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Bounded enumeration reducibility and its degree structure

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Abstract We study a strong enumeration reducibility, called bounded enumeration reducibility and denoted by $\leq_{\mathrm{be}}$, which is a natural extension of s-reducibility $\leq_{\mathrm{s}}$. We show that $\leq_{\mathrm{s}}$, $\leq_{\mathrm{be}}$, and enumeration reducibility do not coincide on the Π^0_1–sets, and the structure $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ of the be-degrees is not elementarily equivalent to the structure of the s-degrees. We show also that the first order theory of $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ is computably isomorphic to true second order arithmetic: this answers an open question raised by Cooper (Z Math Logik Grundlag Math 33:537–560, 1987).

Keywords Strong enumeration reducibilities · Be-reducibility · Global properties

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1 Introduction

In this paper we study the degree structure of a strong enumeration reducibility, called *bounded enumeration reducibility*, or simply be-*reducibility*. A reducibility $\leq_{\mathrm{r}}$ is a *strong enumeration reducibility* if $\leq_{\mathrm{r}}$ is a proper subset of $\leq_{\mathrm{e}}$ (where $\leq_{\mathrm{e}}$ denotes enumeration reducibility), and $\leq_{\mathrm{r}}$ has a least degree $\mathbf{0}_{\mathrm{r}}$ consisting exactly of the

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computably enumerable (c.e.) sets. We recall that any c.e. set W defines an *enumeration operator* (or, simply, e-*operator*) i.e. a mapping Φ carrying sets of numbers to sets of numbers. More precisely, if $A \subseteq \omega$, then Φ maps A to $\Phi(A)$, where

$$\Phi(A) = \{x : (\exists u)[\langle x, u\rangle \in W \ \&\ D_u \subseteq A]\},$$

and D_u is the finite set with canonical index u. If $W = W_e$ then we write $\Phi = \Phi_e$. If $A = \Phi(B)$ then we say that A is *enumeration reducible to* B (or, simply, A is e-*reducible to* B; in symbols: $A \leq_{\mathrm{e}} B$) *via* Φ. We will often identify finite sets with their canonical indices, thus writing for instance $\langle x, D\rangle$ instead of $\langle x, u\rangle$, if $D = D_u$. (Here and throughout the paper, n-tuples or strings of numbers will be often identified with numbers via suitable codings: see any standard textbook on computability theory for details.) There are many reasons to consider enumeration reducibility as the most comprehensive positive reducibility: indeed, $A \leq_{\mathrm{e}} B$ can be regarded as the formalization of the intuitive notion of the existence of an algorithm that allows us to enumerate A, given any enumeration of B.

There are already several papers dedicated to strong enumeration reducibilities. A good survey on these reducibilities can be found in [2, § 8]; seminal work on strong enumeration reducibilities was laid down by Polyakov and Rozinas, [15,14]; more recent and lengthy papers on the subject are [1] and [13].

An e-operator Φ is said to be an s-*operator* if Φ is defined by a c.e. set W such that

$$(\forall u)(\forall x)[\langle x, u\rangle \in W \Rightarrow |D_u| \leq 1],$$

where $|X|$ denotes the cardinality of a given set X. We say that A is s-*reducible to* B (in symbols: $A \leq_{\mathrm{s}} B$) if $A = \Phi(B)$, for some s-operator Φ. There are several reasons why one should consider s-reducibility as the most important strong enumeration reducibility. First of all, mathematical practice shows that in most practical cases where we have $A \leq_{\mathrm{e}} B$ we do in fact have $A \leq_{\mathrm{s}} B$; moreover $\leq_{\mathrm{e}}$ naturally embeds into $\leq_{\mathrm{s}}$: if X^* denotes the set of all finite strings of elements of X, for a given subset $X \subseteq \omega$, then one has: $A \leq_{\mathrm{e}} B$ if and only if $A^* \leq_{\mathrm{s}} B^*$ (see for instance [5]); finally, s-reducibility is known to have many applications to computability theory (for instance, Marchenkov's solution of Post's problem using Post's methods), and to general mathematics, including word problems and abstract computationally complexity: for these applications of $\leq_{\mathrm{s}}$, or, rather, of its isomorphic copy known as Q-reducibility, see for instance Omanadze's survey paper [12].

Useful though it might be, s-reducibility fails however to handle cases when one would normally expect reducibility. For instance, as we show below (Lemma 2.1), we do not have in general that $A \times A \leq_{\mathrm{s}} A$, or $A \oplus B \equiv_{\mathrm{s}} A \times B$ if A, B are not c.e. (where, given subsets $X, Y \subseteq \omega$, we let $X \times Y = \{\langle x, y\rangle : x \in X \ \&\ y \in Y\}$): we do not have $A \times A \leq_{\mathrm{s}} A$, since in order to enumerate a pair $\langle a, b\rangle \in A \times A$, starting from an enumeration of A, we need, in general, both a and b to appear in the enumeration; on the other hand $A \times A \leq_{\mathrm{e}} A$ via the e-operator $\Phi = \{\langle\langle a, b\rangle, \{a, b\}\rangle : a, b \in \omega\}$, whose axioms $\langle x, D\rangle$ do not satisfy $|D| \leq 1$, but do satisfy $|D| \leq 2$. As justified by Lemma 1.1, this leads to the following natural definition: An e-operator Φ is said to be a *bounded enumeration operator* (or, simply, a be-*operator*) if Φ is defined by a c.e. set W such that

$$(\exists n)(\forall u)(\forall x)[\langle x, u\rangle \in W \Rightarrow |D_u| \leq n].$$

It is easy to see that $\leq_{\mathrm{be}}$ is in fact a reducibility, i.e. reflexive and transitive (see e.g. [13]). We say that A is be-*reducible to* B (in symbols: $A \leq_{\mathrm{be}} B$) if $A = \Phi(B)$, for some be-operator Φ. The corresponding degree structure $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$, of the be-*degrees*, will be denoted by $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$. (In general for a given reducibility r, we use the symbol $\boldsymbol{\mathcal{D}}_{\mathbf{r}}$ to denote the structure of the r-degrees, and $\mathbf{0}_{\mathrm{r}}$ to denote the least element, if any, of $\boldsymbol{\mathcal{D}}_{\mathbf{r}}$.)

Lemma 1.1 *The following hold:*

(1) $\leq_{\mathrm{be}}$ *lies in between* $\leq_{\mathrm{s}}$ *and* $\leq_{\mathrm{e}}$*, i.e.* $\leq_{\mathrm{s}} \subseteq \leq_{\mathrm{be}} \subseteq \leq_{\mathrm{e}}$. *Thus* $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ *has least element* $\mathbf{0}_{\mathrm{be}} = \{W : W \text{ c.e.}\}$. *Hence,* $\leq_{\mathrm{be}}$ *is a strong enumeration reducibility;*
(2) $\leq_{\mathrm{be}}$ *is the reducibility generated by* $\leq_{\mathrm{2e}}$*, where we define* $X \leq_{\mathrm{2e}} Y$ *if* $X = \Psi(Y)$ *for some* e-operator Ψ *such that* $|D| \leq 2$ *for every* $\langle x, D\rangle \in \Psi$ *(we call* Ψ *a* 2e-operator*);*
(3) $\leq_{\mathrm{e}}$ *is the smallest reducibility* $\leq_{\mathrm{r}}$ *such that*
   (a) $\leq_{\mathrm{s}} \subseteq \leq_{\mathrm{r}}$*;*
   (b) $X^* \leq_{\mathrm{r}} X$*, for every set* X*, where* X^* *is the set of all finite strings of elements of* X.

Proof We prove the items one by one:

(1) This follows immediately from the fact that $\mathbf{0}_{\mathrm{s}} = \mathbf{0}_{\mathrm{e}} = \{W : W \text{ c.e.}\}$;
(2) For every $n \geq 1$, let $X \leq_{\mathrm{ne}} Y$ if there exists an e-operator Ψ such that $|D| \leq n$ for every $\langle x, D\rangle \in \Psi$. (Thus $\leq_{\mathrm{s}} = \leq_{\mathrm{1e}}$). Let $\leq^*_{\mathrm{2e}}$ be the reducibility generated by $\leq_{\mathrm{2e}}$: thus $X \leq^*_{\mathrm{2e}} Y$ if there exists a sequence $Z_1, \ldots, Z_n$ of sets such that $X = Z_1, Y = Z_n$ and $Z_1 \leq_{\mathrm{2e}} \cdots \leq_{\mathrm{2e}} Z_n$. It is clearly enough to show that $\leq_{\mathrm{be}} \subseteq \leq^*_{\mathrm{2e}}$. To this end, assume that $A \leq_{\mathrm{be}} B$: then there is some $n \geq 1$ such that $A \leq_{\mathrm{ne}} B$. We show by induction on $n \geq 1$ that $A \leq^*_{\mathrm{2e}} B$.
   The claim is trivial for $n = 1, 2$. Thus suppose that $n > 2$, and $\leq_{\mathrm{(n-1)e}} \subseteq \leq^*_{\mathrm{2e}}$. If $A \leq_{\mathrm{ne}} B$ then clearly $A \leq_{\mathrm{s}} B \times B^{n-1}$, and by induction $B^{n-1} \leq^*_{\mathrm{2e}} B$ as $B^{n-1} \leq_{\mathrm{(n-1)e}} B$: thus there is a sequence $B_1, \ldots, B_m$ such that $B^{n-1} = B_1, B_m = B$ and

$$B_1 \leq_{\mathrm{2e}} \cdots \leq_{\mathrm{2e}} B_m.$$

   We now observe that if $X \leq_{\mathrm{2e}} Y$ then for every Z, we have $Z \times X \leq_{\mathrm{2e}} Z \times Y$. Indeed, assume that $X = \Phi(Y)$ via a 2e-operator Φ. Then $Z \times X = \Psi(Z \times Y)$, where

$$\Psi = \{\langle\langle x, y\rangle, D\rangle : x \in \omega \ \& \ (\exists E)[\langle y, E\rangle \in \Phi \ \& \ D = \{\langle x, e\rangle : e \in E\}]\}.$$

   It follows:

$$B \times B^{n-1} = B \times B_1 \leq_{\mathrm{2e}} \cdots \leq_{\mathrm{2e}} B \times B_m = B \times B,$$

and thus,

$$A \leq_s B \times B^{n-1} \leq^*_{2e} B \times B \leq_{2e} B,$$

which implies $A \leq^*_{2e} B$, as desired.

