Math. Struct. in Comp. Science (2010), *vol.* 20, *pp.* 799–811. © Cambridge University Press 2010
doi:10.1017/S0960129510000216

Diamond embeddings into the enumeration degrees

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Received 22 October 2009; revised 1 May 2010

We show that the diamond lattice can be embedded into the Σ^0_2 enumeration degrees preserving 0 and 1, with atoms one high and Π^0_1, and the other one low.

1. Introduction

When studying a degree structure, a very natural question is to determine which lattices can be embedded into the structure. This is important not only for measuring how complicated the structure is, but also for answering global questions about the structure, including issues such as decidability of fragments of its first-order theory. It is known that there is an embedding ι (preserving 0, sup, and jump operation) of the Turing degrees into the enumeration degrees, which maps the Turing degree of a set A to the enumeration degree of the characteristic function of A. When restricted to the low computably enumerable (c.e.) Turing degrees, ι also preserves infima (McEvoy and Cooper 1985). So all lattices that are known to be embeddable into the low c.e. Turing degrees are also embeddable into the Π^0_1 enumeration degrees, and thus into the enumeration degrees below the first enumeration jump $\mathbf{0}'_e$, which comprise the enumeration degrees that partition the Σ^0_2 sets. Since not all finite lattices are embeddable into the c.e. Turing degrees, the question arises as to what one can say about lattice embeddings that are not induced by ι. The first approach to this problem was through diamond embeddings. The *diamond lattice* (or

[†] Andrea Sorbi is partially supported by the project 'Computability with partial information', sponsored by BNSF, Contract No: D002-258/18.12.08.
[‡] Guohua Wu is partially supported by AcRF grants RG58/06 (M52110023) and RG37/09 (M52110101).
[§] Yue Yang is partially supported by NUS grant R 146-000-114-112; part of this research was carried out while he was an INDAM-GNSAGA Visiting Professor at Siena.

simply *diamond*) is the four-element Boolean algebra $\{0, a, b, 1\}$, with bottom 0, top 1, and a complement of b. Interest in diamond embeddings in the Σ_2^0-enumeration degrees was also originally motivated by Cooper's question, raised in Cooper (1984), asking whether the structure of the Σ_2^0-enumeration degrees and the structure of the c.e. Turing degrees are elementarily equivalent. Ahmad (1991) showed that the diamond can be embedded into the Σ_2^0-enumeration degrees through an embedding that preserves 0 and 1, that is, maps 0 to the least enumeration degree $\mathbf{0}_e$ and 1 to its jump $\mathbf{0}_e'$: since the diamond cannot be embedded into the c.e. Turing degrees preserving 0 and 1 (a classical result known as the Non-Diamond Theorem, which is due to Lachlan (Lachlan 1966)), the two structures are not elementarily equivalent. It is worth noting that the two atoms in the embedded diamond of Ahmad's proof are both low enumeration degrees, which is important for the economy of the proof and makes the construction a finite priority argument. Ahmad's diamond embedding result has since turned out to be a particular case of a much more general phenomenon, which was proved by Lempp and Sorbi (Lempp and Sorbi 2002), *viz.*, that every finite lattice is embeddable into the Σ_2^0 enumeration degrees preserving 0 and 1 through an embedding that maps each element except 1 to a low enumeration degree. Different proofs for diamond embeddings have appeared in the literature since then. Of course, if we are not intererested in preserving 0 or 1, then every incomplete Σ_2^0 enumeration degree $\mathbf{a}$ is the bottom of a diamond, since $\mathbf{a}$ is meet-reducible (Nies and Sorbi 1999), whereas not every non-zero $\mathbf{a} \leqslant_e \mathbf{0}_e'$ is the top of a diamond since there are non-zero join-irreducible Σ_2^0 enumeration degrees (Ahmad and Lachlan 1998). The final word in this direction was perhaps given in Arslanov *et al.* (2003), where it was proved that:

(1) If $\mathbf{a} <_e \mathbf{b}$ and $\mathbf{b}$ is total (that is, $\mathbf{b}$ contains the graph of a total function) and there is some total $\mathbf{c}$ with $\mathbf{a} \leqslant_e \mathbf{c} <_e \mathbf{b}$, then the diamond is embeddable with $\mathbf{a}$ as the bottom and $\mathbf{b}$ as the top.

(2) If $\mathbf{a}$ is total and Δ_2^0, then the diamond is embeddable with $\mathbf{a}$ as the top and $\mathbf{0}_e$ as the bottom.

Ahmad's result is a corollary of both statements.

It should be noted that in all known proofs of 0, 1-preserving diamond embeddings, the atoms are mapped to low enumeration degrees. Very little is known about 0, 1-preserving diamond embeddings in which the atoms do not embed to low enumeration degrees. So a new interesting line of research consists of exploring the possible complexity of the images of the intermediate elements of the diamond. This paper, which can be viewed as a sequel to Sorbi *et al.* (2009), aims to provide a contribution in this direction. In Sorbi *et al.* (2009), we showed that there is a minimal pair of Π_1^0 high enumeration degrees. In Theorem 1, we show that we can stretch the proof in Sorbi *et al.* (2009) from a minimal pair to a diamond embedding in which one of the atoms is still Π_1^0 and high, and the other is low (but, of course, necessarily not Π_1^0 by the Lachlan Non-Diamond Theorem). The construction shows a way of combining highness with the minimal pair argument in a simple way. While one set is constructed to be low, the 'meshing' phenomenon described in McEvoy and Cooper (1985) will not happen, which makes the proof of Theorem 1 in the current paper much simpler than the one given for Sorbi *et al.* (2009, Theorem 1), which involves a $0'''$-priority construction.

