Definability in the local theory of the ω-enumeration degrees

Hristo Ganchev ⋆

Sofia University, Faculty of Mathematics and Informatics
5 James Bourchier blvd. 1164 Sofia, Bulgaria
h.ganchev@gmail.com

Abstract We continue the study of the local theory of the structure of the ω-enumeration degrees, started by Soskov and Ganchev [7]. We show that the classes of 1-high and 1-low ω-enumeration degrees are definable. We prove that a standard model of arithmetic is definable as well.

1 ω-enumeration degrees

In computability theory there is a vast variety of reducibility relations. Informally, the common feature among all of them is that an object A is reducible to an object B (denoted by $A \leq B$), if there is an algorithm for transforming the information contained in B into the information contained in A. For example $A \leq_1 B$, if there is a one-one computable function g, such that $\chi_A = \chi_B \circ g$, (here χ_A and χ_B are the characteristic functions of A and B respectively). As a second example consider Turing reducibility, for which $A \leq_T B$, if there is a Turing machine which transforms χ_B into χ_A. We say that A is enumeration reducible to B ($B \neq \emptyset$), $A \leq_e B$, if there is a Turing machine transforming every function enumerating B into a function enumerating A. Finally, A is c.e. in B if there is a Turing machine transforming χ_B into a function enumerating A.

Now let us try to define a reducibility relation between sequences of sets of natural numbers and sets of natural numbers. This problem may be solved in various ways thus yielding a large number of reducibilities. In this paper, we shall be concerned with the solution proposed by Soskov in [6].

Let us denote

$$\mathcal{S}_\omega = \{\mathcal{A} = \{A_n\}_{n<\omega} \mid A_n \subseteq \mathbb{N}\}.$$

Take an element $\mathcal{A} \in \mathcal{S}_\omega$ and a set $X \subseteq \mathbb{N}$. If we are to say that $\mathcal{A} \leq X$, we should be able to obtain effectively the information contained in $\mathcal{A}$ from the information contained in X. Note, that in order to do this, we should be able to restore each element of the sequence and the order in which they occur. This

⋆ The author was partially supported by the European Operational programm HRD through contract BGO051PO001/07/3.3-02/53 with the Bulgarian Ministry of Education and by BNSF through contract D002-258/18.12.08.

is so, since $\mathcal{A} = \{A_n\}_{n<\omega}$ is a mapping from the set of natural numbers to the power set $2^{\mathbb{N}}$:

$$\begin{array}{cccccc} & 0 & 1 & 2 \ldots & n \ldots \\ \mathcal{A}: & \downarrow & \downarrow & \downarrow \ldots & \downarrow \ldots \\ & A_0 & A_1 & A_2 \ldots & A_n \ldots \end{array}$$

In order to simulate this mapping we shall use the Turing jump J_T. The Turing jump is an unary operation (definable in second order arithmetic) $J_T : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, such that for any X, $X \leq J_T(X)$ and $J_T(X) \not\leq X$. So, in some sense, X gives rise to a "natural" sequence X, $J_T(X)$, $J_T^2(X)$, ..., which may be regarded as copy of $\mathbb{N}$.

$$\begin{array}{cccccc} & X & J_T(X) & J_T^2(X) & \ldots & J_X)\ldots \\ X: & \uparrow & \uparrow & \uparrow & \ldots & \uparrow \ldots \\ & 0 & 1 & 2 & \ldots & n \ldots \end{array}$$

Combining the two mappings we arrive to the following definition:

Definition 1 *Let $\mathcal{A} \in \mathcal{S}_\omega$ and $X \subseteq \mathbb{N}$. We shall say, that $\mathcal{A}$ is uniformly reducible to X and write $\mathcal{A} \preceq_\omega X$, if*

$$\forall n(A_n \text{ c.e. in } J_T^n(X) \text{ uniformly in } n).$$

The uniformity condition is necessary, since it guarantees the existence of one algorithm which reduces the sequence $A_0, A_1, A_2, \ldots$ to the sequence X, $J_T(X)$, $J_T^2(X)$,... (recall that the existence of one algorithm is crucial to all reducibilities considered in computability theory).

