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# Immunity properties of the s-degrees

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**Abstract.** We investigate immunity properties of the s-degrees. In particular we show that neither the immune nor the hyperimmune s-degrees are upwards closed since there exist $\Delta^0_2$ s-degrees $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$ such that $\mathbf{a}$ is hyperimmune, but $\mathbf{b}$ is immune free. We also show that there is no hyperhyperimmune $\Pi^0_2$ set $A$ such that $\overline{K} \leq_{\hat{\mathrm{s}}} A$, where $\overline{K}$ is the complement of the halting set and $\leq_{\hat{\mathrm{s}}}$ denotes the finite-branch version of s-reducibility.

**Keywords.** Immune set, hyperimmune set, hyperhyperimmune set, e-reducibility, s-reducibility.

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## 1 Introduction

Rozinas, [16], shows that the immune and hyperimmune enumeration degrees are upwards closed, namely if $\mathbf{a}$ is immune (hyperimmune) and $\mathbf{a} \leq_{\mathrm{e}} \mathbf{b}$, then so is $\mathbf{b}$. (Here $\leq_{\mathrm{e}}$ denotes enumeration reducibility, as defined below. We also recall that if $P$ is a property of sets, then we say that a degree has property $P$ if some set in the degree has property $P$.) The same holds for the Turing degrees: in fact, Jockusch [7], extends this upwards closure property to the cohesive Turing degrees as well. It is an open problem whether the hyperhyperimmune enumeration degrees are upwards closed. In this paper we consider a stronger version of enumeration reducibility known as s-reducibility, and we show that neither the immune nor the hyperimmune s-degrees are upwards closed, by exhibiting $\Delta^0_2$ s-degrees $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$ such that $\mathbf{a}$ is hyperimmune, but $\mathbf{b}$ does not contain any immune set. (Here $\leq_{\mathrm{s}}$ denotes s-reducibility, as defined below.) We also show that there is no hyperhyperimmune $\Pi^0_2$ set $A$ such that $\overline{K} \leq_{\hat{\mathrm{s}}} A$, where $\overline{K}$ is the complement of

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the halting set and $\leq_{\hat{s}}$ denotes the finite-branch version of s-reducibility, as defined below. In particular, it follows that $\deg_s(\overline{K})$ does not contain any $\Delta^0_2$ hyperhyperimmune set (already proved in [14]), and that $\deg_{\hat{s}}(\overline{K})$ is hyperhyperimmune free.

Our main references for computability theory are [4], or [15]. We only introduce here some of the notions and notations that are most commonly used in the paper. A set $A$ is *enumeration reducible* to a set $B$ (abbreviated by *e-reducible*, and denoted by $A \leq_e B$) if there exists a computably enumerable (abbreviated by c.e.) set $\Phi$ such that

$$A = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in \Phi \text{ and } D \subseteq B]\}$$

(where we identify finite sets with their canonical indices, and $\langle u, v\rangle$ denotes the image of $(u, v)$ under the usual Cantor pairing function from pairs of numbers to numbers): we write in this case $A = \Phi(B)$, so the c.e. set $\Phi$ can also be viewed as an operator on sets of numbers, called an *enumeration operator*, or simply an *e-operator*. The *e-degrees* are the equivalence classes of sets under the equivalence relation $\equiv_e$ generated by $\leq_e$. The e-degrees, under the partial ordering relation $\leq_e$ induced by the reducibility, form an upper semilattice with least element $\mathbf{0}_e$ consisting of the c.e. sets, and with supremum given by the usual join operation on sets.

Particular and important cases of e-operators are provided by the so-called s-operators: an e-operator $\Psi$ is an *s-operator* if for every $\langle x, D\rangle \in \Psi$ we have that $|D| \leq 1$, where $|X|$ denotes the cardinality of a given set $X$. Then we say that $A$ is *s-reducible* to $B$ (denoted by $A \leq_s B$), if there exists an s-operator $\Psi$ such that $A = \Psi(B)$. The *s-degrees* are the equivalence classes of sets under the equivalence relation $\equiv_s$ generated by $\leq_s$: we get again an upper semilattice with least element $\mathbf{0}_s$ consisting of the c.e. sets, and with supremum given by the usual join operation on sets. It is clear that $\leq_s$ is included in $\leq_e$, but not conversely: in fact it is known, [18], that every nonzero e-degree contains at least two s-degrees. Amongst the subreducibilities of $\leq_e$, s-reducibility is perhaps the most important and useful one. In most practical instances of a set $A$ being e-reducible to a set $B$, it is often the case that one can in fact show that $A \leq_s B$: this is perhaps due to the fact that the partial ordering $\leq_e$ naturally embeds into $\leq_s$, via the simple observation that $A \leq_e B$ if and only if $A^* \leq_s B^*$, where for a given set $X$, $X^* = \{D : D \text{ finite and } D \subseteq X\}$. (In fact, [8], the s-degree of $X^*$ is the greatest s-degree inside the e-degree of $X$.) Interest in s-reducibility (very often through its presentation as Q-reducibility, see Lemma1.2 below), derives also from its many applications to computability theory and general mathematics: for instance Q-reducibility plays a key role in Marchenkov's solution of Post's problem using Post's methods [10]; and has applications in the study of word problems (for instance, [1], [9]) and in abstract computational complexity (for instance, [2], [5]). For this, and more appropriate references, see the survey paper [12].

The e-operators and the s-operators can be effectively listed and computably approximated as c.e. sets: a computable approximation $\{\Psi_t\}_t$ to an e-operator $\Psi$ is a computable increasing sequence of finite sets such that $\Psi = \bigcup_t \Psi_t$. If $\Phi$ is an e-operator, then for every $x$ let $\mathrm{Ax}_\Phi(x) = \{\langle y, D\rangle \in \Phi : y = x\}$.