(3) Assume that $\leq_r$ satisfies (3a) and (3b). Since $Y \leq_s X^*$ for every $Y \leq_e X$ (see [7,20]; see also [5] and [4] for applications of this property of *), we have

$$\begin{aligned} A \leq_e B &\Rightarrow A^* \leq_s B^* \\ &\Rightarrow A^* \leq_r B^* \\ &\Rightarrow A^* \leq_r B; \end{aligned}$$

but $A \leq_s A^*$, giving $A \leq_r B$.

□

We investigate some basic properties of $\mathcal{D}_{\mathbf{be}}$, which is an upper semilattice with least element. We adapt to $\leq_{be}$ the machinery developed by Slaman and Woodin in [18], to conclude that every countable antichain is definable from finitely many parameters in $\mathcal{D}_{\mathbf{e}}$, in a uniform way. As a consequence of this, the main result of the paper states that the first order theory of the $\mathcal{D}_{\mathbf{be}}$ is computably isomorphic to true second order arithmetic. This solves an open problem proposed by Cooper in [1], asking whether the first order theory of $\mathcal{D}_{\mathbf{be}}$ is undecidable.

1.1 Terminology and notations

Our notations and terminology for computability theory is standard and can be found in [3,16], or [19]. We will often identify e-operators with their associated c.e. sets. We will use the following notations about strings: If σ is a finite string of numbers then $|\sigma|$ denotes the length of σ. (Notice that we use here the same symbol as the one used to denote the cardinality of a set: the particular meaning of the symbol will always be clear from the context). If σ is a string, then for $i < |\sigma|$, the symbol $(\sigma)_i$ denotes the ith component of σ; likewise, if $x \in X_0 \times \cdots \times X_{n-1}$, then $(x)_i$, for $i < n$, denotes the ith coordinate of x. If $\{C_r\}_{r\in\omega}$ is a family of sets of numbers, then for every $I \subseteq \omega$ we define $\bigoplus_{i\in I} C_i = \bigcup_{i\in I}\{i\} \times C_i$. It is clear that if I is computable and $I \subseteq J$ then $\bigoplus_{i\in I} C_i \leq_m \bigoplus_{j\in J} C_j$. Moreover, if $I = \{0, \dots, n-1\}$, $n > 2$, then $\bigoplus_{i\in I} C_i \equiv_m C_0 \oplus \cdots \oplus C_{n-1}$, where $C_0 \oplus \cdots \oplus C_{n-1}$ results inductively from the standard definition of join of two sets, $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$. For $n = 2$ however, $\oplus$ has the standard meaning.

For future reference, we note:

Lemma 1.2 *There is an effective listing of all* be*-operators.*

Proof Let $\{W_e\}_{e\in\omega}$ be the standard enumeration of the c.e. sets. By the s-m-n theorem, there exists a computable function b such that

$$W_{b(\langle p,n\rangle)} = \{\langle x, u\rangle : \langle x, u\rangle \in W_p \;\&\; |D_u| \leq n\}$$

Let Ω_e be the be-operator defined by $W_{b(e)}$. Then $\{\Omega_e\}_{e\in\omega}$ is an effective enumeration of the be-operators. It is clear that the be-operators are closed under composition (in a uniform way). □

Since, by the *s*-*m*-*n*–Theorem, $\Omega_e = W_{b(e)}$, for some computable function b, we can work with uniformly computable approximations $\{\Omega_{e,s}\}_{s\in\omega}$ to the be-operators, by taking $\Omega_{e,s} = W_{b(e),s}$, where $\{W_{i,s}\}_{s\in\omega}$ gives uniformly computable approximations to the c.e. sets.

2 Basic results about be-reducibility

In this section we review or introduce some basic facts concerning $\leq_{\text{be}}$.

2.1 Relations between $\leq_{\text{s}}$, $\leq_{\text{be}}$, and $\leq_{\text{e}}$

We begin by showing that $\leq_{\text{be}}$ is a proper reducibility lying in between $\leq_{\text{s}}$ and $\leq_{\text{e}}$. We show that differences between $\leq_{\text{s}}$ and $\leq_{\text{be}}$ on one hand, and between $\leq_{\text{be}}$ and $\leq_{\text{e}}$ on the other hand, already appear at the level of Π^0_1 sets. This is in a sense the best result we can obtain since on c.e. sets the three reducibilities coincide, being $\mathbf{0}_{\text{s}} = \mathbf{0}_{\text{be}} = \mathbf{0}_{\text{e}}$.

We recall that, given subsets $X, Y \subseteq \omega$, we define $X \times Y = \{\langle x, y\rangle : x \in X \;\&\; y \in Y\}$, and X^* is the set of all finite strings of elements of X. Notice that $X^* \equiv_{\text{s}} X^\infty$, where

$$A^\infty = \{u : D_u \subseteq A\}.$$

Lemma 2.1 *There exists a Π^0_1–set A such that $A^* \not\leq_{\text{be}} A$.*

Proof We build a computable sequence of sets $\{A_s\}_{s\in\omega}$ such that $A_0 = \omega$ and $A_{s+1} \subseteq A_s$, so that $A = \bigcap_{s\in\omega} A_s$ is the desired set. The requirements are, for every e,

$$P_e : A^* \neq \Omega_e(A),$$

where $\{\Omega_e\}_{e\in\omega}$ is an effective list of all be-operators, see Lemma 1.2 (each index e is regarded as a pair $e \in \omega \times \omega$). Thus, for every e, $\langle x, D\rangle \in \Omega_e$ implies $|D| \leq (e)_1$.

In order to satisfy P_e, for a given e, we use the following Friedberg–Muchnik type of diagonalization strategy:

(1) appoint a "fresh" witness $x \in A^*$ such that $|x| = (e)_1 + 1$. Say that $(e)_1 = n$, thus $x = \langle x_0, \dots, x_n\rangle$: "fresh" means that each x_i has never been so far used in the construction, and $x_i \neq x_j$ if $i \neq j$, for all $i, j < n$;
(2) await $x \in \Omega_e(A)$;
(3) extract x from A^*, and restrain $x \in \Omega_e(A)$: this can be done, since $x \in \Omega_e(A)$ via an axiom $\langle x, D\rangle \in \Omega_e$, with $|D| \leq n$, so there is at least one x_i such that $x_i \notin D$; therefore we can restrain $D \subseteq A$, and extract some x_i from A.

Construction. The construction is by stages. At stage s, we define a computable set A^s, together with a witness $x(e,s)$ for every e. We say that we *reset* P_e at stage s, if we set $x(e,s)$ to be undefined. Unless otherwise specified, the values of A and $x(e)$ at each stage are the same as at the previous stage.

Stage 0: Let $A^0 = \omega$, and reset all P_e;
Stage $s+1$: Pick the least $e \leq s$, if any, that *requires attention at stage* $s+1$, i.e. such that $x(e,s)$ is undefined, or $x(e,s) \in A^* \cap \Omega_e(A)[s]$. (Here, as in [19], given an expression $\mathcal{A}$ which is a function of the stage, by $\mathcal{A}[s]$ we mean the evaluation of the expression at the end of stage s. Thus $\Omega_e(A)[s] = \Omega_{e,s}(A^s)$.)

(a) If $x(e,s)$ is undefined, then choose a fresh x, and let $x(e,s+1) = x$;
(b) if $x(e,s) \in A^* \cap \Omega_e(A)[s]$ then there exists some D with $\langle x(e,s), D\rangle \in \Omega_{e,s}$, and $D \subseteq A^s$: let x_i be the first component of $x(e,s)$ such that $x_i \notin D$, and define $A^{s+1} = A^s \setminus \{x_i\}$. Finally, reset at $s+1$ all P_i, with $i > e$.

This ends the construction.

Verification $A = \bigcap_s A^s$ is Π^0_1: no element that is extracted at some point is ever re-enumerated into A.

Assume by induction that for every $i < e$, $x_i = \lim_t x(i,t)$ exists and P_i requires attention only finitely often: let $s_0 \geq e$ be the least stage such that, for every $s > s_0$, for every $i < e$, $x(i,s) = x(i)$ and P_i does not require attention after s_0. So at stage s_0+1 we define $x(e) = x(e, s_0+1)$, the final value of the witness for P_e, and either P_e will never get attention again, in which case $x(e) \in A^* \setminus \Omega_e(A)$, or there is a least $t > s_0+1$, such that $x(e) \in \Omega(A)[t]$, and we restrain $x(e) \in \Omega(A)$, and extract $x(e)$ from A^*, through extraction of some coordinate of $x(e)$ from A. In either case, the inductive claim extends to P_e, which is eventually satisfied, as $A^*(x(e)) \neq \Omega_i(A)(x(e))$. □

Lemma 2.2 *For any set A, if* $A \times A \leq_{\mathrm{s}} A$ *then* $A^* \leq_s A$.

Proof Assume that $A \times A \leq_s A$. Let us first show that there is a computable function f such that for every n, $\Phi_{f(n)}$ is an s-operator, and $A^n = \Phi_{f(n)}(A)$, where

$$A^n = \{x \in A^* : |x| = n\}.$$

From this it follows that $A^* \leq_{\mathrm{s}} A$, since

$$\begin{aligned} x \in A^* &\Leftrightarrow x \in A^{|x|} \\ &\Leftrightarrow x \in \Phi_{f(|x|)}(A), \end{aligned}$$

thus $A^* = \Gamma(A)$, with

$$\Gamma = \{\langle x, D\rangle : \langle x, D\rangle \in \Phi_{f(|x|)}\}$$

which is an s-operator since each $\Phi_{f(|x|)}$ is.