The relation $\preceq_\omega$ gives a tool for comparing elements of $\mathcal{S}_\omega$, namely

Definition 2 *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\omega$. We shall say that $\mathcal{A} \leq_\omega \mathcal{B}$, if*

$$\forall X \subseteq \mathbb{N}(\mathcal{B} \preceq_\omega X \Longrightarrow \mathcal{A} \preceq_\omega X).$$

The relation $\leq_\omega$ is a preorder, so it generates a nontrivial equivalence relation on $\mathcal{S}_\omega$:

$$\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{A} \leq_\omega \mathcal{B} \;\&\; \mathcal{B} \leq_\omega \mathcal{A}.$$

We call the respective equivalence classes ω-enumeration degrees and denote

$$\mathbf{d}_\omega(\mathcal{A}) = \{\mathcal{B} \in \mathcal{S}_\omega \mid \mathcal{A} \equiv_\omega \mathcal{B}\}.$$

We shall denote the collection of all ω-enumeration degrees by $\mathbf{D}_\omega$. The preorder $\leq_\omega$ on $\mathcal{S}_\omega$ induces a partial order $\leq_\omega$ on $\mathbf{D}_\omega$, namely

$$\mathbf{a} \leq_\omega \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b}(\mathcal{A} \leq_\omega \mathcal{B}).$$

Clearly $\mathbf{0}_\omega = \mathbf{d}_\omega(\emptyset, \emptyset, \ldots, \emptyset, \ldots)$ is the least degree in $\mathbf{D}_\omega$. Also, we have that for arbitrary $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\omega$, $\mathbf{d}_\omega(\mathcal{A} \oplus \mathcal{B}) = \mathbf{d}_\omega(A_0 \oplus B_0, A_1 \oplus B_1, \ldots)$ is the l.u.b. of the set $\{\mathbf{d}_\omega(\mathcal{A}), \mathbf{d}_\omega(\mathcal{B})\}$. Thus $\mathcal{D}_\omega = (\mathbf{D}_\omega, \mathbf{0}_\omega, \leq_\omega, \vee)$ is an upper semi-lattice with least element.

The structure $\mathcal{D}_\omega$ is first introduced by Soskov [6] and is further studied in [1,7].

2 Basic properties of the ω-enumeration degrees

Let $\mathcal{A}$ be an element of $\mathcal{S}_\omega$. We set the jump sequence of $\mathcal{A}$ to be the sequence $P(\mathcal{A}) = (P_0(\mathcal{A}), P_1(\mathcal{A}), \dots, P_n(\mathcal{A}), \dots)$, where the sets $P_i(\mathcal{A})$ are defined by

$$P_0(\mathcal{A}) = A_0;$$
$$P_{n+1}(\mathcal{A}) = J_e(P_n(\mathcal{A})) \oplus A_{n+1},$$

(by J_e we denote the enumeration jump). The sequences $\mathcal{A}$ and $P(\mathcal{A})$ are closely related, since $\mathcal{A} \equiv_\omega P(\mathcal{A})$. Furthermore, using the jump sequences Soskov and Kovachev [8] are able to show that the relation $\leq_\omega$ is a real reducibility relation between sequences.

Theorem 1 (Soskov, Kovachev [8])

$$\mathcal{A} \leq_\omega \mathcal{B} \iff \forall n (A_n \leq_e P_n(\mathcal{B}) \textit{ uniformly in } n)$$

Another important role played by the jump sequences is in the definition of a jump operation on sequences.

Definition 3 (Soskov [6]) *Let $\mathcal{A} \in \mathcal{S}_\omega$. We define the jump of $\mathcal{A}$ to be the sequence $\mathcal{A}' = \{P_{1+n}(\mathcal{A})\}_{n\in\omega}$.*

In other words $\mathcal{A}'$ is obtained from the jump sequence of $\mathcal{A}$ by simply deleting its first element. Besides the simplicity of its definition, the jump operation has another nice property, namely:

$$\mathcal{A}' \preceq_\omega X \iff \exists Y \subseteq \mathbb{N}(\mathcal{A} \preceq_\omega Y \;\&\; J_T(Y) \equiv_T X),$$

for each $\mathcal{A} \in \mathcal{S}_\omega$ and $X \subseteq \mathbb{N}$. In other words, the set X "codes" the jump of the sequence $\mathcal{A}$, if and only if it is equivalent to the Turing jump of a set "coding" $\mathcal{A}$ itself.