**Definition 1.1.** We say that an e-operator $\Phi$ is *finite* if for every $x$, $|\mathrm{Ax}_\Phi(x)| < \infty$. If $\mathrm{r} \in \{\mathrm{s}, \mathrm{e}\}$, we say that $A \leq_{\hat{\mathrm{r}}} B$ if there is a finite r-operator $\Phi$ such that $A = \Phi(B)$.

In other words, $\leq_{\hat{\mathrm{s}}}$ and $\leq_{\hat{\mathrm{e}}}$ are the *finite-branch* versions (using terminology from [3]) of s-reducibility and e-reducibility, respectively.

We also recall that a set $A$ is *Q-reducible* to a set $B$ (denoted by $A \leq_{\mathrm{Q}} B$) if there exists a computable function $f$ such that, for every $x$,

$$x \in A \Leftrightarrow W_{f(x)} \subseteq B.$$

We say in this case that the function $f$ *witnesses* that $A \leq_{\mathrm{Q}} B$. Q-reducibility was introduced by Tennenbaum, as quoted by Rogers [15]. It is easy to see, [5]:

**Lemma 1.2.** *If $B \neq \omega$ then $A \leq_{\mathrm{s}} B$ if and only if $\overline{A} \leq_{\mathrm{Q}} \overline{B}$, or equivalently if there exists a computable function $f$ such that, for every $x$,*

$$x \in A \Leftrightarrow W_{f(x)} \cap B \neq \emptyset.$$

*Proof.* See [5]. Moreover, the proof shows that from an s-operator $\Phi$ such that $A = \Phi(B)$ one can construct a suitable computable function $f$ such that for every $x$, $|W_{f(x)}| = |\mathrm{Ax}_\Phi(x)|$, and vice versa from a computable function $f$, one can construct a suitable s-operator $\Phi$ such that for every $x$, $|\mathrm{Ax}_\Phi(x)| = |W_{f(x)}|$. □

The previous lemma gives a useful characterization of s-reducibility, which will be often used in this paper. In particular, we will refer to the following definition.

**Definition 1.3.** We say that a computable function $f$ *witnesses* that $A \leq_{\mathrm{s}} B$, if $f$ witnesses that $\overline{A} \leq_{\mathrm{Q}} \overline{B}$.

The following useful fact is a refinement of a result in [13, Theorem 4], therein stated when both $A$ and $B$ are $\Delta^0_2$ sets.[1] We recall that a $\Delta^0_2$ approximation to

[1] This extension of [13, Theorem 4] has been also noticed independently by C. Harris.

a set $A$ is a computable sequence of sets $\{A_s\}_s$ such that for every $x$, $A(x) = \lim_s A_s(x)$ (identifying sets with their characteristic functions); on the other hand, we say that a computable sequence of sets $\{A_s\}_s$ is a $\Sigma^0_2$ approximation to $A$, if $A = \{x : \liminf_s A_s(x) = 1\}$.

**Theorem 1.4.** *If $A \leq_{\mathrm{e}} B$, with $A \in \Delta^0_2$ and $B \in \Sigma^0_2$, then $A \leq_{\hat{\mathrm{e}}} B$.*

*Proof.* Suppose that $A \leq_{\mathrm{e}} B$, with $A \in \Delta^0_2$ and $B \in \Sigma^0_2$. Let $\Phi$ be an e-operator such that $A = \Phi(B)$. Start with a $\Delta^0_2$-approximation $\{A_s\}_s$ to $A$, with a $\Sigma^0_2$-approximation $\{B_s\}_s$ to $B$, and with a computable approximation $\{\Phi_s\}_s$ to the e-operator $\Phi$, where we recall that each $\Phi_s$ is finite. We show how to construct a finite e-operator $\Psi$ such that $A = \Psi(B)$.

For every $x$ we give instructions for enumerating, step by step and uniformly in $x$, pairs $\langle x, D\rangle \in \Psi$. We also use a parameter $D(x, s)$ denoting, if defined, a finite set.

Step 0) Do not enumerate any pair; let $D(x, 0)$ be undefined.

Step $s + 1$) We distinguish two cases:

Case 1) if $x \notin A_s$ then we do not enumerate any pair, and we do not change $D(x, s)$ (whether undefined or not);

Case 2) if $x \in A_s$ and $D(x, s)$ is defined and $D(x, s) \subseteq B_{s+1}$, then do not enumerate any pair and do not change $D(x, s)$. Otherwise, let $D$ be such that $\langle x, D\rangle \in \Phi_{s+1}$ and the age of $D$ is least among all such finite sets, where the *age* of $D$ is the least $t$ such that for all $u$ with $t \leq u \leq s + 1$, we have that $D \subseteq B_u$; enumerate $\langle x, D\rangle \in \Psi$ at step $s + 1$, and let $D(x, s + 1) = D$; if no such $D$ exists then do not enumerate any pair, and let $D(x, s + 1)$ be undefined.

This ends the construction. Let $\Psi$ be the e-operator obtained by taking all pairs $\langle x, D\rangle$ which are enumerated this way. Clearly $\Psi \subseteq \Phi$, and thus if $x \notin A$ then $x \notin \Psi(B)$. If $x \notin A$ then after some stage we stop enumerating axioms of the form $\langle x, D\rangle \in \Psi$, so $|\mathrm{Ax}_\Psi(x)| < \infty$. If $x \in A$, whence $x \in \Phi(B)$, then there is an axiom $\langle x, E\rangle \in \Phi$ with $E \subseteq B$ and the age of $E$ becomes constant and least, say $t$: if at bigger and bigger stages $s \geq t$, we keep enumerating axioms $\langle x, D\rangle \in \Psi$ such that at a later $u$, $D \nsubseteq B_u$, then we eventually enumerate $\langle x, E\rangle \in \Psi$, after which we do not enumerate any more axioms of the form $\langle x, D\rangle \in \Psi$. In conclusion, $\Psi$ is finite and $A = \Psi(B)$. □

**Theorem 1.5.** *If $A \leq_{\mathrm{s}} B$, $A \in \Delta^0_2$ and $B \in \Sigma^0_2$ then $A \leq_{\hat{\mathrm{s}}} B$, or equivalently, there is a computable function $f$ such that for every $x$, $W_{f(x)}$ is finite and*

$$x \in A \Leftrightarrow W_{f(x)} \cap B \neq \emptyset.$$

*Proof.* This is an immediate consequence of Theorem 1.4 and (the proof of) Lemma 1.2. □

We conclude this section with the following observation about the structure of $\hat{\mathrm{r}}$-degrees within r-degrees, for $\mathrm{r} \in \{\mathrm{e}, \mathrm{s}\}$. We say that an r-degree is $\hat{\mathrm{r}}$-*contiguous* if it consists of just one $\hat{\mathrm{r}}$-degree.

