_0 \in S$, so that $\langle \sigma_0, \sigma_1 \rangle \in W$.

Now let $\sigma_0, \sigma_1 \notin A$. Then $\sigma_0, \sigma_1 \notin R$. Towards a contradiction assume that $\langle \sigma_0, \sigma_1 \rangle \in W$. Without loss of generality, we may assume $\sigma_0 \in S$ and $\sigma_0 * 1 \subseteq \sigma_1$. Since $\sigma_0 \notin R$, $\sigma_0 = \delta(n)$ for some n. But $\sigma_0 \notin A$ and therefore $\delta(n+1) = \sigma_0 * 0 <_L \sigma_0 * 1 \subseteq \sigma_1$. Thus $\sigma_1 \in R$. A contradiction.

We are ready to define the uniform $\mathcal{K}$-system. For arbitrary i and j set, $A_i = \{\sigma \in A \mid (|\sigma|)_1 = i\}$ and $W_{ij} = \{\langle \sigma_0, \sigma_1 \rangle \in W \mid (|\sigma_0|)_1 = i \,\&\, (|\sigma_1|)_1 = j\}$. It is clear that there is a computable function r, such that $W_{ij} = W_{r(i,j)}$. Furthermore, $A_i \leq_e A$ uniformly in i and $\bigcup_{i<\omega} A_i = A$, so that $\bigoplus_{i<\omega} A_i \equiv_e A$. Note that

$$\delta(\langle e, i\rangle) \in A_i \iff \delta(\langle e, i\rangle) \in A \iff \delta(\langle e, i\rangle) \notin W_e(\emptyset),$$

and hence $A_i \neq W_e(\emptyset)$ for arbitrary i and e.

Thus, it remains to prove that $A_i \times A_j \subseteq W_{ij}$ and $\overline{A}_i \times \overline{A}_j \subseteq \overline{W}_{ij}$ for $i \neq j$. Let $\sigma_0 \in A_i$ and $\sigma_1 \in A_j$. From the definition of A_i, A_j and from $i \neq j$, we obtain $\sigma_0, \sigma_1 \in A$, $(|\sigma_0|)_1 = i$, $(|\sigma_1|)_1 = j$, and $\sigma_0 \neq \sigma_1$. Therefore from (4.1), we obtain $\langle \sigma_0, \sigma_1 \rangle \in W_{ij}$.

Now let $\sigma_0 \notin A_i$ and $\sigma_1 \notin A_j$. If either $(|\sigma_0|)_1 \neq i$ or $(|\sigma_1|)_1 \neq j$, then $\langle \sigma_0, \sigma_1 \rangle \notin W_{ij}$. On the other hand if $(|\sigma_0|)_1 = i$ and $(|\sigma_1|)_1 = j$, then $\sigma_0, \sigma_1 \notin A$ and hence using (4.1) we obtain $\langle \sigma_0, \sigma_1 \rangle \notin W_{ij}$. ■

The uniform $\mathcal{K}$-system $\{A_i\}_{i<\omega}$ constructed in Theorem 4 consists of low Σ^0_2, hence Δ^0_2, and non c.e. sets. Thus, if U is a downwards properly Σ^0_2 set, i.e. for every $X \leq_e U$, X is either c.e. or is not Δ^0_2, then $A_i \not\leq_e U$ for all i. Therefore, Lemma 3 and Theorem 4 imply the following theorem, of which Theorem 1 is a particular case.

THEOREM 5
Let U be downwards properly Σ^0_2. Then every countable distributive lattice is embeddable in the interval $[\mathbf{d}_e(U), \mathbf{0}_e']$, preserving least and greatest elements. Moreover, the range of the embedding contains only degrees quasi-minimal and low over $\mathbf{d}_e(U)$, except for the images of the least and greatest elements.

Lemma 3 and Theorem 4 can be relativized over an arbitrary set V. We first need to relativize the notion of a uniform $\mathcal{K}$-system.