The so defined jump operation is strictly monotone, i.e.

$$\mathcal{A} \leq_\omega \mathcal{B} \Longrightarrow \mathcal{A}' \leq_\omega \mathcal{B}'.$$

$$\mathcal{A}' \not\leq_\omega \mathcal{A};$$

We set

$$\mathbf{a}' = \mathbf{d}_\omega(\mathcal{A}')$$

for any $\mathcal{A} \in \mathbf{a}$. The previous properties guarantee, that this definition is unambiguous.

Soskov and Ganchev [7] prove that the jump operation on the ω-enumeration degrees has a very unexpected property:

Theorem 2 (Soskov, Ganchev [7]) *Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$, be such that $\mathbf{a}^{(n)}$ (the n-th iteration of the jump operation on $\mathbf{a}$) is less or equal to $\mathbf{b}$. Then the set*

$$\{\mathbf{x} \in \mathcal{D}_\omega \mid \mathbf{a} \leq_\omega \mathbf{x} \;\&\; \mathbf{x}^{(n)} = \mathbf{b}\}$$

has a least element. We shall denote this element by $I^n_{\mathbf{a}}(\mathbf{b})$.

Note that this theorem is neither true for the structure of the Turing degrees, $\mathcal{D}_T$, nor the structure of the enumeration degrees, $\mathcal{D}_e$. This suggests that $\mathcal{D}_\omega{}'$ (the structure of the ω-enumeration degrees augmented by the jump operation) is rather different from the structures $\mathcal{D}_T$ and $\mathcal{D}_e$. Nevertheless, it turns out that this is not quite so. First Soskov [6], shows that $\mathcal{D}_e{}'$ (the structure of the enumeration degrees augmented by the jump operation) is embeddable in $\mathcal{D}_\omega{}'$ by the mapping $\kappa : \mathbf{D}_e \to \mathbf{D}_\omega$, acting by the rule:

$$\kappa(\mathbf{d}_e(A)) = \mathbf{d}_\omega(A, \emptyset, \emptyset, \ldots).$$

Then Soskov and Ganchev [7] are able to prove, the set $\mathbf{D}_1 = \kappa[\mathbf{D}_e]$ is first order definable in the theory of $\mathcal{D}_\omega{}'$, and so $Th_1(\mathcal{D}_e{}')$ is interpretable within $Th_1(\mathcal{D}_\omega)$. Furthermore it is shown that that the structures $\mathcal{D}_e$ and $\mathcal{D}_\omega{}'$ have isomorphic automorphism groups.

So, although the structures $\mathcal{D}_\omega$ and $\mathcal{D}_e{}'$ are quite different, the first being far richer then the second one, they are closely related.

In the next section, we shall obtain some nice results for the locale theory of the ω-enumeration degrees, using results about Σ^0_2 enumeration degrees and degrees c.e. and above a Turing degree $\mathbf{a}$.

3 The local theory

Let us denote

$$\mathcal{G}_\omega = ([\mathbf{0}_\omega, \mathbf{0}'_\omega], \mathbf{0}_\omega, \leq_\omega).$$

By local theory we mean the theory of the structure $\mathcal{G}_\omega$. From now on we shall restrict our considerations only to ω-enumeration degrees belonging to $\mathcal{G}_\omega$. Thus from now on, unless explicitly otherwise stated, by an arbitrary omega-enumeration degree we will mean an arbitrary omega enumeration degree below $\mathbf{0}'_\omega$.

In the local theory the classes of the n-high and n-low degrees are of particular interest. They are defined by

$$H_n = \{\mathbf{a} \mid \mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n+1)}\}$$
$$L_n = \{\mathbf{a} \mid \mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n)}\}$$

Further more, we set $H = \bigcup H_n$, $L = \bigcup L_n$ and $I = [\mathbf{0}_\omega, \mathbf{0}'_\omega] \setminus (L \cup H)$. The last three classes are studied in [7]. It is shown, that they have a strong connection with the class of the, so called, almost zero degrees.