1.1. *The theorem*

We will prove the following theorem.

Theorem 1. There are non-zero enumeration degrees **a** and **b** below $\mathbf{0}'_e$ such that **a** is low, **b** is high and Π^0_1, and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$, $\mathbf{a} \cap \mathbf{b} = \mathbf{0}_e$. That is, the diamond lattice can be embedded into the Σ^0_2 enumeration degrees preserving 0 and 1, with two atoms, one low and the other high and Π^0_1.

Our notation and terminology for computability theory are standard and follow, unless otherwise specified, the textbook Soare (1987). Good introductions to enumeration reducibility and related notions can be found in Cooper (2003) and Odifreddi (1999). If $\langle z, F \rangle \in \Phi$, with Φ an enumeration operator, F is said to be a *use-neighbourhood* of z (relative to Φ).

2. Requirements

To prove Theorem 1, we will construct two Σ^0_2 sets A and B, via suitable Σ^0_2 approximations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$, and an enumeration operator Γ satisfying the following requirements:

$\mathscr{S}$: $\overline{K} = \Gamma^{A,B}$
$\mathscr{P}_e$: $A \neq W_e$
$\mathscr{L}_{\langle e,i\rangle}$: $i \in^\infty \Phi^A_e \;\Rightarrow\; i \in \Phi^A_e$
$\mathscr{M}_e$: $\Phi^A_e = \Psi^B_e \;\Rightarrow\; \Phi^A_e$ is c.e.
$\mathscr{H}_e$: φ_e total $\Rightarrow C_B$ dominates φ_e,

where C_B is the computation function of B, that is, for any x, we have

$$C_B(x) = \mu s > x[B_s \upharpoonright x \subseteq B \upharpoonright x],$$

where K is a creative set and $\{(W_e, \Phi_e, \Psi_e)\}_{e\in\omega}$ is an effective list of all triples (W, Φ, Ψ), where W is a c.e. set and Φ and Ψ are enumeration operators. Here $i \in^\infty \Phi^A_e$ means that there are infinitely many stages s such that $i \in \Phi^A_e[s]$, where, as in Soare (1987), the '$[s]$' denotes the fact that the expression is evaluated at the end of Stage s in some fixed uniform computable approximation of the enumeration operators using finite sets. In particular, $\Phi^A_e[s]$ stands for $\Phi^{A_s}_{e,s}$, where $\Phi_{e,s}$ and A_s denote the approximations at Stage s of Φ_e and A, respectively. Finally, $\Gamma^{A,B}$ denotes the application of the enumeration operator Γ to the set $A \oplus B$.

Let **a** and **b** be the enumeration degrees of A and B, respectively. Then the $\mathscr{L}$-requirements ensure that **a** is low (as proved in McEvoy and Cooper (1985)), and the $\mathscr{H}$-requirements ensure that **b** is high (refer to Shore and Sorbi (1999) to see why the domination property implies highness). The $\mathscr{P}$-requirements ensure that **a** is non-zero. Together with this, the $\mathscr{M}$-requirements ensure that **a** and **b** form a minimal pair, which means **b** is incomplete. The $\mathscr{S}$-requirement ensures **a** and **b** has join $\mathbf{0}'_e$. Therefore,

$\{\mathbf{0}_e, \mathbf{a}, \mathbf{b}, \mathbf{0}'_e\}$ is a diamond in the Σ^0_2 enumeration degrees. It will follow from the construction that B can be taken to be Π^0_1.

3. Strategies

In this section, we describe how each strategy works and consider the interactions between various strategies. In the following, when clarity is not compromised by the context, we will frequently not specify the stage at which the various expressions are evaluated: so, for instance, at Stage s, we may write $\overline{K}$ instead of $\overline{K}_s$ (where we refer to some fixed c.e. approximation $\{K_s\}_s$ to K using a computable increasing sequence of finite sets), and $\Gamma^{A,B}$ instead of $\Gamma^{A,B}[s]$.

3.1. *The $\mathscr{S}$-strategy*

$\mathscr{S}$ is a global requirement. That is, in the whole construction, we will construct an enumeration operator Γ such that $\overline{K} = \Gamma^{A,B}$. The construction of Γ follows the following rules:

(1) At Stage s, for the least $z \in \overline{K}$ but $z \notin \Gamma^{A,B}$, enumerate z into $\Gamma^{A,B}$ with a use-neighbourhood $\gamma(z)$ containing a big number – we enumerate this number into A and B at this stage (in fact, we may assume that the number is in B already, starting with $B_0 = \omega$) and later, when this number is removed from one of A and B, then z is automatically removed from $\Gamma^{A,B}$. This number in $\gamma(z)$ is called the *crucial element* of the use-neighbourhood $\gamma(z)$.
(2) If $z \in \Gamma^{A,B}$ at Stage s and we want to enumerate $z' > z$ into $\Gamma^{A,B}$ at this stage, we require that the selected use-neighbourhood of z' contains all numbers in the use-neighbourhood of z. Thus, whenever z is removed from $\Gamma^{A,B}$ later, z' is removed from $\Gamma^{A,B}$ automatically.
(3) If $z \in \Gamma^{A,B}$ at Stage s and $z \notin \overline{K}$, we remove the crucial element in the use-neighbourhood $\gamma(z)$ from B.