DEFINITION 3
We say that the system of sets $\{A_i\}_{i<\omega}$ is a uniform $\mathcal{K}$-system over V, if and only if for every natural number i, $A_i \not\leq_e V$ and there is a function r, such that $\text{Graph}(r) \leq_e V$ and whenever $i \neq j$

$$A_i \times A_j \subseteq W_{r(i,j)}(V) \,\&\, \overline{A}_i \times \overline{A}_j \subseteq \overline{W_{r(i,j)}(V)}.$$

Making a slight modification to the proof of Theorem 4 (we just need to substitute $\emptyset$ by V), we can prove that for every set V, there is a uniform $\mathcal{K}$-system $\{A_i\}_{i<\omega}$ over V, such that $\bigoplus_{i<\omega} A_i \equiv_e V'$. On the other hand, Lemma 3 is valid even for the relativized $\mathcal{K}$-system and hence every countable distributive lattice is embeddable in the interval $[\mathbf{d}_e(V), \mathbf{d}_e(V')]$. Furthermore, from the properties of $\mathcal{K}$-pairs, we obtain that the range of the embedding , except for 0 and 1, consists of low and quasi-minimal over $\mathbf{d}_e(V)$ degrees. In other words, we have the following theorem.

THEOREM 6
Every countable distributive lattice is embeddable preserving least and greatest elements in arbitrary interval $[\mathbf{v}, \mathbf{v}']$.

5 Bounding uniform $\mathcal{K}$-systems

The rest of this article is devoted to the proof of Theorem 2. Our goal is to show that every non-trivial Δ^0_2 set bounds a uniform $\mathcal{K}$-system. Before we can do this, we shall need to introduce some more notation.

We will be working with Δ^0_2 approximations to sets. Recall that a Δ^0_2 approximation to a set A is a uniform sequence of finite sets $\{A^{\{s\}}\}_{s<\omega}$ such that for every n we have that $\lim_n A^{\{s\}}(n)$ exists and is equal to $A(n)$. We shall use and respect the convention that for every s, $A^{\{s\}} \subseteq \mathbb{N} \upharpoonright s$. Furthermore, we shall say that a Δ^0_2 approximation has index e if e is an index of the computable function $\rho : \mathbf{N} \to \mathbf{N}$ defined by $\rho(s) = u_s$, where u_s is the canonical index of the finite set $A^{\{s\}}$.

DEFINITION 4
Let A be a set of natural numbers and i be a natural number:

(1) $A^{[i]} = \{\langle i, x\rangle \mid \langle i, x\rangle \in A\}$;
(2) For $R \in \{\leq, <, \geq, >\}$ we set $A^{[Ri]} = \{\langle j, x\rangle \mid \langle j, x\rangle \in A \,\&\, (jRi)\}$.
(3) $A[i] = \{x \mid \langle i, x\rangle \in A\}$.

We start with a dynamic property of sets A and B, a property of the approximations to sets A and B, which ensures that the enumeration degrees of A and B form a $\mathcal{K}$-pairs. This property originates from Kalimullin [5].

LEMMA 4
Let A_0 and A_1 be Δ^0_2 sets with respective Δ^0_2 approximations $\{A_0^{\{s\}}\}_{s<\omega}$ and $\{A_1^{\{s\}}\}_{s<\omega}$ such that for every $i \in \{0, 1\}$, every s and every x:

$$x \in (A_i^{\{s\}} \setminus A_i^{\{s+1\}}) \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq A_{1-i}.$$

Then $d_e(A_0)$ and $d_e(A_1)$ form a $\mathcal{K}$-pair. An index of a c.e. set W such that

$$A_0 \times A_1 \subseteq W \text{ and } \overline{A_0} \times \overline{A_1} \subseteq \overline{W}$$

is uniformly computable from the indices of the approximations to A_0 and A_1.