In order to define the notion of an almost zero degree for every $n \in \mathbb{N}$ set $\mathbf{o}_n$ to be the least degree, satisfying the equality

$$\mathbf{x}^{(n)} = \mathbf{0}_\omega^{(n+1)},$$

i.e., $\mathbf{o}_n = I^n_{\mathbf{0}_\omega}(\mathbf{0}_\omega^{(n+1)})$. Clearly, the degrees $\mathbf{o}_n$ form a strictly decreasing sequence

$$\mathbf{0}'_\omega = \mathbf{o}_0 >_\omega \mathbf{o}_1 >_\omega \mathbf{o}_2 >_\omega \ldots$$

The first natural question to ask is whether this sequence converges to $\mathbf{0}_\omega$, i.e., is it true that

$$\forall n(\mathbf{x} \leq_\omega \mathbf{o}_n) \Longrightarrow \mathbf{x} = \mathbf{0}_\omega.$$

The answer to this question is negative. In fact, the degrees below all $\mathbf{o}_n$ form a countable ideal, whose elements are called almost zero (*a.z.*) degrees. A remarkable property of this ideal is that it has no minimal upper bound (beneath $\mathbf{0}'_\omega$). In addition the a.z. degrees give a nice characterisation for the classes H and L, namely

$$\begin{array}{l} \mathbf{x} \in H \Longleftrightarrow \forall a.z.\ \mathbf{y}(\mathbf{y} \leq_\omega \mathbf{x}) \\ \mathbf{x} \in L \Longleftrightarrow \forall a.z.\ \mathbf{y}(\mathbf{y} \wedge \mathbf{x} = \mathbf{0}_\omega). \end{array} \tag{1}$$

We can reformulate (1) in the terms of the degrees $\mathbf{o}_n$:

$$\begin{array}{l} \mathbf{x} \in H \Longleftrightarrow \forall \mathbf{y}(\forall n(\mathbf{y} \leq_\omega \mathbf{o}_n) \Rightarrow \mathbf{y} \leq_\omega \mathbf{x}) \\ \mathbf{x} \in L \Longleftrightarrow \forall \mathbf{y}(\forall n(\mathbf{y} \leq_\omega \mathbf{o}_n) \Rightarrow \mathbf{y} \wedge \mathbf{x} = \mathbf{0}_\omega) \end{array}$$

This suggests, that for every n we can use $\mathbf{o}_n$ as a parameter to obtain a first order definition for each class H_n and L_n. Indeed, we have

$$\begin{array}{l} \mathbf{x} \in H_n \Longleftrightarrow \mathbf{o}_n \leq_\omega \mathbf{x} \\ \mathbf{x} \in L_n \Longleftrightarrow \mathbf{o}_n \wedge \mathbf{x} = \mathbf{0}_\omega. \end{array}$$

In particular

$$\begin{array}{l} \mathbf{x} \in H_1 \Longleftrightarrow \mathbf{o}_1 \leq_\omega \mathbf{x} \\ \mathbf{x} \in L_1 \Longleftrightarrow \mathbf{o}_1 \wedge \mathbf{x} = \mathbf{0}_\omega. \end{array} \tag{2}$$

Our next goal is to prove the following theorem:

Theorem 3 *The degree $\mathbf{o}_1$ is definable in $\mathcal{G}_\omega$.*

Proof. We shall denote the enumeration jump of the set A by $J_e(A)$. Recall that $J_e(A) = E_A \oplus \overline{E_A}$, where $E_A = \{\langle x, i\rangle \mid x \in W_i(A)\}$[1]. According to [7]

$$\mathbf{o}_1 = (\emptyset, J_e(\emptyset), J_e^2(\emptyset), \ldots) \tag{3}$$

From here, we conclude that $\mathbf{o}_1$ is a non-cuppable degree, i.e.,

$$\forall \mathbf{y}(\mathbf{y} \vee \mathbf{o}_1 = \mathbf{0}'_\omega \Longrightarrow \mathbf{y} = \mathbf{0}'_\omega).$$