Note that in the construction, the use-neighbourhood $\gamma(z)$ not only contains crucial elements, but also contains other numbers, such as those numbers that have been selected as witnesses for $\mathscr{P}$-strategies, and removing such numbers (to satisfy $\mathscr{P}$-strategies) will also remove numbers from $\Gamma^{A,B}$ automatically. We will see this in both the construction and the verification.

3.2. *A $\mathscr{P}$-strategy*

A single $\mathscr{P}$-strategy is simply a Friedberg–Muchnik's diagonalisation argument. Let α be a $\mathscr{P}$-strategy. For convenience, we will sometimes use W_α to denote $W_{e(\alpha)}$ (where $e(\alpha) = e$, with $\mathscr{P}_e$ being the requirement assigned to α), and we will also apply this convention to other parameters and strategies. Strategy α works as follows:

(1) Choose a witness x_α and enumerate x_α into A.
(2) Wait for x_α to enter W_α.
(3) Extract x_α from A.

α has two possible outcomes:

(1) α waits forever at Step 2, in which case $A(x_\alpha) = 1 \neq 0 = W_\alpha(x_\alpha)$, and $\mathscr{P}_e$ is satisfied. Let w denote this outcome.
(2) α reaches Step 3 at some stage. Then $A(x_\alpha) = 0 \neq 1 = W_\alpha(x_\alpha)$, and $\mathscr{P}_e$ is satisfied. Let d denote this outcome.

3.3. *An $\mathscr{L}$-strategy*

Fix e and i. We satisfy the $\mathscr{L}_{\langle e,i\rangle}$-requirement by giving higher priority to the $\mathscr{L}_{\langle e,i\rangle}$-strategies – once i is found in Φ_e^A, we will protect this enumeration (in the construction, we will let the (already visited) $\mathscr{L}_{\langle e,i\rangle}$-strategy with the highest priority act). To be consistent with the $\mathscr{S}$-strategy, we set a threshold, p, for an $\mathscr{L}_{\langle e,i\rangle}$-strategy. That is, when an $\mathscr{L}_{\langle e,i\rangle}$-strategy is first visited, we set p to be a big number in $\overline{K}$, and this strategy can act to protect an enumeration as above only when $\gamma(p)$ is selected (it may happen that p enters K, in which case, we just update it as the next number in $\overline{K}$). Whenever a number $\leqslant p$ enters K, we reset this strategy by removing any restraint set by this strategy. To protect an enumeration from the construction of Γ, we need to remove the crucial element in $\gamma(p)$ from B, so that no further extractions for correcting $\Gamma^{A,B}(p')$ with $p' > p$ can injure this enumeration.

3.4. *An $\mathscr{M}$-strategy*

An $\mathscr{M}_e$-strategy η is a standard minimal pair strategy. That is, we define the length of agreement functions $l(\eta, s)$ and $m(\eta, s)$ as follows:

$$l(\eta, s) = \max\{x < s : \text{ for all } y < x, \Phi_\eta^A(y) = \Psi_\eta^B(y)\}$$
$$m(\eta, s) = \max\{l(\eta, t) : t < s \text{ is an } \eta\text{-stage}\}.$$

We say that a stage s is *η-expansionary* if $s = 0$ or $l(\eta, s) > m(\eta, s)$. At η-expansionary stages, we enumerate all $y < l(\eta, s)$ in Φ_η^A but not yet in V_η at Stage s, into V_η.

In the construction, we will ensure that after y enters V_η, then at any later stage y will be in at least one of Φ_η^A and Ψ_η^B (but not necessarily both). V_η is constructed as a c.e. set and if $\Phi_\eta^A = \Psi_\eta^B$, then V_η will be equal to these two sets, which shows that the $\mathscr{M}_e$-requirement is satisfied at η.

Unlike the high minimal pair construction of Sorbi *et al.* (2009, Theorem 1), since A is constructed as low, the 'meshing' phenomenon will not happen in this diamond embedding, and we do not need to worry about Σ_3-outcomes satisfying the $\mathscr{M}$-strategies. In other words, lowness of A eliminates the risk that there exists some x such that x is in neither Φ_η^A nor Ψ_η^B, but at each stage s, x appears to be in either $\Phi_\eta^A[s]$ or $\Psi_\eta^B[s]$. While lowness of either side of the minimal pair is an immediate antidote to prevent this phenomenon (called 'meshing' in McEvoy and Cooper (1985)) from appearing, other constructions, such as the one in Sorbi *et al.* (2009), require a more careful analysis of interactions between strategies: in particular, in building a minimal pair in which both A and B are high, satisfaction of η would usually be delegated to a lower priority highness strategy demanding that x be removed from Φ_η^A or Ψ_η^B infinitely many times, that is, η

would be satisfied through what is usually called a Σ_3^0-outcome in the jargon of priority arguments.