PROOF. Let $W = \bigcup_{s<\omega} A_0^{\{s\}} \times A_1^{\{s\}}$. The set W is c.e. and its index is obviously computable from the indices of the approximations to A_0 and A_1.

It follows from the properties of a Δ^0_2 approximation that $A_0 \times A_1 \subseteq W$. Fix $(a_0, a_1) \in \overline{A_0} \times \overline{A_1}$. We will prove that for all stages s we have $(a_0, a_1) \notin A_0^{\{s\}} \times A_1^{\{s\}}$ and hence $\overline{A_0} \times \overline{A_1} \subseteq \overline{W}$. Assume towards a contradiction that there is a stage s such that $\langle a_0, a_1\rangle \in A_0^{\{s\}} \times A_1^{\{s\}}$. Then $a_0 < s$ and can be represented as $a_0 = \langle k_0, y_0\rangle$ for some natural numbers k_0, y_0. Similarly $a_1 < s$ and can be represented as $a_1 = \langle k_1, y_1\rangle$ for some natural numbers k_1, y_1. Let $i \in \{0,1\}$ be such that $k_i = \min\{k_0, k_1\}$. As $a_i \notin A_i$ there will be a least stage $s' > s$ such that $a_i \in A_i^{\{s'-1\}} \setminus A_i^{\{s'\}}$. By the property of the approximations $\omega^{[\geq k_i]} \upharpoonright s \subseteq A_{1-i}$. By our choice of i it follows that $a_{1-i} \in A_{1-i}$, contradicting the assumption that $\langle a_0, a_1\rangle \in \overline{A_0} \times \overline{A_1}$. ■

THEOREM 7
Let A be a Δ^0_2 set and let B be a low Δ^0_2 set such that $A \not\leq_e B$. There is a uniform $\mathcal{K}$-system $\{A_i\}_{i<\omega}$ which is uniformly enumeration reducible to A and for every i, $A_i \not\leq_e B$.

PROOF. Fix a Δ^0_2 set A and a low Δ^0_2 set B such that $A \not\leq_e B$. Let $\{A^{\{s\}}\}_{s<\omega}$ be a Δ^0_2 approximation to A and let $\{B^{\{s\}}\}_{s<\omega}$ be a low Δ^0_2 approximation to B. Recall that a low Δ^0_2 approximation has the additional property that for every enumeration operator W with standard Σ^0_1 approximation $\{W^{\{s\}}\}_{s<\omega}$, the approximation $\{W^{\{s\}}(B^{\{s\}})\}_{s<\omega}$ to the set $W(B)$ is also Δ^0_2.

We shall construct a monotone uniform sequence of computable sets $\{V^{\{s\}}\}_{s<\omega}$ and let $V = \bigcup_{s<\omega} V^{\{s\}}$. The constructed set V is c.e., hence an enumeration operator. We set $A_i = V(A)[i]$. This definition automatically ensures that the system $\{A_i \mid i \in \omega\}$ is uniformly enumeration reducible to A. A Σ^0_2 approximation to the set A_i can be obtained by setting for every stage s, $A_i^{\{s\}} = V^{\{s\}}(A^{\{s\}})$. We will ensure that the following three requirements are satisfied:

- For every natural number i:

$$\mathcal{D}_i : \{A_i^{\{s\}}\}_{s<\omega} \text{ is a } \Delta^0_2 \text{ approximation.}$$

- For every pair of distinct natural numbers $i \neq j$:

$$\mathcal{K}_{\langle i,j\rangle} : \forall s, x (x \in (A_i^{\{s\}} \setminus A_i^{\{s+1\}}) \cap \omega^{[k]} \Rightarrow \omega^{[\geq k]} \upharpoonright s \subseteq A_j).$$

- For every pair of natural numbers i and e:

$$\mathcal{N}_{\langle i,e\rangle} : W_e(B) \neq A_i.$$

Where W_e is the e-th enumeration operator in some standard listing of all c.e. set.