**Theorem 1.6.** *If* $\mathrm{r} \in \{\mathrm{e}, \mathrm{s}\}$*, then every* $\Sigma^0_2$ r-*degree containing some* $\Delta^0_2$ *set contains a least* $\hat{\mathrm{r}}$-*degree, comprising all* $\Delta^0_2$ *sets lying in the given r-degree. As a consequence, a* $\Delta^0_2$ *r-degree consists of only* $\Delta^0_2$ *sets if and only if it is* $\hat{\mathrm{r}}$-*contiguous.*

*Proof.* If $A \in \deg_{\hat{\mathrm{r}}}(a)$ and $A$ is $\Delta^0_2$, then for every $B \in \deg_{\mathrm{r}}(A)$ by the previous theorems we have $A \leq_{\hat{\mathrm{r}}} B$. Moreover if $B \in \Delta^0_2$ then $A \equiv_{\hat{\mathrm{r}}} B$. □

## 2 Immunity properties of the s-degrees

In this section we show that the immune $\Delta^0_2$ s-degrees and the hyperimmune $\Delta^0_2$ s-degrees are not upwards closed. We recall that an infinite set $A$ is called:

(i) *immune* if it does not contain any infinite c.e. set;

(ii) *hyperimmune* if for every disjoint strong array $\{D_{f(x)}\}_x$ (meaning a sequence of finite sets given by a computable function $f$ listing their canonical indices, such that $D_{f(x)} \cap D_{f(y)} = \emptyset$ if $x \neq y$), there exists $x$ such that $D_{f(x)} \cap A = \emptyset$;

(iii) *hyperhyperimmune* if for every disjoint weak array $\{W_{f(x)}\}_x$ (meaning a sequence of finite sets given by a computable function $f$ listing c.e. indices for them, such that $W_{f(x)} \cap W_{f(y)} = \emptyset$ if $x \neq y$), there exists $x$ such that $W_{f(x)} \cap A = \emptyset$.

The following definition arises from the notion of a nowhere simple set, due to Shore, [17]: the complement of a noncomputable nowhere simple set is nowhere immune.

**Definition 2.1.** We say that a set $A$ is *nowhere immune* if $A$ is not c.e. and for every c.e. set $B$ with $A \cap B$ infinite, there is an infinite c.e. set $W$ such that $W \subseteq A \cap B$.

We can now show:

**Lemma 2.2.** *Let* $A$ *be a nowhere immune* $\Sigma^0_2$ *set, and let* $B \in \Delta^0_2$*. If* $B \leq_{\mathrm{s}} A$*, then* $B$ *is nowhere immune.*

*Proof.* Assume that $A$ and $B$ are as in the statement of the lemma. By Theorem 1.5, let $f$ be a computable function such that for every $x$, $W_{f(x)}$ is finite and

$$x \in B \Leftrightarrow W_{f(x)} \cap A \neq \emptyset.$$

Let $C$ be an infinite c.e. set such that $B \cap C$ is infinite. Consider the set

$$X = \Big(\bigcup_{x \in C} W_{f(x)}\Big) \cap A.$$

We distinguish two cases:

(i) $X$ c.e.: in this case we have

$$B \cap C = \{x : W_{f(x)} \cap X \neq \emptyset\} \cap C,$$

showing that $B \cap C$ itself is c.e.;

(ii) $X$ not c.e.: then $X$ is infinite and since $A$ is nowhere immune, let $W \subseteq X$ be an infinite c.e. set. Then the set

$$\{x : W_{f(x)} \cap W \neq \emptyset\} \cap C$$

is an infinite c.e. subset of $B \cap C$: infinity follows from the fact that the finite sets $\{W_{f(x)} : x \in B \cap C\}$ cover $W$. □

**Theorem 2.3.** *Let $A \in \Delta^0_2$, and $B, C \in \Sigma^0_2$ be such that $A$ is immune, $C$ is nowhere immune and $A \leq_s B \oplus C$. Then $A \leq_s B$.*

*Proof.* We first show two preliminary lemmata.

**Lemma 2.4.** *Let $A \leq_s B \oplus C$, $A$ immune, $C$ nowhere immune, and suppose that the reduction is witnessed by a computable function $f$ such that, for every $x$, $W_{f(x)}$ is finite and*

$$x \in A \Leftrightarrow W_{f(x)} \cap (B \oplus C) \neq \emptyset.$$

*Suppose that $W$ is a c.e. set such that $|W \cap A| = \infty$, and let*

$$V = \bigcup_{x \in W} W_{f(x)}.$$

*Then*

(i) $|V \cap (\emptyset \oplus C)| < \infty$.

(ii) $|V \cap (B \oplus \emptyset)| = \infty$.