It remains only to show the existence of such a function f. Clearly we can take $f(2) = e$ where Φ_e is the s-operator such that $A \times A = \Phi_e(A)$. Assume by induction that we know how to compute $f(n)$. Let l be a computable function such that, for $n \geq 2$, $l(\langle x_0, \ldots, x_n \rangle) = \langle x_0, \ldots, x_{n-1} \rangle$. Then

$$\begin{aligned} x \in A^{n+1} &\Leftrightarrow \langle l(x), (x)_n \rangle \in A^n \times A \\ &\Leftrightarrow \langle l(x), (x)_n \rangle \in \Phi_{f(n)}(A) \times A. \end{aligned}$$

Thus $A^{n+1} = \Psi(A \times A)$, where

$$\Psi = \{\langle \langle u, v \rangle, D \rangle : (\exists E)[D = E \times \{v\} \,\&\, \langle u, E \rangle \in \Phi_{f(n)}]\}$$

which is clearly an s-operator. Hence $A^{n+1} \leq_{\mathrm{s}} A$ via the composition of the s-operator $\Phi_{f(n+1)} = \Psi \circ \Phi_e$. □

Corollary 2.3 *There exists a Π^0_1–set A such that*

$$A \times A \not\leq_{\mathrm{s}} A \,\&\, A^* \not\leq_{\mathrm{be}} A.$$

Proof Let A be the Π^0_1–set A provided by Lemma 2.1. For this set we also have that $A \times A \not\leq_{\mathrm{s}} A$ by Lemma 2.2. □

Therefore we obtain the following refinement of Lemma 1.1:

Corollary 2.4 *We have $\leq_{\mathrm{s}} \subset \leq_{\mathrm{be}} \subset \leq_{\mathrm{e}}$ with proper inclusions: counterexamples to the reverse inclusions of Lemma 1.1 are provided by Π^0_1 sets.*

Proof Let A be as in Corollary 2.3: then $A \times A \leq_{\mathrm{be}} A$, but $A \times A \not\leq_s A$; and $A^* \leq_{\mathrm{e}} A$, but $A^* \not\leq_{\mathrm{be}} A$. All the sets involved are Π^0_1 since if $X, Y \in \Pi^0_1$ then also $X \times Y, X^* \in \Pi^0_1$. □

In the following corollary (where functions are identified with their graphs), we observe that differences between these reducibilities appear also on total functions, where e-reducibility coincides with Turing reducibility.

Corollary 2.5 *There are total functions f such that $f \times f \not\leq_{\mathrm{s}} f$, and $f^* \not\leq_{\mathrm{be}} f$.*

Proof The result is an immediate consequence of the fact that if $A \in \Pi^0_1$ then $A \equiv_{\mathrm{s}} \chi_A$, where χ_A is the characteristic function of A. On the other hand $\times$ and * preserve $\leq_{\mathrm{s}}$ (for * we even have $X \leq_{\mathrm{e}} Y$ implies $X^* \leq_{\mathrm{s}} Y^*$, see [7,20]; see also [5] and [4] for applications of this property of *). □

We conclude this section with the following simple application of Lemma 2.2. Remember that a set $A = \{a_0 < a_1 < \cdots < a_n < \cdots\}$ is *retraceable* if there is a partial computable function ψ such that, for all n, $\psi(a_{n+1}) \downarrow = a_n$, and $\psi(a_0) \downarrow = a_0$.

Corollary 2.6 *If A is retraceable then $A^* \leq_{\mathrm{s}} A$.*

Proof By Lemma 2.2 it suffices to show that if A is retraceable (as witnessed, say, by a partial computable function ψ) then $A \times A \leq_{\mathrm{s}} A$. For this, consider the following s-operator Γ:

$$\begin{aligned}\Gamma = \{\langle\langle a, b\rangle, D\rangle : \quad & [a < b \ \& \ (\exists n)[\psi^n(b) \downarrow = a] \ \& \ D = \{b\}] \text{ or} \\ & [b < a \ \& \ (\exists n)[\psi^n(a) \downarrow = b] \ \& \ D = \{a\}] \text{ or} \\ & [a = b \ \& \ D = \{a\}]\ \}.\end{aligned}$$

□

Given a poset $\mathcal{P}$, let us indicate with $Th(\mathcal{P})$ the set of sentences (in the language, with equality, of partial orders, i.e. with signature $\leq$) that are true in $\mathcal{P}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partial orders we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are *elementary equivalent* (in symbols, $\mathcal{P}_1 \equiv_{\mathrm{ee}} \mathcal{P}_2$) if $Th(\mathcal{P}_1) = Th(\mathcal{P}_2)$.

Theorem 2.7 *$\mathcal{D}_{\mathbf{bs}}$ and $\mathcal{D}_{\mathbf{be}}$ are not elementarily equivalent.*

Proof Zacharov, [20, Theorem 3], shows that the following sentence σ (which can be translated into the language of posets),

$$\sigma := (\exists a \neq 0)(\forall b)(\forall c)(a \leq b \vee c \rightarrow a \leq b \text{ or } a \leq c)$$

is true in $\mathcal{D}_{\mathbf{s}}$ (i.e., $\mathcal{D}_{\mathbf{s}} \models \sigma$) but not in $\mathcal{D}_{\mathbf{e}}$. Indeed, for every retraceable set A, we have that if $A \leq_{\mathrm{s}} B \oplus C$ then $A \leq_{\mathrm{s}} B$ or $A \leq_{\mathrm{s}} C$. To show that $\mathcal{D}_{\mathbf{e}} \not\models \sigma$, Zacharov proves that for every non-c.e. set A one can find B, C such that $A = B \cap C$, and $A \not\leq_{\mathrm{e}} B$ and $A \not\leq_{\mathrm{e}} C$. But clearly $A \leq_{\mathrm{be}} B \oplus C$, via the be-operator $\Phi = \{\langle x, \{2x, 2x+1\}\rangle : x \in \omega\}$. Hence $\mathcal{D}_{\mathbf{be}} \not\models \sigma$. □

We do not know yet whether $\mathcal{D}_{\mathbf{be}} \not\equiv_{ee} \mathcal{D}_{\mathbf{e}}$, although we conjecture that this is so.

2.2 Structural properties of $\mathcal{D}_{\mathbf{be}}$

In this section we point out some basic properties of the poset $\mathcal{D}_{\mathbf{be}}$. Theorem 2.8 and Corollary 2.9 can be derived as particular cases of more general results of Polyakov and Rozinas, [14]. Since the proof in [14] is only sketched, for completeness, we reproduce full proofs here. Given an upper semilattice $\langle U, \leq, \vee\rangle$, we say that a nonempty $I \subseteq U$ is an *ideal* if for all $x, y \in U$,

(1) $x \leq y \ \& \ y \in I \Rightarrow x \in I$;
(2) $x, y \in I \Rightarrow x \vee y \in I$.

Theorem 2.8 (Exact Pair Theorem, [14]) *Given a countable ideal* $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots\}$ *of* be-*degrees, there exist* be-*degrees* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *such that:*

(i) *for every* n, $\boldsymbol{a}_n <_{\mathrm{be}} \boldsymbol{b}, \boldsymbol{c}$ *and*
(ii) *if* $\boldsymbol{d} \leq_{\mathrm{be}} \boldsymbol{b}, \boldsymbol{c}$ *then for some* n, $\boldsymbol{d} \leq_{\mathrm{be}} \boldsymbol{a}_n$.

Proof Let $A_n \in \boldsymbol{a}_n$ for every n. Define $B = \bigoplus_n A_n$, and build $C = \bigoplus_n C_n$ by infinite extensions: at stage $s+1$ we define approximations C_n^{s+1} to each C_n so that for every $t \geq s+1$ and for every $n \leq s$, $C_n^t = C_n^{s+1}$ (in other words, by the end of stage $s+1$ we have entirely defined all C_n for every $n \leq s$), whereas for $n > s$, C_n^{s+1} is finite.

The idea is to copy each A_n into C_n except for a finite part of it, which is used to guarantee condition (ii). More precisely, at stage $s+1$, with $s = \langle i, j \rangle$ we satisfy the requirement

$$\Omega_i(B) = \Omega_j(C) \Rightarrow (\exists n)[\Omega_j(C) \leq_{\mathrm{be}} A_n]$$

by looking for a finite set F *compatible with* C^s ("compatible with C^s" means that if $y \in F$ and $y \in \bigoplus_{n<s} \omega$ then $y \in \bigoplus_{n<s} C_n$) such that there exists an x with

$$x \in \Omega_j(C^s \cup F) \setminus \Omega_i(B),$$

with $C^s = \bigoplus_n C_n^s$. We choose such a finite set F (if any, otherwise we take $F = \emptyset$), we extend $C^{s+1} = C^s \cup F$, and then fill up with $A_{\langle i,j \rangle}$ the rest of $C_{\langle i,j \rangle}$, by defining, where $C_{\langle i,j \rangle}$ is still undefined, $C_{\langle i,j \rangle}(x) = A_{\langle i,j \rangle}(x)$ (we identify here sets with their characteristic functions).

Now if no such F can be found at stage $s+1 = \langle i, j \rangle + 1$, then $x \in \Omega_j(C)$ if and only if there exists a finite set F such that $\langle x, F \rangle \in \Omega_j$ and F is compatible with C^s, hence

$$\Omega_j(C) = \Omega_j(\bigoplus_n X_n),$$

where

$$X_n = \begin{cases} C_n & \text{if } n < \langle i, j \rangle, \\ \omega, & \text{if } n \geq \langle i, j \rangle. \end{cases}$$

Since C_n differs only finitely from A_n for every $n < s = \langle i, j \rangle$, we have also that

$$\bigoplus_n X_n \equiv_{\mathrm{m}} \bigoplus_n Y_n \equiv_{\mathrm{m}} A_0 \oplus \cdots \oplus A_{\langle i,j \rangle - 1},$$

where

$$Y_n = \begin{cases} A_n & \text{if } n < \langle i, j \rangle, \\ \omega, & \text{if } n \geq \langle i, j \rangle. \end{cases}$$

But $\leq_{\mathrm{m}} \subseteq \leq_{\mathrm{be}}$, thus if $D = \Omega_i(B) = \Omega_j(C)$ then $D \leq_{\mathrm{be}} A_0 \oplus \cdots \oplus A_{\langle i,j \rangle - 1}$. By definition of an ideal, this implies that there exists some n with $D \leq_{\mathrm{be}} A_n$. □

Corollary 2.9 ([14]) *$\mathcal{D}_{\mathbf{be}}$ is not a lattice.*

Proof Take any countable strictly ascending chain of be-degrees

$$\boldsymbol{a}_0 <_{\mathrm{be}} \boldsymbol{a}_1 <_{\mathrm{be}} \cdots :$$

for instance, consider any family $\{B_i : i \in \omega\}$ of sets such that, for every finite $F \subseteq \omega$, if $i \notin F$, then $B_i \not\leq_{\mathrm{be}} \bigoplus_{j \in F} B_j$: any family of sets which are computably independent with respect to $\leq_{\mathrm{be}}$, whose existence is proved in Lemma 2.12, satisfies this property. Finally, take $\boldsymbol{a}_i$ to be the be-degree of $\bigoplus_{i \leq n} B_i$. Then no exact pair $\boldsymbol{b}, \boldsymbol{c}$ for the ideal generated by this chain has greatest lower bound. □

Remark 2.10 As observed in [14], it is clear from the proof that Theorem 2.8 and Corollary 2.9 hold of any reducibility $\leq_{\mathrm{r}}$ such that $\leq_{\mathrm{m}} \subseteq \leq_{\mathrm{r}} \subseteq \leq_{\mathrm{e}}$, originating a degree structure which is an upper semilattice, with join operation on degrees given by the join operation on sets.