Furthermore, since $\mathbf{0}'_\omega = \mathbf{d}_\omega(J_e(\emptyset), J_e^2(\emptyset), J_e^3(\emptyset)\ldots)$, we may conclude, that for arbitrary $\mathbf{x}$ the g.l.b. of $\mathbf{x}$ and $\mathbf{o}_1$ exists and is exactly $I^1_{\mathbf{0}_\omega}(\mathbf{x}')$, i.e.,

$$\mathbf{x} \wedge \mathbf{o}_1 = I^1_{\mathbf{0}_\omega}(\mathbf{x}'). \tag{4}$$

Consider the formula

$$\mathcal{K}(\mathbf{x}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \overset{def}{\Longleftrightarrow} \mathop{\&}_{1 \leq i < j \leq 3} \mathbf{x} = (\mathbf{a}_i \vee \mathbf{x}) \wedge (\mathbf{a}_j \vee \mathbf{x}).$$

Kalimullin [2] shows, that for each enumeration degree $\mathbf{u}$, there are enumeration degrees $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ such that

[1] Here W_i stands for the c.e. set with Gödel index i. Furthermore, $W_i(A) = \{x \mid \langle x, u\rangle \in W_i \ \&\ D_u \subseteq A\}$, where D_u is the finite set with canonical index u.

(K1) $\mathbf{u} \lneq \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$;
(K2) $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \lneq \mathbf{u}'$;
(K3) $\mathbf{u}' = \mathbf{a}_1' = \mathbf{a}_2' = \mathbf{a}_3'$;
(K4) $\mathcal{K}(\mathbf{x}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is true for all $\mathbf{x} \in [\mathbf{u}, \mathbf{u}']$;
(K5) $\mathbf{u}' = \mathbf{a}_1 \vee \mathbf{a}_2 \vee \mathbf{a}_3$

Fix enumeration degrees $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ satisfying (K1) – (K5) for $\mathbf{u} = \mathbf{0}_e$ and $\widetilde{\mathbf{a}}_1$, $\widetilde{\mathbf{a}}_2$ and $\widetilde{\mathbf{a}}_3$ satisfying (K1) – (K4) for $\mathbf{u} = \mathbf{0}_e'$. Set

$$\begin{aligned}\mathbf{b}_1 &= \kappa(\mathbf{a}_1) \vee I^1_{\mathbf{0}_\omega}(\kappa(\widetilde{\mathbf{a}}_1));\\ \mathbf{b}_2 &= \kappa(\mathbf{a}_1) \vee I^1_{\mathbf{0}_\omega}(\kappa(\widetilde{\mathbf{a}}_2));\\ \mathbf{b}_3 &= \kappa(\mathbf{a}_1) \vee I^1_{\mathbf{0}_\omega}(\kappa(\widetilde{\mathbf{a}}_3)).\end{aligned}$$

It is easy to see that $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ satisfy (K1), (K2), (K4), (K5) for $\mathbf{u} = \mathbf{0}_\omega$. On the other hand, it is true that $\mathbf{b}_i' = \kappa(\widetilde{\mathbf{a}}_i)$, for $1 \leq i \leq 3$, and hence according to (4) and (K5)

$$(\mathbf{o}_1 \wedge \mathbf{b}_1) \vee (\mathbf{o}_1 \wedge \mathbf{b}_2) \vee (\mathbf{o}_1 \wedge \mathbf{b}_3) = \mathbf{o}_1.$$

Now suppose that $\mathbf{x}$ is a non-cuppable ω-enumeration degree for which there are degrees $\widetilde{\mathbf{b}}_1$, $\widetilde{\mathbf{b}}_2$ and $\widetilde{\mathbf{b}}_3$ satisfying (K1), (K2), (K4) and (K5) for $\mathbf{u} = \mathbf{0}_\omega$, such that

$$(\mathbf{x} \wedge \widetilde{\mathbf{b}}_1) \vee (\mathbf{x} \wedge \widetilde{\mathbf{b}}_2) \vee (\mathbf{x} \wedge \widetilde{\mathbf{b}}_3) = \mathbf{x}. \tag{5}$$