*Proof.* We first observe that the set

$$E = V \cap (B \oplus C)$$

is infinite, in fact not even c.e. Indeed, if $E$ were c.e. then the following set $\hat{W}$,

$$\hat{W} = \{x : W_{f(x)} \cap E \neq \emptyset\},$$

would be c.e. and $W \cap A \subseteq \hat{W} \subseteq A$, giving an infinite c.e. subset of $A$, contrary to the fact that $A$ is immune. In order to show the claim it is therefore sufficient to show (i). Assume by contradiction that (i) does not hold, i.e.

$$|V \cap (\emptyset \oplus C)| = \infty.$$

Then clearly

$$|\{x : 2x + 1 \in V \cap (\emptyset \oplus C)\}| = \infty$$

and thus the set

$$\{x : 2x + 1 \in V\} \cap C$$

is infinite as well. Since $\{x : 2x + 1 \in V\}$ is a c.e. set and $C$ is nowhere immune, there is an infinite c.e. set $\tilde{W}$ such that

$$\tilde{W} \subseteq \{x : 2x + 1 \in V\} \cap C.$$

We have

$$W^* = \{2x + 1 : x \in \tilde{W}\} \subseteq V \cap (\emptyset \oplus C)$$

and $W^*$ is an infinite c.e. set, but then, since the sets $W_{f(x)}$ are finite and cover $W^*$, it follows that the set

$$\{y : W_{f(y)} \cap W^* \neq \emptyset\}$$

is an infinite c.e. subset of $A$, contrary to immunity of $A$. □

**Lemma 2.5.** *Suppose that $A \leq_s B \oplus C$ as witnessed by a computable function $f$, and let*

$$V = \bigcup_{x \in \omega} W_{f(x)}.$$

*If $V \cap (\emptyset \oplus C)$ is c.e. then $A \leq_s B$.*

*Proof.* Suppose that $f$ and $V$ are as in the statement of the Lemma. Without loss of generality we may suppose that $B \neq \emptyset$. Let $R = V \cap (\emptyset \oplus C)$ and take $b \in B$: define

$$W_{g(x)} = \{y : 2y \in W_{f(x)} \text{ or } [y = b \text{ and } W_{f(x)} \cap R \neq \emptyset]\}.$$

Then for every $x$,

$$x \in A \Leftrightarrow W_{g(x)} \cap B \neq \emptyset.$$ □

Let us now go back to the proof of the theorem. Let $A, B, C$ be as an in the statement of the theorem, and by Theorem 1.5, let $f$ be a computable function such that for every $x$, $W_{f(x)}$ is finite and

$$x \in A \Leftrightarrow W_{f(x)} \cap (B \oplus C) \neq \emptyset.$$

In Lemma 2.4 take $W = \omega$, then it follows that

$$R = V \cap (\emptyset \oplus C)$$

is finite, hence c.e. By Lemma 2.5 it then follows that $A \leq_{\mathrm{s}} B$. □

**Remark 2.6.** From the proof of Theorem 2.3 we also get: if in addition $A \leq_{\mathrm{bs}} B \oplus C$ then $A \leq_{\mathrm{bs}} B$, where $\leq_{\mathrm{bs}}$ is the bounded version of $\leq_{\mathrm{s}}$, i.e. $X \leq_{\mathrm{bs}} Y$ if there exist a finite s-operator $\Phi$, and a number $n$, such that $X = \Phi(Y)$ and for every $x$, $|\mathrm{Ax}_{\Phi}(x)| \leq n$.

An important consequence of Theorem 2.3 is:

**Theorem 2.7.** *Let $A \in \Delta^0_2$ and $B \in \Sigma^0_2$ be such that $A$ is not nowhere immune, $B$ is nowhere immune and $B \not\leq_{\mathrm{s}} A$. Then the s-degree of the set $A \oplus B$ contains neither nowhere immune sets nor $\Delta^0_2$ immune sets.*

*Proof.* Let $A$ and $B$ be as above, and suppose that $C \equiv_{\mathrm{s}} A \oplus B$. The set $C$ cannot be nowhere immune since otherwise from $A \leq_{\mathrm{s}} C$ it would follow that $A$ is nowhere immune by Lemma 2.2. If $C$ is $\Delta^0_2$ then $C$ cannot be immune either since by Theorem 2.3 it would follow that $C \leq_{\mathrm{s}} A$, but $B \leq_{\mathrm{s}} C$, so $B \leq_{\mathrm{s}} A$, a contradiction. □

It is well known that one can define a jump-operation (e-jump) on the e-degrees. McEvoy and Cooper [11] have characterized the e-low e-degrees (i.e. the e-degrees whose jump is the least possible jump) as follows:

**Lemma 2.8.** *The following are equivalent of an e-degree* **a***:*

(i) **a** *is* e-low*;*

(ii) **a** *contains only $\Delta^0_2$ sets;*

(iii) *all e-degrees* **b** $\leq_{\mathrm{e}}$ **a** *contain only $\Delta^0_2$ sets;*

(iv) *for any fixed uniform computable approximation $\{\Phi_{e,s}\}_{e,s}$ of the enumeration operators $\{\Phi_e\}_e$, each set $A$ in* **a** *can be equipped with a $\Delta^0_2$ approximation $\{A_s\}_s$ such that for every $e, x$, $\lim_s \Phi_{e,s}(A_s)(x)$ exists.*

We suppose to fix once and for all a uniform computable approximation of the e-operators $\{\Phi_{e,s}\}_{e,s}$. A $\Delta^0_2$ approximation $\{A_s\}_s$ (relative to $\{\Phi_{e,s}\}_{e,s}$) as in Lemma 2.8(iv) will be called an *e-low* approximation. A set $A$ is called *e-low* if its e-degree is e-low.