Definition 2.11 A countable family of sets $\{C_n\}_{n \in \omega}$ is *computably independent* with respect to a reducibility $\leq_{\mathrm{r}}$ if for every computable set I and $n \notin I$,

$$C_n \not\leq_{\mathrm{r}} \bigoplus_{m \in I} C_m.$$

Note that $\{C_n\}_{n \in \omega}$ is computably independent if and only if $C_n \not\leq_{\mathrm{r}} \bigoplus_{m \neq n} C_m$ for every $n \in \omega$.

Lemma 2.12 *There exists a countable family of sets computably independent with respect to* $\leq_{\mathrm{e}}$.

Proof This can be proved in many ways. For instance, it is known (see e.g. [9]), that there exists a countable family of c.e. sets which are computably independent with respect to Turing reducibility: taking complements, we get a family of Π^0_1 sets that are computably independent with respect to e-reducibility, since Π^0_1e-degrees are total (see proof of Corollary 2.5), and on total functions e-reducibility and Turing reducibility coincide. □

Corollary 2.13 *There exists a countable family of sets computably independent with respect to* $\leq_{\mathrm{be}}$.

Proof This follows trivially from the fact that $\leq_{\mathrm{be}} \subseteq \leq_{\mathrm{e}}$. □

Theorem 2.14 *Every countable partial order is embeddable in* $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$.

Proof The proof follows the pattern of the analogous result for most known reducibilities: see for instance the presentation in [11] of this result for Turing reducibility, due to Sacks [17]. It is known, Mostowski [8], that there exists a computable partial order in which every countable partial ordering is embeddable. Therefore it suffices to prove that every computable partial ordering $\preceq$ is embeddable in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$. By Corollary 2.13

let $\{C_n\}_{n\in\omega}$ be computably independent with respect to $\le_{\text{be}}$. We associate a set B_a to every a in the domain of $\preceq$ (which we may assume to be ω) as follows:

$$B_a = \bigoplus_{n\preceq a} C_n.$$

We have to prove that

$$a \preceq b \;\Leftrightarrow\; B_a \le_{\text{be}} B_b.$$

This follows easily by the properties of $\bigoplus$ mentioned in Sect. 1.1. Suppose that $a \preceq b$: since $\{n : n \preceq a\}$ is computable, it follows that $B_a \le_{\text{m}} B_b$. Suppose now that $B_a \le_{\text{be}} B_b$ and $a \not\preceq b$, hence $a \notin \{n : n \preceq b\}$: then

$$C_a \le_{\text{m}} B_a \le_{\text{be}} B_b \le_{\text{m}} \bigoplus_{n\neq a} C_n,$$

giving that $C_a \le_{\text{be}} \bigoplus_{n\neq a} C_n$, a contradiction, since $\{C_n\}_{n\in\omega}$ is computably independent with respect to $\le_{\text{be}}$. □

With a taste for generalization, as in [14], we can extend the previous result to:

Corollary 2.15 *Let $\le_{\text{r}}$ be a reducibility such that $\le_{\text{m}} \subseteq \le_{\text{r}} \subseteq \le_{\text{e}}$. Then every countable partial order is embeddable into $\mathcal{D}_{\mathbf{r}}$.*

Proof The proof is as in Theorem 2.14, by observing that in fact if $a \preceq b$ then $B_a \le_{\text{m}} B_b$; moreover, $C_a \le_{\text{m}} B_a$ and $B_b \le_{\text{m}} \bigoplus_{n\neq a} C_n$. □

We conclude the section by observing that it is possible to define a suitable jump operation on the be-degrees: see for instance [1, Definition 4.5]. To this jump operation and to the local structure of the be-degrees (i.e. the be-degrees below the first jump) and its first order theory will be devoted a future paper by the authors, [6].

3 Global properties

Global results about a degree structure include, for example, the characterization of the complexity of the first order theory of the degree structure. Here, by the first order theory of a given degree structure $\mathcal{P} = \langle P, \le\rangle$, we mean the set $Th(\mathcal{P})$ of all first order sentences σ in the language of posets such that $\mathcal{P} \models \sigma$. It was proved by Slaman and Woodin [18] that $Th(\mathcal{D}_{\mathbf{e}})$ is computably isomorphic to true second order arithmetic. In particular $Th(\mathcal{D}_{\mathbf{e}})$ is undecidable and not axiomatizable. Cooper [1] raised as an open problem whether the first order theory of $\mathcal{D}_{\mathbf{be}}$ is undecidable. Using the machinery in [18] we can prove something stronger, namely that also the first order theory of $\mathcal{D}_{\mathbf{be}}$ is computably isomorphic to true second order arithmetic. The proof is obtained by adapting the proof of [18, Theorem 2.11] to the be-degrees. We do not claim any essential originality in the proof, except perhaps that our proof is formulated as a plain step-by-step extension argument rather than using the language of forcing.

It has also a more general character: indeed, by replacing $\leq_{\mathrm{be}}$ with $\leq_{\mathrm{r}}$, we easily conclude that the proof works for all reducibilities with certain reasonable properties (see Theorem 3.21), including of course $\leq_{\mathrm{r}} = \leq_{\mathrm{e}}$.

3.1 Definability of antichains

We recall the following:

Definition 3.1 In a poset, a set of elements is an *antichain* if its members are pairwise incomparable with respect to the partial order relation.

Theorem 3.2 *Every countable antichain is definable from finitely many parameters, in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$, in a uniform way; namely, there exists a first order formula $\varphi(x, a, b, c)$ with four free variables in the language of posets such that for every countable antichain $\mathcal{C}$ in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ there exist three degrees $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, such that for every degree $\boldsymbol{x}$,*

$$\boldsymbol{x} \in \mathcal{C} \Leftrightarrow \boldsymbol{\mathcal{D}}_{\mathbf{be}} \models \varphi(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}).$$

Proof As in [18], we show that a countable antichain in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ can be viewed as the set of the minimal solutions of a property which is first order definable in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ with parameters. Let $\mathcal{C} = \{\boldsymbol{c}_n\}_{n\in\omega}$ be a countable antichain in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$. Given $\boldsymbol{x} \in \boldsymbol{\mathcal{D}}_{\mathbf{be}}$, let $(\boldsymbol{x}]$ denote the ideal of $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ generated by $\boldsymbol{x}$, i.e $(\boldsymbol{x}] = \{\boldsymbol{y} \in \boldsymbol{\mathcal{D}}_{\mathbf{be}} : \boldsymbol{y} \leq_{\mathrm{be}} \boldsymbol{x}\}$. Consider the following first order property in the language of posets:

$$P(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \;\Leftrightarrow\; (\boldsymbol{x}] \neq (\boldsymbol{x} \cup \boldsymbol{a}] \cap (\boldsymbol{x} \cup \boldsymbol{b}].$$

We want to prove that, for every countable antichain $\mathcal{C}$, there exists three parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that

$$\boldsymbol{x} \in \mathcal{C} \;\Leftrightarrow\; \boldsymbol{x} \leq_{\mathrm{be}} \boldsymbol{c} \;\&\; P(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \;\&\; \neg(\exists \boldsymbol{z} \leq_{\mathrm{be}} \boldsymbol{c})(\boldsymbol{z} <_{\mathrm{be}} \boldsymbol{x} \;\&\; P(\boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b})).$$

It is important to observe that the property P is fixed. If we change the antichain, only the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ change. This fact ensures the uniformity of our result.

We define the parameters by constructing three sets A, B, C such that $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$, $C \in \boldsymbol{c}$. Let $C = \bigoplus_{n\in\omega} C_n$, with $C_n \in \boldsymbol{c}_n$. Then $\boldsymbol{c}_n \leq \boldsymbol{c}$, for every $n \in \omega$. Let $\{E_u\}_{u\in\omega}$ be the following enumeration of the sets with cardinality less or equal to 1:

$$E_u = \begin{cases} \emptyset & \text{if } u = 0, \\ \{u-1\} & \text{if } u > 0. \end{cases}$$

We will write $\langle x, E_u\rangle$, instead of $\langle x, u\rangle$ or $\langle x, \{u\}\rangle$, to emphasize that we view the second component of a pseudopair as always coding a finite set with cardinality ≤ 1. Given a set W, define

$$W^{\leq 1}(X) = \{x : (\exists E_u)[\langle x, E_u\rangle \in W \;\&\; E_u \subseteq X]. \tag{a}$$

Thus, $W^{\leq 1}(X) \leq_{\mathrm{be}} X \oplus W$.

Finally, let $\{\Omega_e\}_{e\in\omega}$ be an effective listing of the be-operators: see Lemma 1.2.

The sets A and B have to satisfy the following conditions:

(i) For every n, $\boldsymbol{c}_n$ is a solution of $P(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})$, i.e. $\mathcal{D}_{\mathbf{be}} \models P(\boldsymbol{c}_n, \boldsymbol{a}, \boldsymbol{b})$. Since $(\boldsymbol{x}] \subseteq (\boldsymbol{x} \cup \boldsymbol{a}] \cap (\boldsymbol{x} \cup \boldsymbol{b}]$ always holds, we require that $(\boldsymbol{x} \cup \boldsymbol{a}] \cap (\boldsymbol{x} \cup \boldsymbol{b}] \nsubseteq (\boldsymbol{x}]$. Indeed, we prove that:

$$(\forall n)[D_n \leq_{\mathrm{be}} C_n \oplus A, C_n \oplus B \;\&\; D_n \not\leq_{\mathrm{e}} C_n]$$

where the set D_n is defined by:

$$x \in D_n \;\Leftrightarrow\; (\exists u)[\langle x, E_u\rangle \in A^{[n]} \;\&\; E_u \subseteq C_n],$$

or, in accordance with the notation introduced in (a),

$$D_n = (A^{[n]})^{\leq 1}(C_n),$$

where as usual $A^{[n]}$ denotes the nth column of A, i.e.

$$A^{[n]} = \{\langle n, x\rangle : \langle n, x\rangle \in A\}.$$

Clearly $D_n \leq_{\mathrm{be}} C_n \oplus A$, since $|E_u| \leq 1$ for every u.
Furthermore A and B will be constructed so that

$$(A^{[n]})^{\leq 1}(C_n) = (B^{[n]})^{\leq 1}(C_n).$$

Hence $D_n \leq_{\mathrm{be}} C_n \oplus B$ too. It remains to ensure that $D_n \not\leq_{\mathrm{e}} C_n$ by diagonalization.