Let $\mathcal{X} \in \mathbf{x}$ and consider $P_0(\mathcal{X})$. According to Cooper, Sorbi and Yi [5], every non-trivial Δ^0_2 enumeration degree is cuppable. Hence, either $P_0(\mathcal{X})$ is enumeration equivalent to $\emptyset$ or no non-computably enumerable Δ^0_2 set is enumeration reducible to it:

$$P_0(\mathcal{X}) \equiv_e \emptyset \quad \text{or} \quad \forall Y (Y \leq_e P_0(\mathcal{X}) \ \&\ Y \text{ is } \Delta^0_2 \Longrightarrow Y \leq_e \emptyset). \tag{6}$$

Suppose that $P_0(\mathcal{X}) \not\equiv_e \emptyset$. Fix $\widetilde{\mathbf{b}}_1$, $\widetilde{\mathbf{b}}_2$ and $\widetilde{\mathbf{b}}_3$ satisfying (K1), (K3), (K4) and (5).

Fix $\widetilde{\mathcal{B}}_i \in \widetilde{\mathbf{b}}_i$, for $1 \leq i \leq 3$, and consider $\mathbf{d}_e(P_0(\widetilde{\mathcal{B}}_1))$, $\mathbf{d}_e(P_0(\widetilde{\mathcal{B}}_2))$ and $\mathbf{d}_e(P_0(\widetilde{\mathcal{B}}_3))$. We may conclude the last three satisfy (K2), (K4) and (K5) for $\mathbf{u} = \mathbf{0}_e$. Thus at least two of them are different from $\mathbf{0}_e$ (this is implied by (K5)). Suppose, that these are $\mathbf{d}_e(P_0(\widetilde{\mathcal{B}}_1))$ and $\mathbf{d}_e(P_0(\widetilde{\mathcal{B}}_2))$. According to Kalimullin [2], we may conclude that these two degrees are Δ^0_2, so that without loss of generality we may assume that $P_0(\widetilde{\mathcal{B}}_1)$ and $P_0(\widetilde{\mathcal{B}}_2)$ are Δ^0_2 sets. Now from (6), for $i = 1, 2$

$$\forall Y (Y \leq_e P_0(\mathcal{X}) \ \&\ Y \leq_e P_0(\widetilde{\mathcal{B}}_i) \Longrightarrow Y \leq_e \emptyset).$$

Since $\mathbf{x} = (\mathbf{x} \wedge \widetilde{\mathbf{b}}_1) \vee (\mathbf{x} \wedge \widetilde{\mathbf{b}}_2) \vee (\mathbf{x} \wedge \widetilde{\mathbf{b}}_3)$ it must be the case that $P_0(\mathcal{X}) \equiv_e P_0(\widetilde{\mathcal{B}}_3)$. Hence, $P_0(\widetilde{\mathcal{X}}_3) \not\equiv_e \emptyset$. Now applying Kalimullin's theorem once again (this time for $P_0(\widetilde{\mathcal{B}}_1)$ and $P_0(\widetilde{\mathcal{B}}_3)$) we obtain, that $P_0(\widetilde{\mathcal{B}}_3)$ is Δ^0_2. But this contradicts the assumption about $P_0(\mathcal{X})$ and thus $P_0(\mathcal{X}) \equiv_e \emptyset$.

So, we have proved that whenever $\mathbf{x}$ is a non-cuppable ω-enumeration degree for which there are degrees $\widetilde{\mathbf{b}}_1$, $\widetilde{\mathbf{b}}_2$ and $\widetilde{\mathbf{b}}_3$ satisfying (5), (K1), (K2), (K4) and (K5) for $\mathbf{u} = \mathbf{0}_\omega$, it is true that for each $\mathcal{X} \in \mathbf{x}$, $\mathcal{X} \equiv_e \emptyset$.

However $\mathbf{o}_1$ is the biggest degree generated by a sequence beginning with $\emptyset$. Thus $\mathbf{o}_1$ is definable in the first order language of $\mathcal{G}_\omega$.

$\square$

Corollary 1 *The classes H_1, L_1 and $\mathbf{D}_1$ are definable by a first order formula in $\mathcal{G}_\omega$.*

Proof. The degrees in $\mathbf{D}_1$ satisfy the condition:

$$\mathbf{x} \in \mathbf{D}_1 \iff \forall \mathbf{z}(\mathbf{z} \vee \mathbf{o}_1 = \mathbf{x} \vee \mathbf{y} \Longrightarrow \mathbf{x} \leq_\omega \mathbf{z}).$$

$\square$

The last corollary will enable us to prove that there is a standard model of arithmetic definable in $\mathcal{G}_\omega$. Note, that this does not tell us which is the degree of the first order theory of $\mathcal{G}_\omega$, since it is not clear whether $\mathcal{G}_\omega$ is inerpretable in first order arithmetic.