**Corollary 2.9.** *If a low e-degree* **a** *contains an immune set $A$ and a nowhere immune set $B$ with $B \not\leq_{\mathrm{s}} A$, then* **a** *contains an s-degree that contains neither immune nor nowhere immune sets.*

*Proof.* Let **a** be e-low, and let $A, B \in$ **a** be such that $A$ is immune, $B$ is nowhere immune, and $B \not\leq_{\mathrm{s}} A$. Then $\deg_{\mathrm{s}}(A \oplus B)$ is contained in **a**, and by Theorem 2.7 $\deg_{\mathrm{s}}(A \oplus B)$ contains neither nowhere sets nor immune sets, since by e-lowness $\deg_{\mathrm{s}}(A \oplus B)$ contains only $\Delta^0_2$ sets. □

One can also define a jump operation on the s-degrees (see for instance [13]), and derive a characterization of the low s-degrees similar to that in Lemma 2.8. Without entering into the details of the definition of the s-jump, we directly take the following as the definition of an s-low s-degree:

**Definition 2.10.** An s-degree is *s-low* if and only if it contains only $\Delta^0_2$ sets.

In view of Theorem 1.6, we may observe:

**Corollary 2.11.** *For* $\mathrm{r} \in \{\mathrm{s}, \mathrm{e}\}$*, if an r-degree* **a** *is r-low then* **a** *consists of only one* $\hat{\mathrm{r}}$*-degree.*

*Proof.* By Theorem 1.6. □

Following Lemma 2.8, it is easy to see, [13], that **a** is *s-low* if and only if all sets $A$ in **a** can be equipped with a $\Delta^0_2$ approximation $\{A_s\}_s$ (called an *s-low approximation*) such that for every $e, x$, $\lim_s \Psi_{e,s}(A_s)(x)$ exists, where $\{\Psi_{e,s}\}_{e,s}$ is some fixed uniform computable approximation to the s-operators. We fix our uniform computable approximation to the s-operators $\{\Psi_{e,s}\}_{e,s}$ as follows: we start up with some uniform computable approximation $\{\Omega_{e,s}\}_{e,s}$ to all s-operators; let $l$ and $r$ be computable functions such that

$$\Omega_{l(z)} = \{\langle x, D\rangle : \langle x, D \oplus \emptyset\rangle \in \Omega_z\}$$
$$\Omega_{r(z)} = \{\langle x, E\rangle : \langle x, \emptyset \oplus E\rangle \in \Omega_z\};$$

then define

$$\Psi_{e,s} = \begin{cases} \Omega_{z,s}, & \text{if } e = 3z; \\ \Omega_{l(z),s}, & \text{if } e = 3z+1; \\ \Omega_{r(z),s}, & \text{if } e = 3z+2. \end{cases}$$

**Lemma 2.12.** *The s-low s-degrees form an ideal, in particular if* $\deg_s(A)$ *and* $\deg_s(B)$ *are s-low, then so is* $\deg_s(A \oplus B)$.

*Proof.* Let $A$ and $B$ be s-low, and take s-low $\Delta^0_2$ approximations $\{A_s\}_s$ and $\{B_s\}_s$ to $A$ and $B$ respectively, witnessing this fact. We aim at showing that if $C \leq_s A \oplus B$ then $C \in \Delta^0_2$. Assume that $C = \Psi_{3z}(A \oplus B)$ for some s-operator $\Psi_{3z}$ (recall that $\{\Psi_{3z}\}_z$ is a listing of all s-operators), and that $\lim_s \Psi_{3z,s}(A_s \oplus B_s)(x)$ does not exist. Since $\{A_s\}_s$ and $\{B_s\}_s$ are $\Delta^0_2$ approximations, this means that there are infinitely many axioms $\langle x, D \oplus E\rangle \in \Psi_{3z}$ each of which applies at finitely many stages but $D \oplus E \not\subseteq A \oplus B$. But since $\Psi_{3z}$ is an s-operator (thus $\langle x, D \oplus E\rangle \in \Psi_{3z}$ implies $D = \emptyset$ or $E = \emptyset$), by definitions of $\Psi_{3z+1,s}$ and $\Psi_{3z+2,s}$ it follows that there exist infinitely many axioms $\langle x, D\rangle \in \Psi_{3z+1}$ each of which applies at finitely many stages but $D \not\subseteq A$, or there exist infinitely many axioms $\langle x, E\rangle \in \Psi_{z+2}$ which apply at some stage but $E \not\subseteq B$. In the former case we have that $\lim_s \Psi_{3z+1,s}(A_s)(x)$ does not exist, and in the latter case we have that $\lim_s \Psi_{3z+2,s}(B_s)(x)$ does not exist, contrary to the assumption that we work with s-low approximations to $A$ and $B$. □

The following is the final preliminary result that we need for the main result of this section:

**Lemma 2.13.** *For every s-low set* $A$ *there exists an e-low set* $B$ *such that* $B$ *is nowhere immune, and* $B \not\leq_s A$.

*Proof.* Suppose that $A$ is given with the required properties, and let $\{A_s\}_s$ be an s-low approximation to $A$. We build an e-low approximation $\{B_s\}_s$ to a set $B$ such that $B$ is nowhere immune and $B \not\leq_s A$.

We aim at satisfying the following requirements:

$$\begin{aligned} P_e &: |W_e| = \infty \Rightarrow (\exists V)[V \text{ c.e. and } V \subseteq W_e \cap B \text{ and } |V| = \infty] \\ I_e &: B \neq \Psi_e(A) \\ L_{\langle e,x\rangle} &: \lim_s \Phi_{e,s}(B_s)(x) \text{ exists,} \end{aligned}$$

where $\{\Psi_e\}_e$ and $\{\Phi_e\}_e$ are effective listings of the s-operators and the e-operators, respectively, equipped with their fixed uniform computable approximations.

Notice that satisfaction of all $I_e$ makes $B \not\leq_s A$, in particular $B$ is not c.e.; satisfaction of all requirements $P_e$ (together with the fact that $B$ is not c.e.) makes $B$ nowhere immune; and finally satisfaction of all $L_{\langle e,x\rangle}$ makes $B$ e-low.

In the course of the construction, at stage $s$ we define $B_s$, and the values of parameters $w(e, n, s)$, $u(e, s)$, and $D(e, x, s)$: we say that we *reset* the parameters

$u(e,s)$ and $D(e,x,s)$ if we define them to be undefined. If not otherwise specified each parameter retains the same value as at the previous stage.