(ii) For every n, $\boldsymbol{c}_n$ is a minimal solution of $P(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})$.
Let $\{X_m\}_{m\in\omega}$ be an enumeration of the sets be-reducible to C. For example, let $X_m = \Omega_m(C)$, for every $m \in \omega$. We require that:

$$(\forall m)\big[(\exists D)[D \leq_{\mathrm{be}} X_m \oplus A, X_m \oplus B \;\&\; D \not\leq X_m] \;\Rightarrow\; (\exists n)[C_n \leq_{\mathrm{be}} X_m]\big].$$

The requirements. In conclusion, the requirements are, for all e, i, n, m:

$$P_{e,n} : D_n \neq \Phi_e(C_n),$$
$$R_{e,i,m} : \Omega_e(X_m \oplus A) = \Omega_i(X_m \oplus B) \not\leq_{\mathrm{be}} X_m \;\Rightarrow\; (\exists n)[C_n \leq_{\mathrm{be}} X_m],$$

where Φ_e denotes the eth enumeration operator: in other words, as anticipated above, we aim at $D_n \not\leq_{\mathrm{e}} C_n$, rather than simply $D_n \not\leq_{\mathrm{be}} C_n$

The construction. We first introduce some definitions, from [18]. Although our proof is organized as a classical step-by-step extension argument rather than as a forcing argument, we will keep some of the terminology of [18] that is typical of the language of forcing.

Definition 3.3 A *condition* is a triple $p = \langle A_p, B_p, k_p \rangle$ where:

(1) A_p and B_p are finite sets;
(2) for every n, $(A_p^{[n]})^{\leq 1}(C_n) = (B_p^{[n]})^{\leq 1}(C_n)$;
(3) $k_p \geq \max\{x : (\exists n)(\exists u)[\langle n, \langle x, E_u \rangle\rangle \in A_p \cup B_p]\}$, i.e.

$$k_p \geq \max \bigcup_n \left((A_p^{[n]})^{\leq 1}(\omega) \cup (B_p^{[n]})^{\leq 1}(\omega) \right).$$

Definition 3.4 If p and q are conditions, q *extends* p (in symbols: $q \sqsupseteq p$) if:

(1) $k_p \leq k_q$;
(2) $A_p \subseteq A_q$ and $B_p \subseteq B_q$;
(3) for every x,

$$x \in \left(\bigcup_n (A_q^{[n]})^{\leq 1}(\omega) \cup (B_q^{[n]})^{\leq 1}(\omega) \right) \Big\backslash \left(\bigcup_n (A_p^{[n]})^{\leq 1}(\omega) \cup (B_p^{[n]})^{\leq 1}(\omega) \right) \Rightarrow x > k_p.$$

At stage s we build a condition p_s, so that $p_{s+1} \sqsupseteq p_s$. In the end, we set $A = \bigcup_{s \in \omega} A_{p_s}$ and $B = \bigcup_{s \in \omega} B_{p_s}$.

Step 0: $p_0 = \langle \emptyset, \emptyset, 0 \rangle$.
Step $s+1$: We distinguish three cases:

(1) $s = 3\langle e, n \rangle$.
We diagonalize D_n against $\Phi_e(C_n)$, i.e. $D_n \neq \Phi_e(C_n)$. Pick $x > k_s$, thus $\langle n, \langle x, E_u \rangle\rangle \notin A_{p_s} \cup B_{p_s}$, for every u.
If $x \in \Phi_e(C_n)$, then set $p_{s+1} = \langle A_{p_s}, B_{p_s}, x \rangle$. Otherwise $p_{s+1} = \langle A_{p_s} \cup \{\langle n, \langle x, E_0 \rangle\rangle\}, B_{p_s} \cup \{\langle n, \langle x, E_0 \rangle\rangle\}, x \rangle$. We have that:

$$\begin{aligned} x \in \Phi_e(C_n) &\Rightarrow \langle x, E_u \rangle \notin A^{[n]}, \quad \text{for every } u \\ &\Rightarrow x \notin D_n. \\ x \notin \Phi_e(C_n) &\Rightarrow \langle x, E_0 \rangle \in A^{[n]} \\ &\Rightarrow x \in D_n \quad (\text{since } \emptyset \subseteq C_n). \end{aligned}$$

(2) $s = 3\langle e, i, m \rangle + 1$.
We look for a condition $q \sqsupseteq p_s$ such that, for some x,

$$x \in \Omega_e(X_m \oplus A_q) \;\&\; (\forall q' \sqsupseteq q)[x \notin \Omega_i(X_m \oplus B_{q'})]$$

or

$$x \in \Omega_i(X_m \oplus B_q) \;\&\; (\forall q' \sqsupseteq q)[x \notin \Omega_e(X_m \oplus A_{q'})].$$

If such condition exists, then set $p_{s+1} = q$. Otherwise $p_{s+1} = p_s$.

(3) $s = 3\langle e, x, m\rangle + 2$.
We look for a condition $q \sqsupseteq p_s$ such that $x \in \Omega_e(X_m \oplus A_q)$. If such condition exists, then set $p_{s+1} = q$. Otherwise $p_{s+1} = p_s$.

Verification. It remains to prove that the requirements $R_{e,i,m}$ are satisfied. It is useful to give the following definitions.

Definition 3.5 Let $p = \langle A_p, B_p, k_p\rangle$ be a condition. A finite set F is *compatible* with $\langle A_p, k_p\rangle$ (in symbols: $\mathrm{cpt}(F, \langle A_p, k_p\rangle)$) if

(1) $A_p \subseteq F$;
(2) $x \in \left(\bigcup_n (F^{[n]})^{\leq 1}(\omega)\right) \setminus \left(\bigcup_n (A_p^{[n]})^{\leq 1}(\omega)\right) \Rightarrow x > k_p$.

Lemma 3.6 *If $q \sqsupseteq p$, with $q = \langle A_q, B_q, k_q\rangle$ and $p = \langle A_p, B_p, k_p\rangle$, then for every finite F,*

$$\mathrm{cpt}(F, \langle A_q, k_q\rangle) \Rightarrow \mathrm{cpt}(F, \langle A_p, k_p\rangle).$$

Moreover A_p and A_q are compatible with $\langle A_p, k_p\rangle$.

Proof The claims are straightforward consequences of the definitions. □

Definition 3.7 Let $q \sqsupseteq p$ and assume that $\mathrm{cpt}(F, \langle A_p, k_p\rangle)$. The *amalgamation* of q with F (in symbols: $q * F$) is the condition defined as follows:

(1) $A_{q*F} = A_q \cup F$;
(2) $B_{q*F} = B_q \cup (F \setminus A_q)$;
(3) $k_{q*F} = \max \bigcup_n \left((A_{q*F}^{[n]})^{\leq 1}(\omega) \cup (B_{q*F}]^{[n]})^{\leq 1}(\omega)\right) \cup \{k_q\}$.

Given p and q as above and a finite set G, we define G *compatible* with $\langle B_p, k_p\rangle$ (in symbols: $\mathrm{cpt}(G, \langle B_p, k_p\rangle)$) and the *amalgamation* of q with G similarly, with the roles of F and G interchanged. Note that

$$\mathrm{cpt}(F, \langle A_p, k_p\rangle) \;\&\; q \sqsupseteq p \Rightarrow q * F \sqsupseteq p.$$

In particular, $p * F \sqsupseteq p$.
Suppose now that

$$\Omega_e(X_m \oplus A) = \Omega_i(X_m \oplus B) \not\leq_{\mathrm{be}} X_m. \tag{b}$$

Then:

Lemma 3.8 *Let $s = 3\langle e, i, m\rangle + 1$. If there are no conditions $q_0, r_0 \sqsupseteq p_s$ such that, for some x,*

$$x \in \Omega_e(X_m \oplus A_{q_0}) \;\&\; (\forall r' \sqsupseteq r_0)[x \notin \Omega_e(X_m \oplus A_{r'})]$$

then, for every x,

$$x \in \Omega_e(X_m \oplus A) \;\Leftrightarrow\; (\exists F)[\mathrm{cpt}(F, \langle A_{p_s}, k_{p_s}\rangle) \;\&\; x \in \Omega_e(X_m \oplus F)].$$

Proof (⇒): The left-to-right implication follows immediately from Lemma 3.6.
(⇐): Assume that F is a finite set, F compatible with $\langle A_{p_s}, k_{p_s}\rangle$, and $x \in \Omega_e(X_m \oplus F)$. Let $s' = 3\langle e, x, m\rangle + 2$. If $x < i$, then $p_s \sqsupseteq p_{s'}$, hence by Lemma 3.6 F is compatible with $\langle A_{p_{s'}}, k_{p_{s'}}\rangle$ too. As we have that $p_{s'} * F \sqsupseteq p_{s'}$ and $x \in \Omega_e(X_m \oplus A_{p_{s'}*F})$, the construction at stage $s' + 1$ ensured that $x \in \Omega_e(X_m \oplus A)$. If $x \geq i$ then $p_{s'} \sqsupseteq p_s$. As we have that $p_s * F \sqsupseteq p_s$ and $x \in \Omega_e(X_m \oplus A_{p_s*F})$, there exists an $r' \sqsupseteq p_{s'}$ such that $x \in \Omega_e(X_m \oplus A_{r'})$ (otherwise $p_s * F$ and $p_{s'}$ would be two extensions of p_s contradicting our hypothesis). The construction at stage $s' + 1$ ensures that $x \in \Omega_e(X_m \oplus A)$.