η has two possible outcomes $i <_L f$, where i denotes the case where there are infinitely many η-expansionary stages and f denotes the case where there are only finitely many η-expansionary stages.

3.5. *An $\mathscr{H}$-strategy*

Let ζ be a strategy working for $\mathscr{H}_e$. As usual, ζ has a fixed infinite computable set $E_\zeta = \{x_0 < x_1 < \dots\}$ at its disposal, and we start with $E_\zeta \subseteq B$. Strategy η waits for a stage s at which $\varphi_\zeta(y)\downarrow$ for all y such that $x_0 \leqslant y < x_1$ (without loss of generality, we may assume that $\varphi_\zeta(y) < s$). If no such s exists, φ_ζ is not total. So we suppose that such an s exists. So ζ extracts x_0 from B forever at the first stage $t > s$ that we visit ζ again. Thus, for all y with $x_0 \leqslant y < x_1$, we have $C_B(y) \geqslant t > s > \varphi_\zeta(y)$. We then work on x_1 in the same way, and so on. We define the length of agreement function $l(\zeta, s)$ as follows:

$$l(\zeta, s) = \max\{x < s : \text{ for all } y < x, \varphi_\zeta(y) \text{ converges}\}$$
$$m(\zeta, s) = \max\{l(\zeta, t) : t < s \text{ is a } \zeta\text{-stage}\}.$$

We say that a stage s is ζ-*expansionary* if $s = 0$ or $l(\zeta, s) > m(\zeta, s)$. At ζ-expansionary stages, we extract the least number x in $E_\zeta[s]$ (which is E_ζ minus the elements that have been extracted at stages $< s$) from B, provided there exists a bigger number in $E_\zeta[s]$ less than $l(\zeta, s)$. Thus, if φ_ζ is total, ζ will extract all numbers in E_ζ from B, and for any $y \geqslant x_0$, we have $C_B(y) > \varphi_\zeta(y)$.

ζ has two possible outcomes $i <_L f$, where i denotes the case where there are infinitely many ζ-expansionary stages, and f denotes the case where there are only finitely many ζ-expansionary stages.

Notte that the basic idea of an $\mathscr{H}$-strategy does not differ from similar constructions that appear in the literature. In particular, the handling of the $\mathscr{H}$-strategies in this construction is quite similar to the one used in the construction of a minimal pair of high enumeration degrees (Sorbi *et al.* 2009), that is, at any expansionary stage, at most one element is extracted from A and B.

3.6. *Interactions between strategies*

We now discuss possible interactions between the strategies, and describe how to solve any conflicts that may occur.

We will first consider the case where a $\mathscr{P}$-strategy α is below the infinitary outcome i of an $\mathscr{M}$-strategy η, and there exist $\mathscr{H}$-strategies between them. Without loss of generality, we can assume that $\eta^\frown i \subseteq \zeta^\frown i \subseteq \alpha$, where ζ is an $\mathscr{H}$-strategy. Then ζ and η can prevent a $\mathscr{P}$-strategy α from being satisfied. Suppose α selects a number x_α at Stage s_0 (so x_α is enumerated into A at this stage) and sees that x_α is in W_α at Stage s_1. So α wants to remove x_α from A now, but it also notices that if x_α is removed from A, then some z can also be removed from Φ_η^A. It can also happen that z is also removed from Ψ_η^B by ζ's action, so when η is visited again, z is in neither Φ_η^A nor Ψ_η^B, so our strategy for η

fails if $z \in V_\eta$. To avoid this, as in the standard construction of a high minimal pair of enumeration degrees (Sorbi *et al.* 2009), only one element (the least number of a suitably appointed finite set F) can be extracted from either A or B between two consecutive η-expansionary stages. So, after seeing that x_α is in W_α, α waits until all numbers less than x_α in E_ζ have been extracted from B, and then at the next η-expansionary stage, α extracts x_α from A. Hence, α's action is delayed at most finitely many times, and such an action is consistent with ζ.

In general, if there are many $\mathscr{H}$-strategies between η and α, we extract only the least number from $\{x_\alpha\} \cup (\bigcup_\zeta E_\zeta)$ at each η-expansionary stage. Again, α's action is delayed in this way at most finitely many times, and is thus consistent with η, and with the $\mathscr{H}$-strategies between η and α too.

We will now consider whether any consistency problems arise between an $\mathscr{L}$-strategy β and other strategies, such as a $\mathscr{P}$-strategy α, an $\mathscr{M}$-strategy η or the splitting-strategy $\mathscr{S}$. Suppose β has higher priority than α and η has higher priority than both α and β. Suppose that at Stage s_1, α extracts a number x from A (so at this point, α's extraction does not injure η). Then, before the next η-expansionary stage (that is, before we see new axioms enumerating numbers in V_η into Φ_η^A), β (working for requirement $\mathscr{L}_{\langle e,i\rangle}$) sees that i is now in Φ_e^A, so to keep i in Φ_e^A, and to ensure consistency between β and the $\mathscr{S}$-strategy, β extracts the crucial element of $\gamma(p_\beta)$ from B, as described in the $\mathscr{L}$-strategy. Then at the next η-expansionary stage $s_3 > s_2$, both A and B-sides have changed, which can result in the failure of η.

