Some applications of the Jump Inversion Theorem for the Degree Spectra

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Abstract. In the present paper we give two applications of the Jump inversion theorem for the degree spectra [12], which says that every jump spectrum is also a spectrum and that if a spectrum $\mathcal{A}$ is contained in the set of the jumps of the degrees in some spectrum $\mathcal{B}$ then there exists a spectrum $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{A}$ is equal to the set of the jumps of the degrees in $\mathcal{C}$. In the first application we give a method of constructing a structure, possessing an nth - jump degree equal to $\mathbf{0}^{(n)}$ and which has no kth -jump degree for $k < n$. In the second result we relativize Wehner's construction [13] and obtain a structure whose nth -jump spectrum contains all degrees above an arbitrary fixed degree.
Key words: Turing degrees; degree spectra; forcing; Marker's extensions; enumerations.

1 Degree spectra and jump spectra

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$ be a countable structure, where the set A is infinite, each $R_i \subseteq A^{r_i}$ and the equality $=$ is among $R_1, \dots, R_s$.

The notion of a degree spectrum of a countable structure is introduced by Richter [9] and further studied by Ash, Downey, Jockush and Knight [1, 2, 6].

An *enumeration* f of $\mathfrak{A}$ is a total mapping of $\mathbb{N}$ onto A.

Given a set $R \subseteq A^a$ and an enumeration f of $\mathfrak{A}$, let

$$f^{-1}(R) = \{\langle x_1, \dots, x_a \rangle \mid (f(x_1), \dots, f(x_a)) \in R\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_s)$.

Definition 1. *The degree spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\} .$$

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Here by $d_{\mathrm{T}}(B)$ we denote the Turing degree of the set B.

For every structure $\mathfrak{A}$ the degree spectrum $\mathrm{DS}(\mathfrak{A})$ is closed upwards [11], i.e. for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ & $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathrm{DS}(\mathfrak{A})$.

Definition 2. *The jump spectrum* of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A}) = \{\mathbf{a}' \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

Theorem 3. *[12] For every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$.*

The structure $\mathfrak{B}$ is constructed in two stages. First, we define the least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ which we call *Moschovakis' extension* of $\mathfrak{A}$. Roughly speaking $\mathfrak{A}^*$ is an extension of $\mathfrak{A}$ with additional coding machinery. Using this coding machinery we define the set $K_{\mathfrak{A}}$ which is an analogue of Kleene's set K. Finally we set $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.

Theorem 4. *[12] Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$. Then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.*

The structure $\mathfrak{C}$ is obtained as a *Marker's extension* of $\mathfrak{A}$ [8], coding $\mathfrak{B}$ in $\mathfrak{C}$. In the construction we use a relativized variant of the representation of Σ^0_2 sets of Goncharov and Khoussainov [3].

Definition 5. Let $n \geq 1$. *The nth jump spectrum* of $\mathfrak{A}$ is the set $\mathrm{DS}_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

One can easily see by induction on n that for every n there exists a structure $\mathfrak{A}^{(n)}$ such that $\mathrm{DS}_n(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^{(n)})$.

Theorem 6. *[12] Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. Then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.*

2 Some Applications

Definition 7. A degree $\mathbf{a}$ is said to be the *nth jump degree* of a structure $\mathfrak{A}$ if $\mathbf{a}$ is the least element of $\mathrm{DS}_n(\mathfrak{A})$.

Notice that if $\mathbf{a}$ is the nth jump degree of $\mathfrak{A}$ then for all k, $\mathbf{a}^{(k)}$ is the $(n+k)th$ jump degree of $\mathfrak{A}$. Hence if a structure $\mathfrak{A}$ possesses an nth jump degree then it possesses $(n+k)$th jump degrees for all k.

The definitions above can be naturally generalized for all recursive ordinals α. In [2] Downey and Knight proved with a fairly complicated construction that for every recursive ordinal α there exists a linear ordering $\mathfrak{A}$ such that $\mathfrak{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathfrak{A}$.