The first two groups of requirements ensure that for every $i \neq j$ the pair (A_i, A_j) is a $\mathcal{K}$-pair. This together with Lemma 4 ensures that the system $\{A_i\}_{i<\omega}$ is a uniform $\mathcal{K}$-system. Indeed for every i an index of the approximation $\{A_i^{\{s\}}\}_{s<\omega}$ is uniformly computable from the index of $\{A^{\{s\}}\}_{s<\omega}$ and the index which will be produced by the construction of the c.e. set V. From this by Lemma 4, we can obtain uniformly in i and j an index of a c.e. set $W_{i,j}$ such that $A_i \times A_j \subseteq W_{i,j}$ and $\overline{A_i} \times \overline{A_j} \subseteq \overline{W_{i,j}}$. Finally, the third group of requirements ensures that for every i, $A_i \not\leq_e B$.

CONSTRUCTION

The construction is in stages. At stage 0, we set $V^{\{0\}} = \emptyset$. At stage $s > 0$, we construct $V^{\{s+1\}}$ from its value constructed at the previous stage, by allowing certain requirements to enumerate new axioms in it.

Step 1. *Satisfying the $\mathcal{K}$-requirements.*

If $V^{\{s\}}(A^{\{s\}}) \setminus V^{\{s\}}(A^{\{s+1\}}) = \emptyset$ then set $\hat{V}^{\{s+1\}} = V^{\{s\}}$. Otherwise, we represent every natural number z as $z = \langle i, \langle k, y \rangle \rangle$ for some numbers i, k, y. Choose the number z such that $z \in V^{\{s\}}(A^{\{s\}}) \setminus V^{\{s\}}(A^{\{s+1\}})$ with least k, say $z_0 = \langle i_0, \langle k_0, y_0 \rangle \rangle$. Although we do not know yet what $A_{i_0}^{\{s+1\}}$ will be, as this depends on what new axioms we will enumerate in $V^{\{s+1\}}$, it is quite possible that ultimately we will have:

$$\langle k_0, y_0 \rangle \in (A_{i_0}^{\{s\}} \setminus A_{i_0}^{\{s+1\}}) \cap \omega^{[k_0]}.$$

To ensure that the requirements $\mathcal{K}_{\langle i_0, j \rangle}$ for every j are satisfied, we need to enumerate $\omega^{[\geq k_0]} \upharpoonright s$ in A_j for every $j \neq i$. So we set:

$$\hat{V}^{\{s+1\}} = V^{\{s\}} \cup \{\langle \langle j, x \rangle, \emptyset \rangle \mid x \in \omega^{[\geq k_0]} \upharpoonright s \,\&\, j \neq i_0\}.$$

Note that for every i, we are adding finitely many axioms for elements in $\omega^{[i]}$. Hence, $\hat{V}^{\{s+1\}}$ is a computable set. Furthermore, for every i, we have $\hat{V}^{\{s+1\}}(A^{\{s+1\}})[i] \subseteq \omega \upharpoonright s$.

Step 2. *Satisfying the $\mathcal{N}$-requirements.*

For every $k = \langle i, e \rangle$ define $l(k,s) = l(A_i^{\{s\}}, W_e^{\{s\}}(B^{\{s\}}))$, the length of agreement between A_i and $W_e(B)$, measured at stage s. Here, W_e is approximated by its standard Σ_1^0 approximation. Choose the least $k \leq s$ such that $l(k,s) > \max\{l(k,t) \mid t < s\}$. In other words choose the least $k \leq s$ such that s is an expansionary stage for the requirement $\mathcal{N}_k$. We will call such stages s, k-expansionary. If there is no such number k, set $V^{\{s+1\}} = \hat{V}^{\{s+1\}}$ and end this stage.