To avoid this, at Stage s_2, when β acts, we also put the number x back into A to make sure that the A-side is kept the same as before, and hence to keep η working as before. β is happy with this enumeration as i is kept in Φ_e^A. And α is also happy with the re-enumeration of x back into A as it is initialised at Stage s_2, and when it is visited again, it will select another number, x' say, as a witness.

4. Construction

We are now ready to give the full construction. First we assign the following priority ranking to the requirements:

$$\mathscr{S} < \mathscr{P}_0 < \mathscr{L}_0 < \mathscr{H}_0 < \mathscr{M}_0 < \mathscr{P}_1 < \mathscr{L}_1 < \mathscr{H}_1 < \mathscr{M}_1 < \cdots$$

$$< \mathscr{P}_e < \mathscr{L}_e < \mathscr{H}_e < \mathscr{M}_e < \cdots .$$

The construction makes use of a priority tree: see Soare (1987) for details of the tree method in computability theory. The priority tree T is built as follows. First note that $\mathscr{S}$ is a global strategy, so we do not put it on T. Assume that τ is a node on T. Then:

(1) If $|\tau| = 4e$, let τ be a $\mathscr{P}_e$-strategy. τ has outcomes $w <_L d$, where w denotes the outcome that τ waits for x_τ to appear in W_τ, and w denotes the outcome that τ extracts x_τ from A eventually.

(2) If $|\tau| = 4e + 1$, let τ be a $\mathscr{L}_e$-strategy. τ has outcome 1.

(3) If $|\tau| = 4e + 2$, let τ be an $\mathscr{H}_e$-strategy. τ has outcomes $i <_L f$, where i denotes the outcome that there are infinitely many τ-expansionary stages, and f denotes the outcome that there are only finitely many τ-expansionary stages.

(4) If $|\tau| = 4e + 3$, let τ be an $\mathscr{M}_e$-strategy. τ has outcomes $i <_L f$, where i denotes the outcome that there are infinitely many τ-expansionary stages, and f denotes the outcome that there are only finitely many τ-expansionary stages.

We assume that at each stage $s > 0$, exactly one number, k_s, is extracted from $\overline{K}$.

We now describe the full construction. At Stage s, we first define the current true path σ_s with $|\sigma_s| \leqslant s$. In the following, when a strategy is initialised all parameters associated with it will be cancelled and can only be redefined at a later stage when it is visited again.

Stage 0**:** Initialise all the nodes on T and let A, V_η, where $\eta \in T$ is any $\mathscr{M}$-strategy, and all constructed functionals, be $\emptyset$; let $B = \omega$. Let $\sigma_0 = \lambda$, the root of T, and go to Stage 1.

Stage $s > 0$**:** Stage s consists of five phases:

Phase 1. Extract the crucial element of $\gamma(k_s)$ from B, enumerate into A all the numbers that have been extracted from A after the stage at which $\gamma(k_s)$ is defined, and reset all strategies ξ with threshold $p_\xi \geqslant k_s$: if k_s is a threshold of ξ and ξ is not satisfied yet, choose the least number in $\overline{K}_s$, bigger than k_s, of course, and define it as p_ξ. If ξ is reset and ξ' is a strategy with lower priority, then ξ' is initialised automatically.

Phase 2. Find the highest $\mathscr{L}$-strategy β that is not satisfied yet but sees $i(\beta)$ entering $\Phi^A_{e(\beta)}$ at Stage s. Extract $\gamma(p_\beta)$ (or, rather, the crucial element of $\gamma(p_\beta)$) from B, and enumerate into A all the numbers that have been extracted from A after the stage at which p_β is defined. Declare that β is satisfied at Stage s and initialise all the strategies with lower priority.

Phase 3. Define σ_s inductively starting from λ with $|\sigma_s| \leqslant s$. At the end of Stage s, initialise all the strategies with priority lower than σ_s.

Substage 0**:** Let $\sigma_s(0) = \lambda$. Recall that λ is a $\mathscr{P}_0$ strategy. If p_λ and x_λ are not defined (this happens only at Stage 1), define them as fresh numbers, enumerate x_λ into A and stop Stage s. If p_λ, x_λ are defined and x_λ is not in W_λ, let $\lambda^\frown w$ act at the next substage. If λ has been satisfied before, let $\lambda^\frown d$ act at the next substage. Otherwise, that is, λ has not been satisfied so far and x_λ is in W_λ, extract x_λ from A, declare that λ is satisfied at Stage s and stop Stage s.

Substage $t > 0$**:** Assume that $\sigma_s(t)$ has already been constructed. If $|\sigma_s(t)| = s$ then set $\sigma_s = \sigma_s(t)$ and go to the next stage. If $|\sigma_s(t)| < s$, find the outcome for $\sigma_s(t)$ as follows:

Case 1: $\sigma_s(t) = \alpha$ is a $\mathscr{P}$-strategy:

(α1): x_α and p_α are not defined.
Define each of them as fresh numbers, enumerate x_α into A, let $\sigma_s = \sigma_s(t)$ and go to the next stage.