Step 0. Let $B_0 = \emptyset$; reset all parameters.

Step $s+1$. By substages: At substage $i < s$, we distinguish the following cases:

(i) $i = 3e$ (we work towards satisfying $P_e$): assume that $n$ is the least number such that $w(e,n)$ is undefined. If all $u(j,s)$ with $j \leq e+n$ are defined and there exists a least $w \in W_{e,s}$ such that $w > u(j,s)$ for all such $j$, then let $w(e,n,s+1) = w$ and let $w \in B_{s+1}$. Reset all parameters $u(j,s)$ with $j > e+n$, and move to Step $s+2$. Otherwise move to substage $i+1$.

(ii) $i = 3e+1$ (we work towards satisfying $I_e$): if $u(e,s)$ is undefined, then choose a big number $u$ (never used so far in the construction), let $u(e,s+1) = u$, let $u \in B_{s+1}$ and move to substage $i+1$. Otherwise, let $u(e,s) \in B_{s+1}$ if and only if $u(e,s) \notin \Psi_{e,s}(A_s)$. If $u(e,s) \in B_s \setminus B_{s+1}$ then reset all $D(j,x,s)$ with $e < \langle j,x \rangle$ and move to Step $s+2$; otherwise go to substage $i+1$.

(iii) $i = 3\langle e,x \rangle + 2$ (we work towards satisfying $L_{\langle e,x \rangle}$): if $D(e,x,s)$ is undefined and there exists $D$ such that $x \in \Phi_{e,s}(D)$, and $D \cap \{u(j,s) : j \leq \langle e,x \rangle\} \cap \overline{B_s} = \emptyset$, then choose the least such $D$, set $D = D(e,x,s+1)$, let $D \subseteq B_{s+1}$, reset all parameters $u(j,s)$ with $j > \langle e,x \rangle$, and go to Step $s+2$. Otherwise, move to substage $i+1$.

If $i+1 = s$ then we move to Step $s+2$.
Finally let $B = \{x : (\exists t)(\forall s \geq t)[x \in B_s]\}$. This ends the construction.

We now show that the construction works. First of all we notice that each $w(e,n,s)$ never changes after being defined for the first time. Also, one can argue by induction on $j$ that $u(j) = \lim_s u(j,s)$ exists: assume that this is true of every $i < j$, and assume also that $t$ is a stage such that $B_s(u(i)) = B_t(u(i))$ for all $s \geq t$ and $i < j$. After its definition $u(j,s)$ can be set undefined again either because of some pair $e,n$ with $e+n < j$ (due to the action of the strategy for $P_e$ in relation to the definition of $w(e,n)$ for the first time), but this can happen only once for each such pair $e,n$; or $u(j,s)$ can be set again undefined because of some number $\langle e,x \rangle < j$ and a new definition of $D(e,x,s)$ (due to the action of the strategy for $L_{\langle e,x \rangle}$), but at stages $s > t$, $D(e,x,s)$ can be redefined at most once: indeed, at $s+1$ with $s \geq t$, no $i < \langle e,x \rangle$ resets $D(e,x,s)$, since there is no change $B_{s+1}(u(i)) \neq B_s(u(i))$, and on the other hand $D(e,x,s+1)$ does not change any more if defined for the first time. So eventually $u(j,s) = u(j)$ remains unchanged, and by s-lowness of $A$, $\lim_s \Psi_{j,s}(u(j))$ exists, so there is some $t' \geq t$ such that $B_s(u(j))$ never changes at stages $s \geq t'$. This shows also that the requirement $I_j$ is satisfied since $\Psi_{j,s}(A_s)(u(j))$ can change only finitely many times, and at each change we change $B_s(u(j))$ accordingly in order to diagonalize: eventually we get $B(u(j)) \neq \Psi_j(A)(u(j))$.

To show that each requirement $P_e$ is satisfied, suppose that we have already defined $w(e,m)$ for every $m < n$. Let $t$ be a stage such that no $u(j,s) = u(j)$ changes at any stage $s \geq t$, for all $j \leq e + n$. If $W_e$ is infinite, then at some stage $s \geq t$ there is a number $w \in W_{e,s}$ such that $w > u(j)$ for all such $j$, so we can define the final value $w(e,n)$ to be such a $w$, and let $w(e,n) \in B$. It follows that if $W_e$ is infinite then $B$ contains an infinite c.e. subset $V \subseteq W_e$, namely $V = \{w(e,n) : n \in \omega\}$.

It remains to show that each $L_{\langle e,x\rangle}$ is satisfied. Suppose that $t$ is the least stage so that at all $s \geq t$, each $u(j,s)$, with $j \leq \langle e,x\rangle$, has reached its limit $u(j)$, and $B_s(u(j))$ does not change. If there are infinitely many stages $s$ such that $x \in \Phi_{e,s}(B_s)$ then at the first such stage $s \geq t$ we can find a finite set $D$ such that $x \in \Phi_e(D)$ and $D \cap \{u(j) : j \leq \langle e,x\rangle\} \cap \overline{B} = \emptyset$: as argued earlier, this set $D$ is the final value of $D(e,x,s)$, and the construction ensures that at all big enough stages $s$ we have that $D \subseteq B_s$, giving that $x \in \Phi_{e,s}(B_s)$. □

**Theorem 2.14.** *There exist $\Delta^0_2$ s-degrees* **a** *and* **b** *such that* $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$*, with* **a** *hyperimmune and* **b** *immune-free. In fact, for every s-low and immune* **a***, there exists a $\Delta^0_2$ immune-free s-degree* **b** *such that* $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$.