Otherwise for the least k such that s is k-expansionary, say $k = \langle i, e \rangle$, we try to code the set A in the set A_i. We define

$$V^{\{s+1\}} = \hat{V}^{\{s+1\}} \cup \{\langle \langle i, \langle k, y \rangle \rangle, \{y\} \rangle \mid \langle k, y \rangle < s\}.$$

Note that again we are adding finitely many axioms to $V^{\{s+1\}}$. It follows that $V^{\{s+1\}}$ is computable and that for every i, $V^{\{s+1\}}(A^{\{s+1\}})[i] \subseteq \omega \upharpoonright s$.

This completes the construction.

We prove that the constructed set V satisfies all requirements in three steps.

PROPOSITION 2

For all $i \in \omega$ the sequence $\{A_i^{\{s\}}\}_{s<\omega}$ is a Δ_2^0 approximation.

PROOF. Fix i and a natural number x. We will prove that all axioms enumerated in V for $\langle i, x \rangle$ are enumerated at stages $s > x$ and are either valid at all but finitely many stages or invalid at all but finitely many stages. Fix an axiom $\langle\langle i,x\rangle,D\rangle$, enumerated in $V^{\{s+1\}}$ at stage s. If this axiom is enumerated under Step (1). of the construction then $x<s$ and $D=\emptyset$. As $V^{\{s+1\}}\subseteq V^{\{t\}}$ at all $t\geq s+1$ it follows that $x\in A_i^{\{t\}}$ at all $t\geq s+1$.

If the axiom is enumerated under Step (2). of the construction then $x=\langle k,y\rangle<s$, where k and y are natural numbers, and $D=\{y\}$. As $\{A^{\{s\}}\}_{s<\omega}$ is a Δ^0_2 approximation to A there is a stage s_y such that at all $t\geq s_y$ we have $A^{\{t\}}(y)=A(y)$ and hence if $A(y)=1$, the axiom is valid at all stages $t\geq s_y$ and if $A(y)=0$, the axiom is invalid at all stages $t\geq s_y$.

It follows that for all s, $A_i^{\{s\}}\subseteq\omega\upharpoonright s$ and that for all x, $\lim_s A_i^{\{s\}}(x)$ exists (by definition it is of course equal to $A_i(x)$). ■

Proposition 3
For every $i\neq j$ the sets A_i and A_j form a $\mathcal{K}$-pair.

Proof. Assume towards a contradiction that for some i and j the requirement $\mathcal{K}_{\langle i,j\rangle}$ is not satisfied, i.e. there is a stage s and numbers x and k such that:

$$x\in(A_i^{\{s\}}\setminus A_i^{\{s+1\}})\cap\omega^{[k]}\text{ and }\omega^{[\geq k]}\upharpoonright s\nsubseteq A_j.$$

Then $x=\langle k,y\rangle$ for some number y and:

$$\langle i,\langle k,y\rangle\rangle\in V^{\{s\}}(A^{\{s\}})\setminus V^{\{s+1\}}(A^{\{s+1\}}).$$

As $V^{\{s\}}(A^{\{s+1\}})\subseteq V^{\{s+1\}}(A^{\{s+1\}})$, it follows that:

$$\langle i,\langle k,y\rangle\rangle\in V^{\{s\}}(A^{\{s\}})\setminus V^{\{s\}}(A^{\{s+1\}}).$$

At stage s under Step (1). of the construction, we select $\langle i_0,\langle k_0,y_0\rangle\rangle$ as the number with least second coordinate that belongs to the set $V^{\{s\}}(A^{\{s\}})\setminus V^{\{s\}}(A^{\{s+1\}})$. Hence, $k_0\leq k$ and:

$$V^{\{s+1\}}\supseteq\hat{V}^{\{s+1\}}=V^{\{s\}}\cup\{\langle\langle j,z\rangle,\emptyset\rangle\mid z\in\omega^{[\geq k_0]}\upharpoonright s\,\&\,j\neq i_0\}.$$

If $i_0=i$ then $j\neq i_0$ and an axiom $\langle\langle j,z\rangle,\emptyset\rangle$ is enumerated in $V^{\{s+1\}}$ for every $z\in\omega^{[\geq k_0]}\upharpoonright s$. As $k_0\leq k$ and hence $\omega^{[\geq k_0]}\upharpoonright s\subseteq\omega^{[\geq k]}\upharpoonright s$ it follows that $\omega^{[\geq k]}\upharpoonright s\subseteq A_j$ contradicting our assumption.