(α2): x_α and p_α are defined.
If α has been satisfied before and has not been initialised since then, let $\sigma_s(t + 1) = \alpha^\frown d$ and go to the next substage.

Otherwise, check whether x_α is in W_α or not. If it is not in W_α, let $\sigma_s(t+1) = \alpha^\frown w$ and go to the next substage. If it is in W_α, enumerate x_α into F_{s+1} and stop Stage s by letting $\sigma_s = \alpha$.

Case 2: $\sigma_s(t) = \beta$ is an $\mathscr{L}$-strategy:

Check whether p_β is defined or not. If p_β is not defined yet, define it as a big number in $\overline{K}$ that is bigger than s, and stop Stage s. Otherwise, that is, either β has been satisfied before or $i(\beta) \notin \Phi^A_{e(\beta)}$, let $\sigma_s(t+1) = \alpha^\frown 1$ and go to the next substage. *Note that β may act at Phase 2.*

Case 3: $\sigma_s(t) = \zeta$ is an $\mathscr{H}$-strategy:

Check whether s is ζ-expansionary or not. If it is not ζ-expansionary, let $\sigma_s(t+1) = \alpha^\frown f$ and go to the next substage. If it is ζ-expansionary, let $\sigma_s(t+1) = \alpha^\frown i$, enumerate those numbers less than $l(\zeta, s)$ that are in $E_{\zeta,s}$, but not the biggest one, into F_{s+1}, and go to the next substage.

Case 4: $\sigma_s(t) = \eta$ is an $\mathscr{M}$-strategy.

There are two cases:

(η1): s is not η-expansionary.
Let $\sigma_s(t+1) = \eta^\frown f$, and go to the next substage.

(η2): s is η-expansionary.
Let $\sigma_s(t+1) = \eta^\frown i$, enumerate all $y < l(\eta, s)$ in Φ^A_η, but not yet in $V_\eta[s]$, into V_η, and go to the next substage.

Phase 4 (Extraction). Extract the least number in F_{s+1} from A or B. If the number being extracted is from A, then this number is selected by a $\mathscr{P}$-strategy α (actually, $\sigma_s = \alpha$), and we declare at this phase that α is satisfied at Stage s.

Phase 5 (Extending Γ). Find the least number n in $\overline{K}$ with $n \notin \Gamma^{A,B}$, and enumerate n into $\Gamma^{A,B}$ with use-neighbourhood consisting of the following numbers:

(1) a single crucial number, which is a fresh number;

(2) all numbers in $\gamma(n')$, if $n' < n$ and $n' \in \Gamma^{A,B}$;

(3) if $n \geqslant p_\alpha$, where α is a $\mathscr{P}$-strategy, and α is active at Stage s (that is, α of higher priority has been visited at a stage $t \leqslant s$ and not initialised since then), then $\gamma(n)$ contains x_α. In particular, if n is a threshold of a $\mathscr{P}$-strategy α, then $\gamma(n)$ contains x_α.

This completes the full construction. Take

$$A = \{x : (\exists t)(\forall s \geqslant t)[x \in A_s]\}$$
$$B = \{x : (\exists t)(\forall s \geqslant t)[x \in B_s]\}.$$

In fact, it follows from the construction that $B = \{x : (\forall s)[x \in B_s]\}$.)

5. Verification

We will now prove that the constructed A, B and Γ satisfy all the requirements. We say that a strategy ξ is visited at Stage s if $\xi = \sigma_s(t)$ for some t. Define TP, the true path, as $\liminf_s \sigma_s$. That is, λ is on TP (λ is visited at every stage s), and if ξ is on TP, then for an outcome O of ξ, we have $\xi^\frown O$ is on TP if and only if there are infinitely many $\xi^\frown O$-stages, and for any outcome $O' <_L O$, there are at most finitely many $\xi^\frown O'$-stages. The following is a crucial lemma for the verification.

Lemma 5.1. For each ξ on TP:

(i) ξ can be reset or initialised at most finitely often.
(ii) There is an outcome O of ξ such that $\xi^\frown O$ is on TP.
(iii) If O is as in the previous item, ξ can initialise $\xi^\frown O$ at most finitely often.

Therefore, TP is infinite.

Proof. We prove this lemma by induction. Let ξ^- be the immediate predecessor of ξ, and assume that (i)–(iii) are all true for ξ^-. That is, there is a least stage s_{ξ^-} after which:

(i) ξ^- cannot be reset or initialised,
(ii) no nodes to the left of ξ can be visited again,
(iii) ξ can be initialised by ξ^- at most finitely many times.

We now prove that the lemma is true for ξ.

From the assumption on ξ^- and the choice of s_{ξ^-}, after Stage s_{ξ^-}, we know ξ can never be initialised by strategies with higher priority. Let s_0 denote s_{ξ^-}, and let $s_1 \geqslant s_0$ be the stage at which p_ξ is selected (if ξ is an $\mathscr{L}$-strategy or a $\mathscr{P}$-strategy). Then ξ can be reset only when some number $k \leqslant p_\xi$ leaves $\overline{K}$ (as $\overline{K}$ is infinite, p_ξ settles down eventually). This shows that (i) is true for ξ.