□

Since the compatibility condition defined in Definition 3.5 is decidable, we have:

Corollary 3.9 *Under the assumptions of Lemma 3.8 we have:*

$$\Omega_e(X_m \oplus A) \leq_{\mathrm{be}} X_m.$$

Proof Let

$$G = A_{p_s} \cup \{\langle n, \langle x, E_u\rangle\rangle : n, u \in \omega \ \&\ x > k_{p_s}\}.$$

By the previous lemma (and by the definition of a condition, and that of a finite set F compatible with $\langle A_{p_s}, k_{p_s}\rangle$), it is not difficult to see that

$$\Omega_e(X_m \oplus A) = \Omega_e(X_m \oplus G).$$

Since G is computable, we also have that $X_m \oplus G \leq_{\mathrm{m}} X_m$, and thus $\Omega_e(X_m \oplus A) \leq_{\mathrm{be}} X_m$, as $\leq_{\mathrm{be}}$ extends $\leq_{\mathrm{m}}$. □

But the conclusion of the previous corollary is a contradiction, since in (b) we have supposed that $\Omega_e(X_m \oplus A) = \Omega_i(X_m \oplus B) \not\leq_{\mathrm{be}} X_m$.

Hence there must exist two conditions $q_0, r_0 \sqsupseteq p_s$ and a number x_0 such that

$$x_0 \in \Omega_e(X_m \oplus A_{q_0}) \ \&\ (\forall r' \sqsupseteq r_0)[x_0 \notin \Omega_e(X_m \oplus A_{r'})]. \tag{c}$$

We can suppose that $r_0 = \langle A_{p_s}, B_{p_s}, k_{q_0}\rangle$: indeed, (c) remains true if we extend q_0 and r_0 to two conditions in which k_{q_0} and k_{r_0} are replaced with $k = \max\{k_{q_0}, k_{r_0}\}$, and then of course we can replace the new r_0 with the extension of p_s, given by $\langle A_{p_s}, B_{p_s}, k\rangle$. Hence

Lemma 3.10 *Let $x_0 \in \omega$ and $q_0, r_0 \sqsupseteq p_s$ be such that:*

- $r_0 = \langle A_{p_s}, B_{p_s}, k_{q_0}\rangle$;
- $x_0 \in \Omega_e(X_m \oplus A_{q_0}) \ \&\ (\forall r' \sqsupseteq r_0)[x_0 \notin \Omega_e(X_m \oplus A_{r'})]$.

Then there exist $q, r \sqsupseteq p_s$ such that, for some n, z, u,

- $A_q = A_r \cup \{\langle n, \langle z, E_u\rangle\rangle\}$ *and* $B_q = B_r \cup \{\langle n, \langle z, E_u\rangle\rangle\}$;
- $k_r = k_q$;

– $x_0 \in \Omega_e(X_m \oplus A_q)$ & $(\forall r' \sqsupseteq r)[x_0 \notin \Omega_e(X_m \oplus A_{r'})]$.

Further, $A_{r_0} \subseteq A_r$ *and* $B_{r_0} \subseteq B_r$*; consequently* $A_{r_0} \subseteq A_q$ *and* $B_{r_0} \subseteq B_q$.

Proof Let $\langle n_0, \langle z_0, E_{u_0}\rangle\rangle, \langle n_1, \langle z_1, E_{u_1}\rangle\rangle, \ldots, \langle n_h, \langle z_h, E_{u_h}\rangle\rangle$ be a 1-1 enumeration of $A_{q_0} \setminus A_{r_0}$. Clearly $k_{r_0} > z_i$, for every $0 \leq i \leq h$. We build by induction a finite sequence $\{r_i\}$ of conditions extending p_s, starting with the given r_0. Suppose that we have defined $r_i = \langle A_{r_i}, B_{r_i}, k_{r_0}\rangle$, with $i \leq h$, and

$$(\forall r' \sqsupseteq r_i)[x_0 \notin \Omega_e(X_m \oplus A_{r'})].$$

Let $F_i = A_{r_i} \cup \{\langle n_i, \langle z_i, E_{u_i}\rangle\rangle\}$. Consider the condition $r_i * F_i$. (Notice that by assumptions and definition of amalgamation, $k_{r_i * F_i} = k_{r_0}$.) If

$$(\forall r' \sqsupseteq r_i * F_i)[x_0 \notin \Omega_e(X_m \oplus A_{r'})],$$

then define $r_{i+1} = r_i * F_i$ and proceed with the recursion. Otherwise stop the recursion, extend $r_i * F_i$ to a condition q such that $x_0 \in \Omega_e(X_m \oplus A_q)$ and let $r = \langle A_r, B_r, k_q\rangle$, with $A_r = A_q \setminus \{\langle n_i, \langle z_i, E_{u_i}\rangle\rangle\}$ and $B_r = B_q \setminus \{\langle n_i, \langle z_i, E_{u_i}\rangle\rangle\}$. Since $q \sqsupseteq r_i * F_i$, we have that $r \sqsupseteq r_i$, hence

$$(\forall r' \sqsupseteq r)[x_0 \notin \Omega_e(X_m \oplus A_{r'})].$$

Notice also that $A_q = A_r \cup \{\langle n_i, \langle z_i, E_{u_i}\rangle\rangle\}$, $B_q = B_r \cup \{\langle n_i, \langle z_i, E_{u_i}\rangle\rangle\}$ and $k_r = k_q$.

Now we claim that there must be an $i \leq h$ such that the second case holds for $r_i * F_i$. If not, then at the last step of the recursion ($i = h$), we built a condition r_{h+1} such that $A_{r_{h+1}} = A_{q_0}$ and $(\forall r' \sqsupseteq r_{h+1})[x_0 \notin \Omega_e(X_m \oplus A_{r'})]$. This is a contradiction, since $x_0 \in \Omega_e(X_m \oplus A_{q_0})$. □

Let q and r be the conditions provided by the previous lemma, with $k_q = k_r = k$. Let n, z, u be such that $A_q = A_r \cup \{\langle n, \langle z, E_u\rangle\rangle\}$ and $B_q = B_r \cup \{\langle n, \langle z, E_u\rangle\rangle\}$. We want to prove that, for such n, $C_n \leq_{\text{be}} X_m$.

Lemma 3.11 $E_u \subseteq C_n$.

Proof Suppose that $E_u \nsubseteq C_n$. Then $t = \langle A_q, B_r, k\rangle$ is a condition, since r is a condition and

$$(A_q^{[n]})^{\leq 1}(C_n) = (A_r^{[n]})^{\leq 1}(C_n) = (B_r^{[n]})^{\leq 1}(C_n).$$

Furthermore $x_0 \in \Omega_e(X_m \oplus A_t)$. Now suppose that there exists G compatible with $\langle B_r, k\rangle$ such that $x_0 \in \Omega_i(X_m \oplus G)$. Then $r * G \sqsupseteq p_s$ can force $\Omega_e(X_m \oplus A) \neq \Omega_i(X_m \oplus B)$ at stage $s+1$, contradicting our assumption in (b) that $\Omega_e(X_m \oplus A) = \Omega_i(X_m \oplus B)$. Hence

$$(\forall G)[\text{cpt}(G, \langle B_r, k\rangle) \Rightarrow x_0 \notin \Omega_i(X_m \oplus G)].$$

Since $\langle B_t, k_t\rangle = \langle B_r, k\rangle$,we have that

$$(\forall G)[\mathrm{cpt}(G, \langle B_t, k_t\rangle) \Rightarrow x_0 \notin \Omega_i(X_m \oplus G)].$$

Then for every $t' \sqsupseteq t$, $x_0 \notin \Omega_i(X_m \oplus B_{t'})$, and thus $t \sqsupseteq p_s$ can force $\Omega_e(X_m \oplus A) \neq \Omega_i(X_m \oplus B)$ at stage s+1, contradiction. □

Lemma 3.12 *For every y,*

$$y \in C_n \Leftrightarrow (\exists G)[\mathrm{cpt}(G, \langle B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\}, k\rangle) \;\&\; x_0 \in \Omega_i(X_m \oplus G)].$$

Proof (⇒:) Suppose that $y \in C_n$. Let again q, r be the conditions provided by Lemma 3.10. Then the condition

$$t = \langle A_q, B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\}, k\rangle$$

is an extension of p_s such that $x_0 \in \Omega_e(X_m \oplus A_t)$: this is so because by Lemma 3.11, $z \in (A_q^{[n]})^{\leq 1}(C_n)$. Now if there is no G compatible with $\langle B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\}, k\rangle$ such that $x_0 \notin \Omega_i(X_m \oplus G)$, then for every $t' \sqsupseteq t$, $x_0 \notin \Omega_i(X_m \oplus B_{t'})$, and thus t can force $\Omega_e(X_m \oplus A) \neq \Omega_i(X_m \oplus B)$ at stage $s+1$, contradiction.
(⇐): Suppose that $y \notin C_n$. Then $v = \langle A_r, B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\}, k\rangle$ is an extension of p_s such that

$$(\forall v' \sqsupseteq v)[x_0 \notin \Omega_e(X_m \oplus A_{v'})].$$

Now suppose that there exists a finite set G compatibile with

$$\langle B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\}, k\rangle)$$

and $x_0 \in \Omega_i(X_m \oplus G)$. Then $v * G \sqsupseteq p_s$ can force $\Omega_e(X_m \oplus A) \neq \Omega_i(X_m \oplus B)$ at stage s+1, contradiction. □

Corollary 3.13 *For the n provided by Lemma 3.10, we have $C_n \leq_{\mathrm{be}} X_m$.*

Proof Let r, n, z, k be as in Lemma 3.10, and for every y let

$$G_y = B_r \cup \{\langle n, \langle z, E_{y+1}\rangle\rangle\} \cup \{\langle m, \langle x, E_u\rangle\rangle : m, u \in \omega \;\&\; x > k\}.$$

It follows by Lemma 3.12:

$$y \in C_n \Leftrightarrow x_0 \in \Omega_i(X_m \oplus G_y).$$

Since G_y is computable, we have that $X_m \oplus G_y \leq_{\mathrm{m}} X_m$ uniformly in y, hence there is a computable function g such that

$$X_m \oplus G_y = \Omega_{g(y)}(X_m),$$

where each $\Omega_{g(y)}$ is an m-*operator*, i.e. an e-operator Φ of the form $\Phi = \{\langle x, \{f(x)\}\rangle : x \in \omega\}$, for some total computable function f. Notice that an index for such a computable function f_y relative to $\Omega_{g(y)}$, can be uniformly found in y.