*Proof.* If $A$ is s-low and immune, by the previous lemma let $B$ be e-low (hence s-low) such that $B$ is nowhere immune and $B \not\leq_{\mathrm{s}} A$. From Lemma 2.12, it follows that $A \oplus B$ is s-low, hence all sets $C \in \deg_{\mathrm{s}}(A \oplus B)$ are $\Delta^0_2$. Then, by Theorem 2.7 we can conclude that $\deg_{\mathrm{s}}(A \oplus B)$ is immune-free. This shows that for every s-low and immune **a**, there exists a $\Delta^0_2$ immune-free s-degree **b** such that $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$. To conclude the proof of the theorem, let for instance $C$ be e-low and non c.e.: then it is known that the set $A = K_C$ is hyperimmune (see for instance [6]: recall that $K_C = \{x : x \in \Phi_x(C)\}$) and $A \equiv_{\mathrm{e}} C$, hence $A$ is e-low, and therefore s-low. Take $\mathbf{a} = \deg_{\mathrm{s}}(A)$, thus $\mathbf{a} = \deg_{\mathrm{s}}(A)$ is hyperimmune, and by the above argument there exists an immune-free s-degree **b** such that $\mathbf{a} \leq_{\mathrm{s}} \mathbf{b}$. □

## 3 The s-degree of $\overline{K}$

In this final section we take a look at the complete s-degree, that is the s-degree of $\overline{K}$, where $K$ is any creative set. We immediately observe:

**Fact 3.1.** $\deg_{\mathrm{s}}(\overline{K})$ is hyperimmune.

*Proof.* Consider the set $K_{\overline{K}} = \{x : x \in \Phi_x(\overline{K})\}$ (see also proof of Theorem 2.14). As observed in [6], $K_{\overline{K}}$ is hyperimmune. Moreover, [11], $\overline{K} \leq_1 K_{\overline{K}}$, and as the latter set is $\Sigma^0_2$ we also have $K_{\overline{K}} \leq_s \overline{K}$, hence $K_{\overline{K}} \in \deg_s(\overline{K})$. □

Unfortunately, we do not know if $\deg_s(\overline{K})$ is hyperhyperimmune, although we conjecture that it is not so. However, we can prove:

**Theorem 3.2.** *Let $A$ be a $\Pi^0_2$ set and $\overline{K} \leq_{\hat{s}} A$. Then $A$ is not hyperhyperimmune.*

*Proof.* Let $f$ be a computable function such that for every $x$, $W_{f(x)}$ is finite and

$$x \in \overline{K} \Leftrightarrow W_{f(x)} \cap A \neq \emptyset.$$

Suppose that $r$ is a computable function such that the partial computable function $\varphi_{r(i)}$, with index $r(i)$, enumerates $W_i \cap K$ without repetitions, and its domain is an initial segment of $\omega$, see [15, Corollary 5.V(d)]. If the index $i$ is clear from the context, for $j$ in the domain of $\varphi_{r(i)}$ denote $a_j = \varphi_{r(i)}(j)$. Then it is easy to see that there is a computable function $\tau$ of two variables such that for all $i, n$,

$$W_{\tau(i,n)} = \begin{cases} W_i \setminus \{a_0, \ldots, a_{n-1}\}, & \text{if } |W_i \cap K| \geq n; \\ \emptyset, & \text{if } |W_i \cap K| < n. \end{cases}$$

By induction, we now construct a computable function $g$. Suppose that we have already defined $g(0), \ldots, g(n)$ so that, for all $x, y \leq n$,

(i) $W_{g(x)} \cap A \neq \emptyset$;

(ii) $x \neq y \Rightarrow W_{g(x)} \cap W_{g(y)} \cap A = \emptyset$;

(iii) $W_{g(x)}$ is finite.

(It is trivial to observe that we can define $g(0)$ with the above properties.)

Step $n+1$. It is well known that for every $n$,

$$n \in (W_n \setminus \overline{K}) \cup (\overline{K} \setminus W_n):$$

this property states that $\overline{K}$ is completely productive via the identity function. (Completely productive sets are described in [15]; the complement of every creative set is completely productive.) Consider now the c.e. set

$$W_{\alpha(n)} = \left\{x : (\exists s \geq x)\left[W_{f(x),s} \cap A_s \cap \left(\bigcup_{i \leq n} W_{g(i)}\right) \neq \emptyset\right]\right\},$$

where $\{\overline{A_s}\}$ is a $\Sigma^0_2$ approximation to $\overline{A}$.

**Lemma 3.3.** *$W_{\alpha(n)} \cap K$ is finite.*

*Proof.* Suppose that $W_{\alpha(n)} \cap K$ is infinite. Then there is $y$ such that for infinitely many $s$ and $x \in K$,

$$y \in W_{f(x),s} \cap A_s \cap \Big(\bigcup_{i \leq n} W_{g(i)}\Big).$$

But for all $x \in K$, $W_{f(x)} \subseteq \overline{A}$, and therefore $y \in \overline{A}$. Then by $\Sigma^0_2$-ness of $\overline{A}$, we have that there exists $t$ such that for all $s \geq t$, $y \notin A_s$, a contradiction. □

Define

$$W_{g(n+1)} = \bigcup_{j \leq |W_{\alpha(n)} \cap K|} W_{f(\tau(\alpha(n),j))}.$$

**Lemma 3.4.** *For all $k$, $1 \leq k \leq n$,*

$$W_{g(k)} \cap W_{g(n+1)} \cap A = \emptyset.$$

*Proof.* Let $k$, $1 \leq k \leq n$, be such that

$$W_{g(k)} \cap W_{g(n+1)} \cap A \neq \emptyset.$$

Then there is $j$, $0 \leq j \leq |W_{\alpha(n)} \cap K|$, such that

$$W_{g(k)} \cap W_{f(\tau(\alpha(n),j))} \cap A \neq \emptyset.$$

Since the identity function is a completely productive function for $\overline{K}$, we have

$$\tau(\alpha(n), j) \in (K \cap W_{\tau(\alpha(n),j)}) \cup (\overline{K} \setminus W_{\tau(\alpha(n),j)}).$$

If $\tau(\alpha(n), j) \in K \cap W_{\tau(\alpha(n),j)}$ then $W_{f(\tau(\alpha(n),j))} \subseteq \overline{A}$ and

$$W_{g(k)} \cap W_{f(\tau(\alpha(n),j))} \cap A = \emptyset.$$

a contradiction.