If $i_0\neq i$ then, as $x\in\omega^{[\geq k_0]}$, the axiom $\langle\langle i,x\rangle,\emptyset\rangle$ is enumerated in $V^{\{s+1\}}$ and hence $x\in A_i^{\{s+1\}}$ which contradicts the assumption that $x\in A_i^{\{s\}}\setminus A_i^{\{s+1\}}$.

In both cases the assumption that $\mathcal{K}_{\langle i,j\rangle}$ is not satisfied leads to a contradiction and is therefore wrong. ■

Proposition 4
For every i, $A_i\nleq_e B$.

Proof. First, we note that by Proposition 2 and our choice of low approximation to B for every $k=\langle i,e\rangle$ we have that $W_e(B)=A_i$ if and only if there are infinitely many k-expansionary stages. Indeed, we have Δ^0_2 approximations to $W_e(B)$ and A_i hence for every n there is a stage s_n such that at all $t>s_n$ we have $A_i^{\{t\}}\upharpoonright n=A_i\upharpoonright n$ and $W_e^{\{t\}}(B^{\{t\}})\upharpoonright n=W_e(B)\upharpoonright n$. If $A_i=W_e(B)$ then for all n, $l(k,s_n)\geq n$, i.e. the length of agreement grows unboundedly with infinitely many expansionary stages. If $A_i\neq W_e(B)$ then there is a number n such that $A_i(n)\neq W_e(B)(n)$ and the length of agreement is bounded by n, $l(k,t)<n$ at all $t\geq s_{n+1}$.

Assume towards a contradiction that there is an $\mathcal{N}$-requirement which is not satisfied and let k be the least index such that $\mathcal{N}_k$ is not satisfied.

It follows that for all $m = \langle i_m, e_m \rangle < k$ the requirement $\mathcal{N}_m$ is satisfied and there is a stage s_0 such that all stages $t > s_0$ are not m-expansionary for any $m < k$. Hence, during the course of the whole construction each requirement $\mathcal{N}_m$, where $m < k$, adds only finitely many axioms to V. By Proposition 2, each such axiom is valid or invalid at all but finitely many stages. Let $s_1 \geq s_0$ be a stage such that at all $t > s_1$ each axiom added by a requirement $\mathcal{N}_m$, where $m < k$, does not change its state (i.e. it is valid at all $t > s_1$ or invalid at all $t > s_1$).

We now turn to Step (1) of the construction. If at stage $t > s_1$ an element z has the property $z \in V^{\{t\}}(A^{\{t\}}) \setminus V^{\{t\}}(A^{\{t+1\}})$ then $\langle z, \emptyset \rangle \notin V^{\{t\}}$ and an axiom for z enumerated under Step (2) of the construction is valid at stage t and invalid at stage $t+1$. By our choice of stage s_1 this axiom is enumerated by $\mathcal{N}_l$ where $l \geq k$. It follows that z can be represented as $z = \langle j_l, \langle l, y_l \rangle \rangle$; furthermore, l can be represented as $l = \langle j_l, e_l \rangle$. Hence, if at stage $t > s_1$ the number z with least second coordinate such that $z \in V^{\{t\}}(A^{\{t\}}) \setminus V^{\{t\}}(A^{\{t+1\}})$ has second coordinate k then it has first coordinate i. Otherwise z has second coordinate strictly larger than k. In both cases no more axioms of the form $\langle \langle i, \langle k, y \rangle \rangle, \emptyset \rangle$ are enumerated in $V^{\{t+1\}}$ at stages $t > s_1$.