Hence $C_n = \Omega(X_m)$, where

$$\Omega = \{\langle y, D\rangle : \langle x_0, D\rangle \in \Omega_i \circ \Omega_{g(y)}\}.$$

It is clear that Ω is a be-operator, since

$$\begin{aligned}\langle y, D\rangle \in \Omega \ &\Leftrightarrow \langle x_0, D\rangle \in \Omega_i \circ \Omega_{g(y)}\\ &\Leftrightarrow (\exists E)[\langle x_0, E\rangle \in \Omega_i \ \&\ D = f_y[E]] :\end{aligned}$$

Hence $|D| \leq |E|$ and thus Ω is a be-operator since so is Ω_i. □

□

3.2 Definability of countable relations

We now turn to definability of countable relations.

Lemma 3.14 *For every* be-*degree* $\boldsymbol{a}$*, there exists a collection* $\{\boldsymbol{c}_i : i \in \omega\}$ *of* be-*degrees, which does not introduce any new relation below* $\boldsymbol{a}$*, i.e. for every* $\boldsymbol{x}, \boldsymbol{y} \leq_{\text{be}} \boldsymbol{a}$*, every* $i \in \omega$*, and every computable set* I*,*

$$\boldsymbol{x} \cup \boldsymbol{c}_i \leq_{\text{be}} \boldsymbol{y} \cup \bigoplus_{j \in I} \boldsymbol{c}_j \ \Leftrightarrow\ \boldsymbol{x} \leq_{\text{be}} \boldsymbol{y} \ \&\ i \in J.$$

Proof We sketch the proof. Let $\boldsymbol{a} \in \mathcal{D}_{\mathbf{be}}$ be given, $A \in \boldsymbol{a}$, and $X_m = \Omega_m(A)$, where $\{\Omega_m\}_{m\in\omega}$ is a list of be-operators in $\mathcal{E}$. We want to construct a set C, such that, letting $C_i = C^{[i]}$, the ith column of C, we have that $\{\boldsymbol{c_i}\}_{i\in\omega}$ is the desired collection of be-degrees.

The construction of C is by stages, aiming at satisfying the following requirements, where $(X, Y) = (X_m, X_n)$ is any pair of sets $\leq_{\text{be}} A$, $i \in \omega$, $\Omega = \Omega_q$ is any be-operator:

$$\begin{aligned}&P_{\Omega,X,Y} : X = \Omega(Y \oplus C) \Rightarrow X \leq_{\text{be}} Y;\\ &R_{\Omega,i,Y} : C_i \neq \Omega(Y \oplus \bigoplus_{j \neq i} C_j).\end{aligned}$$

Suppose that a stage s we have defined a binary string γ_s which is intended to be a finite initial segment of the characteristic function of the eventual set C: let $\gamma_0 = \emptyset$. We define $\gamma_{s+1} \supset \gamma_s$ as follows.

If $s = 2t + 1$, $t = \langle q, m, n\rangle$ then we distinguish the following two cases (where $\Omega = \Omega_q$, $X = X_m$, $Y = X_n$):

(1) $(\exists x, F)[F$ is finite & $\min F \geq |\gamma_s|$ & $x \in \Omega(Y \oplus (\gamma_s^+ \cup F)) \setminus X]$ (where for a given string γ, $\gamma^+ = \{x < |\gamma| : \gamma(x) = 1\}$).
In this case, let $\gamma_{s+1} \supseteq \gamma_s$ be the least string γ such that $|\gamma| > |\gamma_s|$ and $F \subseteq \gamma^+$. Notice that this entails $X \neq \Omega(Y \oplus C)$.
(2) Otherwise, simply extend γ_{s+1} as above with $F = \emptyset$. In this case it is easy to see that

$$X = \Omega(Y \oplus G),$$

where G is the computable set,

$$G = \gamma_s^+ \cup \{x : x > |\gamma_s|\}.$$

But $Y \oplus G \equiv_{\mathrm{m}} Y$, hence $X \leq_{\mathrm{be}} Y$, since $\leq_{\mathrm{be}}$ extends $\leq_{\mathrm{m}}$.

If $s = 2t + 2$, $t = \langle q, i, n \rangle$, and $\Omega_q = \Omega$, $Y = X_n$, then let $x > |\gamma_s|$. We still distinguish two cases:

(1) $(\exists F)[F$ is finite & $\min F \geq |\gamma_s|$ & $F \subseteq \bigoplus_{j \neq i} \omega$ & $x \in \Omega(Y \oplus ((\gamma_s^+ \cap \bigoplus_{j \neq i} \omega) \cup F)$].
In this case let $\gamma_{s+1} \supseteq \gamma_s$ be the least string γ such that $|\gamma| > |\gamma_s|$, $\gamma(\langle i, x \rangle) = 0$ and $F \subseteq \gamma^+$. It follows that $x \in \Omega(Y \oplus \bigoplus_{j \neq i} C_j) \setminus C_i$.
(2) Otherwise, let $\gamma_{s+1} \supseteq \gamma_s$ be the least string γ such that $\gamma(\langle i, x \rangle) = 1$. It follows that $x \in C_i \setminus \Omega(Y \oplus \bigoplus_{j \neq i} C_j)$.

□

Remark 3.15 Notice that we could strengthen the result as to make $C_i \not\leq_{\mathrm{e}} (Y \oplus \bigoplus_{j \neq i} C_j)$, for every i and X: an argument similar to the one used in the above proof shows that it is in fact possible to satisfy the stronger requirements

$$R_{\Phi,i,Y} : C_i \neq \Phi(Y \oplus \bigoplus_{j \neq i} C_j),$$

for every enumeration operator Φ. Notice also, by the observation made at the end of step $2t + 1$, that the result holds not only for $\leq_{\mathrm{be}}$, but more generally for every $\leq_{\mathrm{r}}$ such that $\leq_{\mathrm{m}} \subseteq \leq_{\mathrm{r}} \subseteq \leq_{\mathrm{e}}$.

Theorem 3.16 *Every countable relation is first order definable from finitely many parameters in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$, in a uniform way.*

Proof First of all, we prove the theorem for countable sets. Given a countable set $\mathcal{A}$ of degrees, let $\boldsymbol{a}$ be above every element of $\mathcal{A}$ ($\boldsymbol{a}$ exists since $\mathcal{A}$ is countable, and $U_i \leq_{\mathrm{be}} \bigoplus_{j \in \omega} U_j$, for every countable collection of sets $\{U_j\}_{j \in \omega}$). Now by Lemma 3.14 consider a set $\mathcal{C}$ of degrees, with the same cardinality of $\mathcal{A}$, not introducing any new relation on the degrees under $\boldsymbol{a}$. Note that $\mathcal{C}$ is antichain (in fact a computably independent set of degrees) and so it is definable from parameters by Theorem 3.2.

Let f be a bijection, mapping $\mathcal{A}$ to $\mathcal{C}$ and consider the set $\mathcal{C}^* = \{\boldsymbol{x} \cup f(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{A}\}$. $\mathcal{C}^*$ is also an antichain, since for every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$

$$\boldsymbol{x} \cup f(\boldsymbol{x}) \leq_{\mathrm{be}} \boldsymbol{y} \cup f(\boldsymbol{y}) \;\Leftrightarrow\; \boldsymbol{x} \leq_{\mathrm{be}} \boldsymbol{y} \;\&\; f(\boldsymbol{x}) = f(\boldsymbol{y}) \;\Leftrightarrow\; \boldsymbol{x} = \boldsymbol{y}$$

and so it is definable from parameters. Now we can show that

- *$\mathcal{A}$ is definable from parameters*: in fact

$$\boldsymbol{x} \in \mathcal{A} \;\Leftrightarrow\; \boldsymbol{x} \leq_{\mathrm{be}} \boldsymbol{a} \;\&\; (\exists \boldsymbol{c} \in \mathcal{C})(\boldsymbol{x} \cup \boldsymbol{c} \in \mathcal{C}^*).$$

  The left to right implication is trivial. Now suppose that $\boldsymbol{x} \leq_{\mathrm{be}} \boldsymbol{a}$ and there exists an element $\boldsymbol{c} \in \mathcal{C}$ such that $\boldsymbol{x} \cup \boldsymbol{c} \in \mathcal{C}^*$. Then $\boldsymbol{x} \cup \boldsymbol{c} = \boldsymbol{y} \cup f(\boldsymbol{y})$, for some $\boldsymbol{y} \in \mathcal{A}$. By definition of $\mathcal{A}$, it follows that $\boldsymbol{x} = \boldsymbol{y}$ and $\boldsymbol{c} = f(\boldsymbol{y})$. Then $\boldsymbol{x} \in \mathcal{A}$.
- *The function f from $\mathcal{A}$ to $\mathcal{C}$ is definable from parameters*: in fact

$$f(\boldsymbol{x}) = \boldsymbol{y} \;\Leftrightarrow\; \boldsymbol{x} \in \mathcal{A} \;\&\; \boldsymbol{y} \in \mathcal{C} \;\&\; \boldsymbol{x} \cup \boldsymbol{y} \in \mathcal{C}^*.$$

Note that, by Theorem 3.2, the formulas defining $\mathcal{A}$ and f are fixed, only the parameters change. This ensures the uniformity of our results.