Theorem 4 *FOA is interpretable in $\mathcal{G}_\omega$.*

Proof. Denote by R_ω the collection of all degrees, that are the g.l.b. of a degree in $\mathbf{D}_1[\mathbf{0}_\omega, \mathbf{0}'_\omega]$ and $\mathbf{o}_1$, i.e.,

$$R_\omega = \{\mathbf{x} \wedge \mathbf{o}_1 \mid \mathbf{x} \in \mathbf{D}_1[\mathbf{0}_\omega, \mathbf{0}'_\omega]\}.$$

Since for $\mathbf{x} \leq_\omega \mathbf{0}'_\omega$,

$$(\mathbf{x} \wedge \mathbf{o}_1)' = \mathbf{x}',$$

it turns out that

$$\{\mathbf{x}' \mid \mathbf{x} \in \mathbf{D}_1[\mathbf{0}_\omega, \mathbf{0}'_\omega]\} = \{\mathbf{x}' \mid \mathbf{x} \in R_\omega\}.$$

According to the results in [7] the jump operation is an isomorphism between the intervals $[\mathbf{0}_\omega, \mathbf{o}_1]_\omega$ and $[\mathbf{0}'_\omega, \mathbf{0}''_\omega]_\omega$. So, we obtain:

$$(R_\omega, \leq_\omega) \cong (\{\mathbf{x}' \mid \mathbf{x} \in \mathbf{D}_1[\mathbf{0}_\omega, \mathbf{0}'_\omega]\}, \leq_\omega) \cong (\{\mathbf{x}' \mid \mathbf{x} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}'_e]\}, \leq_e).$$

McEvoy [3] shows that the elements in $\{\mathbf{x}' \mid \mathbf{x} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}'_e]\}$ are exactly the Π^0_2 enumeration degrees above $\mathbf{0}'_e$. On the other hand these are exactly the degrees to which the c.e. in and above $J_T(\emptyset)$ degrees are mapped by the Roger embedding ι. Thus $(R_\omega, \leq_\omega)$ is isomorphic to the structure of the c.e. in and above $J_T(\emptyset)$ degrees.

Nies, Slaman and Shore [4] prove that for every Turing degree $\mathbf{a}$, there is a standard model of arithmetic definable in the degrees c.e. in and above $\mathbf{a}$.

Now the theorem follows from the fact, that R_ω is first order definable in $\mathcal{G}_\omega$.

$\square$

References

1. H. Ganchev. Exact pair theorem for the ω-enumeration degrees. In B. Loewe S. B. Cooper and A. Sorbi, editors, *Computation and Logic in the Real World, Lecture Notes in Comp. Science*, volume 4497, pages 316–324. Springer-Verlag, 2007.
2. I. Sh. Kalimullin. Definability of the jump operator in the enumeration degrees. *Journal of Mathematical Logic*, 3:257–267, 2003.
3. K. McEvoy. Jumps of quasi-minimal enumeration degrees. *J. Symbolic Logic*, 50:839–848, 1985.
4. André Nies, Richard A. Shore, and Theodore A. Slaman. Interpretability and definability in the recursively enumerable degrees. *Proc. London Math. Soc. (3)*, 77(2):241–291, 1998.
5. S. B. Cooper, A. Sorbi and X. Yi. Cupping and noncupping in the enumeration degrees of σ_2^0 sets. *Ann. Pure Appl. Logic*, 82:317–342, 1997.
6. I. N. Soskov. The ω-enumeration degrees. *Journal of Logic and Computation*, 17:1193–1214, 2007.
7. I. N. Soskov and H. Ganchev. The jump operator on the ω-enumeration degrees. *to appear in Ann. Pure and Appl. Logic.*
8. I. N. Soskov and B. Kovachev. Uniform regular enumerations. *Mathematical Structures in Comp. Sci.*, 16(5):901–924, 2006.