Let D be the finite set of all y, such that $\langle \langle i, \langle k, y \rangle \rangle, \emptyset \rangle \in V$. We will prove that for every natural number y we have $y \in A$ if and only if $\langle k, y \rangle \in A_i$ for all $y \notin D$. Hence $A \leq_e A_i = W_e(B)$, contradicting the fact that $A \not\leq_e B$.

Fix $y \notin D$. The only axiom for $\langle i, \langle k, y \rangle \rangle$ in V (if any) is $\langle \langle i, \langle k, y \rangle \rangle, \{y\} \rangle$. Hence, if $y \notin A$ then $\langle k, y \rangle \notin A_i$. If $y \in A$ then let $s > s_1$ be a stage such that $y < s$ and s is k-expansionary. The assumption that $A_i = W_e(B)$ yields that there are infinitely many k-expansionary stages. Step 2 of the construction enumerates the axiom $\langle \langle i, \langle k, y \rangle \rangle, \{y\} \rangle$ in $V^{\{s+1\}}$ hence $y \in A_i$. □

■

Theorem 2 is now a direct application of Lemma 3 and Theorem 7.

Funding

The authors were supported by a Bulgaria National Science Fund Grant No. D002-258/18.12.08. H.G. was partially supported by the European Operational programme Human Resources Development through contract BGO051PO001/07/3.3-02/53 with the Bulgarian Ministry of Education. M.S. was supported by a Marie Curie European Reintegration Grant No. 239193 within the 7th European Community Framework Programme.

References

[1] S. Ahmad. Embedding the diamond in the Σ_2 enumeration degrees. *Journal of Symbolic Logic*, **50**, 195–212, 1991.

[2] C. Bianchini. *Bounding Enumeration Degrees*, PhD Thesis, University of Siena, 2000.

[3] S. B. Cooper. Enumeration reducibilty, nondeterministic computations and relative computability of partial functions. *Recursion Theory Week, Oberwolfach 1989*, Vol. 1432 of *Lecture Notes in Mathematics*, K. Ambos-Spies, G. Muler, and G. E. Sacks, eds, pp. 57–110. Springer, 1990.

[4] C. M. Harris. *On the Jump Classes of Noncuppable Enumeration Degrees*, Submitted for publication, 2009.

[5] I. Sh. Kalimullin. Definability of the jump operator in the enumeration degrees. *Journal of Mathematical Logic*, **3**, 257–267, 2003.

[6] A. H. Lachlan. Lower bounds for pairs of recursively enumerable degrees. *Proceedings of the London Mathematical Society*, **16**, 537—569, 1966.
[7] S. Lempp and A. Sorbi. Embedding finite lattices into the Σ^0_2-enumeration degrees. *Journal of Symbolic Logic*, **67**, 69–90, 2002.
[8] K. McEvoy and S. B. Cooper. On minimal pairs of enumeration degrees. *Journal of Symbolic Logic*, **50**, 983–1001, 1985.
[9] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Springer, 1987.
[10] A. Sorbi. The enumeration degrees of the Σ^0_2 sets. In *Complexity, Logic and Recursion Theory*, A. Sorbi, ed., pp. 303–330. Marcel Dekker, 1975.
[11] A. Sorbi. Embeddings into the enumeration degrees. In *Proceedings of the International Conference "Logic and Applications"*, honouring Yu. L. Ershov on his 60-th birthday; and of the International Conference on Mathematical Logic, honouring A. L. Malt'sev on his 90-th birthday anniversary and the 275-th anniversary of the Russian Academy of Sciences (Novosibirsk, Russia), S. Goncharov, ed., pp. 144–164. Novosibirsk State University, 2002.

Received 6 November 2009