We now show that (ii) and (iii) are both true for ξ. There are four cases:

Case 1: $\xi = \alpha$ *is a* $\mathscr{P}$*-strategy.*

By the choice of s_0, when α is first visited after s_0, we have that α defines p_α and x_α (x_α is enumerated into A). This x_α will not be cancelled in the remainder of the construction, even though p_α is updated (when the previous p_α enters K, but this can happen at most finitely many times).

If x_α does not enter W_α at any stage, any (further) α-stage will be an $\alpha^\frown w$-stage, and hence according to the definition of TP, we have $\alpha^\frown w$ is on TP.

If x_α enters W_α at some stage, at the next few α-stages $s > s_0$, we know x_α is put into F_s. Note that at such a stage s, σ_s is defined as α, and only the least number from F_s is extracted from A or B. At these stages, numbers in E_ζ, where $\zeta \subset \alpha$ is an $\mathscr{H}$-strategy with infinitary outcome along α, are extracted one by one, so, eventually, at a large enough α-stage s, we know x_α will turn out to be the least number in F_s, thus x_α is extracted from A at this stage, and α is declared to be satisfied. After this stage, any further α-stage is an $\alpha^\frown d$-stage, and (ii) is true. Note that after Stage s, α never acts again, so (iii) is true for α.

Case 2: $\xi = \beta$ *is an* $\mathscr{L}$*-strategy.*

As β has only one outcome, and any β-stage is also a $\beta^\frown 1$-stage, $\beta^\frown 1$ is on TP, thus (ii) is true for β. And (iii) is also true for β since β may only act when $i(\beta)$ enters $\Phi^A_{e(\beta)}$, but by the choice of s_{ξ^-}, such an action can only happen (in Phase 2) at most once after this stage.

Case 3: *$\xi = \zeta$ is an $\mathscr{H}$-strategy.*

If φ_ζ is total, there are infinitely many ζ-expansionary stages, so ζ has outcome i infinitely often. That is, at these stages, $\zeta^\frown i$ is visited. If φ_ζ is not total, there are only finitely many ζ-expansionary stages, and after a large enough stage, ζ always has outcome f. That is, any further ζ stage is a $\zeta^\frown f$-stage. In all cases, (ii) is true for ζ. And (iii) is obviously true for ζ since ζ does no action during the whole construction.

Case 4. *$\xi = \eta$ is an $\mathscr{M}$-strategy.*

The argument for this case is similar to the argument for ζ. We consider whether there are finitely many or infinitely many η-expansionary stages, and show that in either case, η has an outcome O such that $\eta^\frown O$ is on TP, so (ii) is true. And (iii) is also true since, apart from enumerating V_η, we know η does no action during the whole construction.

This shows that (i)–(iii) is true for every node on TP, so TP is infinite. □

Note that the proof of Lemma 5.1 also shows that the $\mathscr{L}$-strategies are satisfied, and whenever an $\mathscr{L}$-strategy β is visited at a stage s in Phase 3, it does not act at all. This means we have the following lemma.

Lemma 5.2. Given a requirement $\mathscr{L}_e$, let β be an $\mathscr{L}_e$-strategy on TP. Then the requirement $\mathscr{L}_e$ is satisfied by β or by an $\mathscr{L}_e$-strategy to the left of β.

Notice that β can act even though it is not on the current true path.

From Lemma 5.1, we can also see that the $\mathscr{P}$-strategies are also all satisfied along the true path. Let α be a $\mathscr{P}_e$-strategy on TP. If x_α never enters W_α, then $A(x_\alpha) = 1 \neq 0 = W_\alpha(x_\alpha)$. This corresponds to outcome w. If x_α enters W_α, then, as argued above, x_α will be extracted from A eventually, and hence $A(x_\alpha) = 0 \neq 1 = W_\alpha(x_\alpha)$. This corresponds to outcome d. Thus, we have the following lemma.

Lemma 5.3. Given a requirement $\mathscr{P}_e$, let α be a $\mathscr{P}_e$-strategy on TP. Then the requirement $\mathscr{P}_e$ is satisfied by α.

Now we show that for any e, the $\mathscr{H}_e$-strategy is also satisfied.

Lemma 5.4. Given a requirement $\mathscr{H}_e$, let ζ be an $\mathscr{H}_e$-strategy on TP. Then the requirement $\mathscr{H}_e$ is satisfied by ζ.

Proof. Without loss of generality, suppose that φ_e is total. Then, as ζ is on TP, there are infinitely many ζ-expansionary stages, and hence each number x in E_ζ is eventually removed. Note that in the construction, a (least) number in E_ζ is put into F_s at a stage s only when $l(\zeta, s)$ is bigger than the next number in E_ζ. So, for any $x \in E_\zeta$, except the least one, we have first convergence of $\varphi_e(x)$ at a stage s, with $\varphi_e(x) < s$, and then $B \upharpoonright x$ changes later. This ensures that $C_B(x) > \varphi_e(x)$, and hence C_B dominates φ_e, which means $\mathscr{H}_e$ is satisfied. □

Now we show that for all e, the $\mathscr{M}_e$-strategy is also satisfied.