Here we shall present a construction which allows us to obtain for every natural number n examples of structures which have $(n+1)$st jump degree but do not have kth jump degree for $k \leq n$.

The idea of this construction is the following. In [12] we give an example of a group $\mathfrak{A}$, a subgroup of the set of rational numbers, satisfying the following conditions:

1. $\mathrm{DS}(\mathfrak{A}) \subseteq \{\mathbf{a} : \mathbf{0^{(n)}} \leq \mathbf{a}\}$.
2. $\mathrm{DS}(\mathfrak{A})$ has no least element.
3. $\mathfrak{A}$ has a first jump degree equal to $\mathbf{0}^{(n+1)}$.

Let $\mathfrak{B} = (N; =)$ be a structure such that $\mathrm{DS}(\mathfrak{B})$ is equal to the set of all Turing degrees. Clearly $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. By Theorem 6, there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$. Therefore $\mathfrak{C}$ does not have an nth jump degree and hence it has no kth jump degree for $k \leq n$. On the other hand $\mathrm{DS}_{n+1}(\mathfrak{C}) = \mathrm{DS}_1(\mathfrak{A})$ and hence the $(n+1)$th jump degree of $\mathfrak{C}$ is $\mathbf{0}^{(n+1)}$.

Our second application is a generalization of results of Slaman [10] and Wehner [13]. They give an example of a structure with degree spectrum consisting of all nonrecursive Turing degrees.

Theorem 8. *[13] There is a family of finite sets, which has no r.e. enumeration, i.e. r.e. universal set, and for each nonrecursive set X there is a enumeration recursive in X.*

First we relativize this theorem.

Theorem 9. *Let $B \subseteq N$. There is a family $\mathcal{F}$ of sets, which has no r.e. in B enumeration, and for each set $X >_T B$ there is a enumeration of the family $\mathcal{F}$, recursive in X.*

Following an idea of Kalimullin [7] we consider the following family of sets

$$\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F \text{ finite set, } F \neq W_n^B\}.$$

Proposition 10. *Let $X \subseteq N$. If a universal for $\mathcal{F}$ set U is r.e. in X then $X >_T B$.*

It is clear that $B \leq_T X$.

If we assume that $B \equiv_T X$, then we can construct a recursive in B function g, such that $(\forall n)(W^B_{g(n)} \neq W^B_n)$. This is a contradiction with the recursion theorem.

Proposition 11. *Let $B <_T X$. There exists a universal set U for the family $\mathcal{F}$, such that $U \leq_T X$.*

Since $X \not\leq B$ then at least one of the sets X or $\overline{X}$ is not r.e. in B. Without loss of generality assume that X is not r.e. in B. Fix an enumeration of $X = \{x_1, \ldots, x_s, \ldots\}$ and denote by $\nu_s = \langle x_1, \ldots, x_s\rangle$.

The set U we construct in stages. At each stage s we find an approximation U^s of U and a witness $x^s_{n,F,i}$ for every finite set F and $i, n \in N$.

Construction

$$\begin{aligned} &U^0 = \{(0,0)\} \cup \{(0, 2x+1) \mid x \in B\} \cup \{(1,2)\} \cup \{(1, 2x+1) \mid x \notin B\} \cup \\ &\{(\langle n, F, i\rangle + 2, 2n+4)\} \cup \{(\langle n, F, i\rangle + 2, 2x+1) \mid x \in F\} \end{aligned} \quad (1)$$

for each finite set F and $i, n \in N$ and let $x^0_{n,F,i} = -1$.

At stage s, denote by $F^s_{\langle n,F,i\rangle} = \{x \mid (\langle n, F, i\rangle + 2, 2x+1) \in U^s\}$.