Finally we how to define countable relations. Let $\mathcal{R}$ be an n-ary relation on degrees. Consider the n projections of $\mathcal{R}$, i.e.

$$\mathcal{R}_i = \{\boldsymbol{x} : (\exists \langle \boldsymbol{x}_1, \ldots \boldsymbol{x}_n \rangle \in \mathcal{R})(\boldsymbol{x} = \boldsymbol{x}_i)\},$$

for each $1 \leq i \leq n$. Let $\boldsymbol{r}$ be above every element of $\bigcup_{1 \leq i \leq n} \mathcal{R}_i$. Now consider a set $\mathcal{C}$, which is a disjoint union of sets $\mathcal{C}_i$ with $1 \leq i \leq n$, such that for every i, $\mathcal{C}_i$ has the same cardinality as $\mathcal{R}_i$ and for every $\boldsymbol{x}, \boldsymbol{y} \leq_{\mathrm{be}} \boldsymbol{r}$ and $\boldsymbol{c}, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_k \in \mathcal{C}$

$$\boldsymbol{x} \cup \boldsymbol{c} \leq_{\mathrm{be}} \boldsymbol{y} \cup \boldsymbol{c}_1 \cup \cdots \cup \boldsymbol{c}_k \;\Leftrightarrow\; \boldsymbol{x} \leq_{\mathrm{be}} \boldsymbol{y} \;\&\; \boldsymbol{c} = \boldsymbol{c}_j,$$

for some $1 \leq j \leq k$. Such a $\mathcal{C}$ exists by Lemma 3.14. Note that $\mathcal{C}$ is an independent antichain. For every i, let f_i be a bijection, mapping $\mathcal{R}_i$ to $\mathcal{C}_i$. By the same arguments used for countable sets, we can prove that each $\mathcal{R}_i$ and f_i are definable from parameters. Now define

$$\mathcal{A} = \{\boldsymbol{c}_1 \cup \cdots \cup \boldsymbol{c}_n : \langle f_1^{-1}(\boldsymbol{c}_1), \ldots, f_n^{-1}(\boldsymbol{c}_n) \rangle \in \mathcal{R}\}.$$

$\mathcal{A}$ is a countable set and so it is definable from parameters. Furthermore, for each element $\boldsymbol{a} \in \mathcal{A}$, there is a unique sequence $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in \mathcal{C}$ such that $\boldsymbol{a} = \boldsymbol{c}_1 \cup \cdots \cup \boldsymbol{c}_n$, since $\mathcal{C}$ is an independent antichain. Then

$$\mathcal{R} = \{\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \rangle : (\forall i \leq n)(\boldsymbol{x}_i \in \mathcal{R}_i) \;\&\; f_1(\boldsymbol{x}_1) \cup \cdots \cup f_n(\boldsymbol{x}_n) \in \mathcal{A}\}$$

gives the desired definition of $\mathcal{R}$ from parameters. The required uniformity follows from the uniformity of Theorem 3.2 and what we proved for countable sets. □

3.3 Classifying the complexity of the first order theory

Let us say that a reducibility $\leq_r$ is *arithmetical* if there exists a formula of second order arithmetic $\varphi(U, V)$ with free set variables amongst U, V and no occurrence of quantified set variables, such that, for all pairs of sets X, Y,

$$X \leq_r Y \Leftrightarrow \mathbb{N} \models \varphi(X, Y).$$

Lemma 3.17 *Let V_2 denote the true sentences of second order arithmetic. If $\leq_r$ is arithmetical and $\mathcal{D}_\mathbf{r}$ is the degree structure of $\leq_r$, then $Th(\mathcal{D}_\mathbf{r}) \leq_1 V_2$.*

Proof Let σ be a sentence in the language of posets. Consider a bijection $x \mapsto X$ mapping each variable in σ to a set variable in the second order language of arithmetic. This gives a way of translating sentences in the language L of posets into second order arithmetical sentences, upon interpretation of $x \leq y$ with $\varphi(X, Y)$, where φ is the arithmetical definition of the reducibility. It is therefore straightforward that $Th(\mathcal{D}_\mathbf{r}) \leq_m V_2$. Finally (using a familiar argument, since V_2 is a cylinder, see [16]) from a computable function f which gives the m-reduction, we can construct a one-one computable function g which gives the same reduction. We define g by induction. Suppose that $g(i)$ is defined on every $i < n$. Let $f(n) = m$, where m codes a sentence σ of second order arithmetic. Then define $g(n)$ to be the Gödel number of the sentence $\sigma \,\&\cdots\&\, \sigma$, where the number of iterations of & is such that the code of the sentence obtained is different from all $g(i)$, for $i < n$. □

Lemma 3.18 *The reducibility $\leq_{\text{be}}$ is arithmetical.*

Proof Notice that $X \leq_{\text{be}} Y$ if and only if

$$(\exists e)[\Phi_e \text{ bounded } \;\&\; X = \Phi_e(Y)],$$

where "Φ_e bounded" means: $(\exists n)(\forall x, D)[\langle x, D\rangle \in \Phi_e \Rightarrow |D| \leq n]$, which is a Σ^0_2 expression; on the other hand, a simple inspection shows that $X = \Phi_e(Y)$ is a Π^0_2 expression, hence $X \leq_{\text{be}} Y$ is Σ^0_3. □

Theorem 3.19 *The first order theory of $\mathcal{D}_\mathbf{be}$ is computably isomorphic to true second order arithmetic.*

Proof We prove that the two theories have the same 1-degree, thus they are computably isomorphic by the Myhill Isomorphism Theorem, see [10]. One direction is given by Lemma 3.17. For the converse, we recall that a *standard model of arithmetic* is a structure $\langle N, 0, s, +, \times\rangle$ where N is a countable set, 0 is a distinguished element, s is a unary function, $+$ and $\times$ are binary functions, such that $\langle N, 0, s, +, \times\rangle$ satisfies finitely many first order axioms (let us call $\mathcal{P}^-$ the collection of these axioms, for instance the axioms of Robinson's arithmetic) together with the second order induction. Now suppose that $\vec{\boldsymbol{n}}$ is a list of given degrees coding a countable set of degrees: we can say, with a first order formula in the language of posets, that given degrees $\vec{\boldsymbol{a}}$ code (on the set coded by $\vec{\boldsymbol{n}}$) an element 0, and the graphs of functions $s, +, \times$, satisfying the requirements to be a standard model of arithmetic: this can be done since every countable relation is first order definable, with parameters, in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$, by Theorem 3.16; $\mathcal{P}^-$ is finite; and second order quantification can be expressed by replacing quantification on subsets of N with quantification on the finitely many parameters defining such sets (using Theorem 3.16 and uniformity). Given a sentence φ of second order arithmetic, let $\hat{\varphi}$ be obtained by replacing quantification on subsets of N with that on the parameters defining such sets, and by replacing in an obvious way the occurrences of $0, s, +, \times$ with their first order definitions with parameters in the language of posets; finally, let φ^* say that there are degrees $\vec{\boldsymbol{n}}, \vec{\boldsymbol{a}}$ as above, which code a standard model of arithmetic, in which $\hat{\varphi}$ holds: then φ is true if and only if the first order sentence φ^*, in the language of posets, is true in $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$. Since φ^* can be effectively obtained from φ, we have that $V_2 \leq_{\mathrm{m}} Th(\boldsymbol{\mathcal{D}}_{\mathbf{be}})$. Finally, as in the proof of Lemma 3.17, using the fact that theories are cylinders, we conclude that in fact $V_2 \leq_1 Th(\boldsymbol{\mathcal{D}}_{\mathbf{be}})$. □

The following answers a question raised by Cooper [1, Question 5.13]:

Corollary 3.20 *The first order theory of $\boldsymbol{\mathcal{D}}_{\mathbf{be}}$ is undecidable and not axiomatizable.*

Proof It follows from the fact that true second order theory of arithmetic is undecidable and not axiomatizable. □

A closer look at its proof shows that Theorem 3.2 holds in fact of every reducibility $\leq_{\mathrm{r}}$ such that $\leq_{2\mathrm{e}} \subseteq \leq_{\mathrm{r}}$ (hence $\leq_{\mathrm{be}} \subseteq \leq_{\mathrm{r}}$), $\leq_{\mathrm{r}} \subseteq \leq_{\mathrm{e}}$, and $\leq_{\mathrm{r}}$ is given by some class $\mathcal{E}_{\mathrm{r}}$ of e-operators with suitable closure properties. For Corollary 3.9 to hold, it is sufficient that $\leq_{\mathrm{r}}$ extends $\leq_{\mathrm{m}}$ but this is ensured by the fact that $\leq_{\mathrm{be}} \subseteq \leq_{\mathrm{r}}$. More delicate is Corollary 3.13: a reasonable sufficient closure property for $\mathcal{E}_{\mathrm{r}}$, is the following:

(∗) For every $\Phi \in \mathcal{E}_{\mathrm{r}}$, for every x_0, and for every total computable function f, such that $\Phi_{f(y)}$ is an m-operator for every y, then $\Omega \in \mathcal{E}_{\mathrm{r}}$, where

$$\Omega = \{\langle y, D\rangle : y \in \omega \ \& \ \langle x_0, D\rangle \in \Phi \circ \Phi_{f(y)}\}.$$

It is not difficult to see that the above closure property is satisfied by any class $\mathcal{E}_{\mathrm{r}}$ which is *closed under smaller axioms*, meaning that for every $\Phi \in \mathcal{E}_{\mathrm{r}}$, and every enumeration operator Ψ, if

$$(\forall x, D)[\langle x, D\rangle \in \Psi \Rightarrow (\exists y, E)[\langle y, E\rangle \in \Phi \ \& \ |D| \leq |E|\,]$$

then $\Psi \in \mathcal{E}_{\mathrm{r}}$. Reducibilities $\leq_{\mathrm{be}}$ and $\leq_{\mathrm{e}}$ are clearly closed under smaller axioms.

Summarizing the various results of this section in a more general setting, we can conclude:

Theorem 3.21 *Let $\leq_{\mathrm{r}}$ be such that:*

(1) $\leq_{\mathrm{be}} \subseteq \leq_{\mathrm{r}} \subseteq \leq_{\mathrm{e}}$*;*
(2) *there is a class of* e*-operators $\mathcal{E}_{\mathrm{r}}$, such that the set $\{e : \Phi_e \in \mathcal{E}_{\mathrm{r}}\}$ is arithmetical, $\mathcal{E}_{\mathrm{r}}$ satisfies closure property* (∗)*, and $X \leq_{\mathrm{r}} Y$ if and only if $X = \Phi(Y)$ for some $\Phi \in \mathcal{E}_{\mathrm{r}}$.*

Then $Th(\boldsymbol{\mathcal{D}}_{\mathbf{r}})$ is computably isomorphic to true second order arithmetic.

Proof The proof is the same as for $\leq_{\mathrm{be}}$, by replacing every occurrence of $\leq_{\mathrm{be}}$ in the proof with $\leq_{\mathrm{r}}$: of course we can not conclude in general that $\leq_{\mathrm{r}}$ is Σ^0_3, but $\leq_{\mathrm{r}}$ is still arithmetical under the assumptions. □

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