If $\tau(\alpha(n), j) \in \overline{K} \setminus W_{\tau(\alpha(n),j)}$, then by definition of $\tau(\alpha(n), j)$, $\tau(\alpha(n), j) \notin W_{\alpha(n)}$, and thus

$$W_{f(\tau(\alpha(n),j))} \cap A \cap \Big(\bigcup_{i \leq n} W_{g(i)}\Big) = \emptyset.$$

It follows that

$$W_{g(k)} \cap W_{f(\tau(\alpha(n),j))} \cap A = \emptyset,$$

a contradiction, again. □

**Lemma 3.5.** $W_{g(n+1)} \cap A \neq \emptyset$.

*Proof.* Since the identity function is a completely productive function for $\overline{K}$, for $j = |W_{\alpha(n)} \cap K|$ we have $W_{\tau(\alpha(n),j} \subseteq \overline{K}$, thus $\tau(\alpha(n), j) \in \overline{K} \setminus W_{\tau(\alpha(n),j)}$, giving

$$W_{f(\tau(\alpha(n),j)} \cap A \neq \emptyset. \qquad \square$$

By generalizing the Reduction Principle for c.e. sets, let $\beta$ be a computable function such that for all $x, y$,

$$W_{\beta(x)} \subseteq W_{g(x)},$$
$$x \neq y \Rightarrow W_{\beta(x)} \cap W_{\beta(y)} = \emptyset,$$
$$\bigcup_{x \in \omega} W_{\beta(x)} = \bigcup_{x \in \omega} W_{g(x)}.$$

Then for all $x, y$,

$$W_{\beta(x)} \cap A \neq \emptyset,$$
$$x \neq y \Rightarrow W_{\beta(x)} \cap W_{\beta(y)} = \emptyset,$$
$$W_{\beta(x)} \text{ is finite.}$$

To see that $W_{\beta(x)} \cap A \neq \emptyset$, notice that $W_{g(x)} \cap A \neq \emptyset$ by Lemma 3.5, and $W_{g(x)} \cap A \subseteq W_{\beta(x)}$: if $W_{g(x)} \cap A \cap W_{\beta(y)} \neq \emptyset$ for some $y \neq x$, then, as $W_{\beta(y)} \subseteq W_{g(y)}$, it would follow that $W_{g(x)} \cap W_{g(y)} \cap A \neq \emptyset$, contrary to the property of $g$ established by Lemma 3.4.

The disjoint weak array $\{W_{\beta(x)}\}_x$ witnesses that $A$ is not hyperhyperimmune. $\square$

We derive as a corollary a result already proved in [14]:

**Corollary 3.6.** $\deg_{\hat{s}}(\overline{K})$ *is hyperhyperimmune-free.*

*Proof.* If $A \in \deg_{\hat{s}}(\overline{K})$, then $A \in \Delta^0_2$ (hence $\overline{K} \leq_{\hat{s}} A$ by Theorem 1.5), so $A \in \Pi^0_2$. $\square$

This leaves open the following question:

**Question 3.7.** Is $\deg_s(\overline{K})$ hyperhyperimmune-free?

## Bibliography


[1] O. Belegradek, Algebraically closed groups, (Russian) *Algebra i Logika* **13** (1974), 239–255.

[2] M. Blum and I. Marques, On complexity properties of recursively enumerable sets, *J. Symbolic Logic* **38** (1973), 579–593.

[3] S. B. Cooper, Enumeration reducibility using bounded information: counting minimal covers, *Z. Math. Logik Grundlag. Math.* **33**:6 (1987), 537–560.

[4] S. B. Cooper, *Computability Theory*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[5] J. T. Gill, III, and P. H. Morris, On subcreative sets and $S$-reducibility, *J. Symbolic Logic* **39** (1974), 669–677.

[6] C. M. Harris, Goodness in the enumeration and singleton degrees, *Arch. Math. Logic* (to appear).

[7] C. G. Jockusch, Jr., Upward closure and cohesive degrees, *Israel J. Math.* **15** (1973), 332–335.

[8] T. F. Kent, $s$-degrees within $e$-degrees, in: *Theory and applications of models of computation,* 579–587, Lecture Notes in Comput. Sci. 4978, Springer, Berlin, 2008.

[9] A. Macintyre, Omitting quantifier-free types in generic structures, *J. Symbolic Logic* **37** (1972), 512–520.

[10] S. S. Marchenkov, A certain class of incomplete sets, (Russian) *Mat. Zametki* **20**:4 (1976), 473–478.

[11] K. McEvoy and S. B. Cooper, On minimal pairs of enumeration degrees, *J. Symbolic Logic* **50**:4 (1985), 983–1001.

[12] R. Sh. Omanadze, Quasi-Degrees of Recursively Enumerable Sets, in: *Computability and models,* 289–319, Univ. Ser. Math., Kluwer/Plenum, New York, 2003.

[13] R. Sh. Omanadze and A. Sorbi. Strong enumeration reducibilities, *Arch. Math. Logic* **45**:7 (2006), 869–912.

[14] R. Sh. Omanadze and A. Sorbi. A characterization of the $\Delta^0_2$ hyperhyperimmune sets, *J. Symbolic Logic* **73**:4 (2008), 1407–1415.

[15] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Book Co., New York–Toronto, Ont.–London, 1967.

[16] M. G. Rozinas, Partial degrees of immune and hyperimmune sets, (Russian) *Sibirsk. Mat. Ž.* **19**:4 (1978), 866–870.

[17] R. A. Shore, Nowhere simple sets and the lattice of recursively enumerable sets, *J. Symbolic Logic* **43**:2 (1978), 322–330.

[18] S. D. Zakharov, e- and s-degrees, (Russian) *Algebra i Logika* 23:4 (1984), 395–406.

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