Lemma 5.5. Given a requirement $\mathcal{M}_e$, let η be the $\mathcal{M}_e$-strategy on TP. Then the requirement $\mathcal{M}_e$ is satisfied at η.

Proof. Let s_η be the last stage at which η is initialised. Let V_η be the set constructed after Stage s_η. We prove that η is satisfied, and, furthermore, if $\Phi_\eta^A = \Psi_\eta^B$, then $\Phi_\eta^A = V_\eta$.

There are two cases:

Case 1: There are only finitely many η-expansionary stages.

In this case, $l(\eta, s)$ has a liminf, which entails a disagreement between Φ_η^A and Ψ_η^B at some number m say, at infinitely many η-stages. Without loss of generality, we can suppose $m \in \Phi_\eta^A[s] - \Psi_\eta^B[s]$, where s is an η-expansionary stage after which η always has outcome f. This disagreement is preserved since every further stage at which η is visited is an $\eta^\frown f$-stage, so m is kept in Φ_η^A and we can argue that m is not in Ψ_η^B since m is not in Ψ_η^B infinitely many times. This shows that $\Phi_\eta^A \neq \Psi_\eta^B$, and η is satisfied.

Case 2: There are infinitely many η-expansionary stages.

In this case, there are infinitely many η-expansionary stages and at these stages, we may put more and more numbers into V_η. We now prove that $V_\eta = \Phi_\eta^A$.

To do this, since A is constructed to be low, we only need to show that at any η-expansionary stage, for each $m \in V_\eta$, we have m is in both Φ_η^A and Ψ_η^B. Let $s_1 < s_2$ be two consecutive η-expansionary stages. If no number is extracted from A at Stage s_1, then m always stays in Φ_η^A up to Stage s_2 since no smaller number can be extracted from A in this period.

So we assume that a number x is extracted from A by a $\mathcal{P}$-strategy α at Stage s_1. Hence, at Stage s_1, no extraction due to $\mathcal{H}$-strategies is performed, so Ψ_η^B is kept the same. Again, if no $\mathcal{L}$-strategy with higher priority acts and no number less than or equal to p_α exits $\overline{K}$ between Stages s_1 and s_2, then B has no change below the corresponding use, so m is in Ψ_η^B up to Stage s_2.

Thus, suppose that between Stages s_1 and s_2, either an $\mathcal{L}$-strategy with higher priority acts or a number less than or equal to p_α exits $\overline{K}$. Then a number will be extracted from B, which perhaps drives m out of Ψ_η^B before Stage s_2. However, if this happens, the number extracted from A at Stage s_1, x_α, will be put into A again, which ensures that m is in Φ_η^A, up to Stage s_2.

Therefore, as A is low, if there are infinitely many η-expansionary stages and at any such stage, s say, for any m, we have $m \in V_\eta[s]$, then m is also in $\Phi_\eta^A[s]$. That is, m is in $\Phi_\eta^A[s]$ for infinitely many s, which ensures that m is in Φ_η^A. Therefore, in this case, we will have $V_\eta = \Phi_\eta^A$, and Φ_η^A is c.e.

This completes the proof of Lemma 5.5. □

We will now prove that the $\mathcal{S}$-requirement is satisfied.

Lemma 5.6. The requirement $\mathcal{S}$ is satisfied. That is, $\overline{K} = \Gamma^{A,B}$.

Proof. Whenever a number k exits $\overline{K}$ at a stage, if k is still in $\Gamma^{A,B}$, we remove k from $\Gamma^{A,B}$ by extracting the crucial element in $\gamma(k)$ from B. (Notice that collaboration between A and B is essential in Γ-correction, since when the B side cannot be changed because of some minimal pair strategy, A has already automatically corrected Γ by the extraction of some x_α, which happens because of the way use-neighbourhoods for Γ are defined.) Thus, to show that the $\mathscr{S}$-requirement is satisfied, we only need to show that for k, k can be added to and later removed from $\Gamma^{A,B}$ at most finitely often, which will ensure that $\Gamma^{A,B}(k) = \overline{K}(k)$ for all k. This is true because for a fixed k, k is removed from $\Gamma^{A,B}$ when one of the following happens:

(1) a number $k' \leqslant k$ exits $\overline{K}$;

(2) an $\mathscr{L}$-strategy β acts to satisfy itself, and thus a crucial number in $\gamma(p_\beta)$ is extracted from B, or a $\mathscr{P}$-strategy α acts to satisfy itself and thus an x_α is extracted from A.

Since k is fixed and after k is extracted from $\Gamma^{A,B}$ it will be enumerated into $\Gamma^{A,B}$ later on provided k is in $\overline{K}$, after a late enough stage, no strategy's action can extract k from $\Gamma^{A,B}$. Now, by induction, we can see that after a late enough stage, k is kept in $\Gamma^{A,B}$ if k is in $\overline{K}$. □

This completes the proof of Theorem 1. Since elements that are extracted from B, which starts as $B_0 = \omega$, are never re-enumerated back into it, so it follows that B is Π^0_1.

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