– If $F^s_{\langle n,F,i\rangle} \neq W^B_{n,s}$ and $x^s_{n,F,i} \neq -1$, we set $x^{s+1}_{n,F,i} = x^s_{n,F,i}$.
– If $F^s_{\langle n,F,i\rangle} = W^B_{n,s}$ and $x^s_{n,F,i} \neq -1$, we set $x^{s+1}_{n,F,i} = -1$ and add $(\langle n,F,i\rangle + 2, 2\nu_s + 1)$ to U^{s+1}.
– If $x^s_{n,F,i} = -1$, we check if there is a z such that $z \in F^s_{\langle n,F,i\rangle} \not\Leftrightarrow z \in W^B_{n,s}$. If there is such a number z, we set $x^{s+1}_{n,F,i}$ to be the least one. If not, we add $(\langle n,F,i\rangle + 2, 2\nu_s + 1)$ to U^{s+1}.

End of construction

Let $U = \bigcup_s U^s$ and $F = \bigcup F^s$.

Consider the sequence $\{x^s_{n,F,i}\}$.

1. If this sequence has a limit a natural number, i.e. it is stable for all $s \geq s_0$ for some s_0, then the index $\langle n,F,i\rangle$ is an index of a finite set from the family $\mathcal{F}$.
2. If the sequence has a limit -1 or it does not have a limit at all, then there exists a monotone sequence of stages $s_1 < s_2 < \ldots < s_k < \ldots$, such that $W^B_{n,s} = \{\nu_{s_k} \mid k \in N\} \cup F$. It follows that the set $\{\nu_{s_k} \mid k \in N\}$ is r.e. in B, and hence X is r.e. in B. A contradiction.

It follows that every set with index greater than 1 in U is finite and belongs to the family $\mathcal{F}$. It is clear that every member of the family $\mathcal{F}$ has an index.

Moreover $(\langle n,F,i\rangle + 2, 2x + 1) \in U$ if and only if one of the following holds:

1. $x \in F$;
2. $x = \langle \nu_0, \ldots, \nu_s\rangle$, for some s.

Hence $U \leq_T X$.

So the constructed set U is universal for the family $\mathcal{F}$ and $U \leq_T X$.

Theorem 12 (Wehner, Slaman). *[13][10] There is a structure $\mathfrak{C}$, for which* $\mathrm{DS}(\mathfrak{C}) = \{x \mid x >_T 0\}$.

The relativized result is the following:

Theorem 13. *For each $n \in N$ and every Turing degree $b \geq 0^{(n)}$ there exists $\mathfrak{C}$, for which* $\mathrm{DS}_n(\mathfrak{C}) = \{x \mid x >_T b\}$.

We construct the structure $\mathfrak{A}$, such that $\mathrm{DS}(\mathfrak{A}) = \{x \mid x >_T b\}$, using the family $\mathcal{F}$ in the same way as is done in [13]. Let $\mathfrak{B} = (N; =)$. It is clear that $b \in \mathrm{DS}_n(\mathfrak{B})$ for each $b \geq 0^{(n)}$. Thus $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. By the Jump inversion Theorem 6 there exists a structure $\mathfrak{C}$, such that $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

Finally we would like to note that there is a relativized variant of Wehner's result for $b = 0^{(n)}$ and for $b = 0''$ as follows:

Theorem 14. *[4] For every n there is a structure $\mathfrak{C}$, such that* $\mathrm{DS}(\mathfrak{C}) = \{x \mid x^{(n)} >_T 0^{(n)}\}$*, i.e. the degree spectrum contains exactly all non-low$_n$ Turing degrees.*

Theorem 15. *[5] There is a structure $\mathfrak{C}$, such that* $\mathrm{DS}(\mathfrak{C}) = \{x \mid x' \geq_T 0''\}$.

And the last authors made a suggestion that they can use an arbitrary Turing degree b in place of $0''$ and thereby building structures with spectrum $\{x \mid x' \geq_T b\}$.

In conclusion would like to point out that the Jump inversion theorem gives a method to lift some interesting results for degree spectra to the nth jump spectra.

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