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The Limitations of Cupping in the Local Structure of the Enumeration Degrees

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Abstract We prove that a sequence of sets containing representatives of cupping partners for every nonzero Δ^0_2 enumeration degree cannot have a Δ^0_2 enumeration. We also prove that no subclass of the Σ^0_2 enumeration degrees containing the nonzero 3-c.e. enumeration degrees can be cupped to $\mathbf{0}'_e$ by a single incomplete Σ^0_2 enumeration degree.

Keywords enumeration degrees, cupping, local structure, difference hierarchy

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1 Introduction

In an upper semi-lattice with greatest element $\langle \mathcal{A}, \leq, \vee, 1\rangle$ we say that an element $\mathbf{a}$ is cuppable if there exists an element $\mathbf{b} \neq \mathbf{1}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{1}$. The element $\mathbf{b}$ is called a *cupping partner* for $\mathbf{a}$. The cupping properties of both the local structure of the Turing degrees reducible to $\mathbf{0}'$, $\mathcal{D}_T(\leq \mathbf{0}')$, and the structure of all c.e. Turing degrees, $\mathcal{R}$, have been thoroughly investigated. Posner and Robinson [15] show that every nonzero degree in $\mathcal{D}_T(\leq \mathbf{0}')$ is cuppable, while Cooper and Yates [6] show the existence of a nonzero c.e. degree which cannot be cupped within $\mathcal{R}$.

In this article we consider cupping properties of the local degree structure of all enumeration degrees reducible $\mathbf{0}'_e$, $\mathcal{D}_e(\leq \mathbf{0}'_e)$. Intuitively we say that a set A is *enumeration reducible* to a set B, denoted as $A \leq_e B$, if there is an effective procedure to enumerate A given any enumeration of B. This relation is a preorder on the sets of natural numbers and induces an equivalence relation $\equiv_e$. The equivalence class of a set A, denoted by $d_e(A)$, is the *enumeration degree* of the set A. The collection of all enumeration degrees together with the relation

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$\leq$, where $d_e(A) \leq d_e(B)$ if $A \leq_e B$, defines the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$. This structure is furthermore an upper semi-lattice with jump operator and least element $\mathbf{0}_e$, the collection of all computably enumerable sets. The semi-lattice of the enumeration degrees can be considered as an extension of the semi-lattice of the Turing degrees, as the second semi-lattice can be embedded in the first, via an order theoretic embedding ι preserving the least upper bound and the jump operator.

Cooper [3] proves that the Σ^0_2 enumeration degrees are exactly the enumeration degrees reducible to $\mathbf{0}'_e$. The images of the c.e. Turing degrees under the embedding ι are exactly the Π^0_1 enumeration degrees. The Δ^0_2 Turing degrees embed onto a proper subset of the Δ^0_2 enumeration degrees, the total Δ^0_2 enumeration degrees. Thus the local structure of the enumeration degrees itself can be considered as a proper extension of the local structure of the Turing degrees.

In [7], Cooper, Sorbi and Yi prove that every nonzero Δ^0_2 enumeration degree can be cupped by a total incomplete Δ^0_2 enumeration degree, in contrast to the Σ^0_2 enumeration degrees where non-cuppable degrees exist. Soskova and Wu [18] examine the cupping properties of the Δ^0_2 enumeration degrees further and show that every nonzero Δ^0_2 enumeration degree can be cupped by a 1-generic Δ^0_2, hence non-total and low, enumeration degree. The latter two results show that one has a certain flexibility when searching for cupping partners of Δ^0_2 enumeration degrees. On the other hand the last result shows that we can limit our search for a cupping partner to a small subclass of the Δ^0_2 enumeration degrees. It would be natural to ask whether or not we can narrow this search even further, perhaps there is a finite set which contains a cupping partner for every nonzero Δ^0_2 enumeration degree. Lewis [13] proves that this is not true for the Δ^0_2 Turing degrees. Our first result shows that the Δ^0_2 enumeration degrees are not any different in this respect.

Given a sequence of sets $\{A_i\}_{i \in C}$, where C is some computable index sets, we shall say that the set $\mathbb{A} = \{\langle i, x\rangle \mid i \in C \wedge x \in A_i\}$ is an *enumeration* of the sequence. For example the universal set $K_1 = \{\langle e, x\rangle \mid x \in W_e\}$ is a Σ^0_1 enumeration of the sequence $\{W_e\}_{e<\omega}$ of all c.e. sets. On the other hand any finite sequence of Δ^0_2 sets has a Δ^0_2 enumeration.

We prove that if a sequence of sets $\{A_i\}_{i<\omega}$ contains within its members a representative of a cupping partner for every nonzero Δ^0_2 enumeration degree, then this sequence does not have a Δ^0_2 enumeration.

Theorem 1 *Let $\{A_i\}_{i<\omega}$ be a sequence of sets with a Δ^0_2 enumeration $\mathbb{A}$. There exists a nonzero Δ^0_2 enumeration degree $\mathbf{b}$ such that for every $i < \omega$ if A_i is incomplete then $d_e(A_i) \vee \mathbf{b} \neq \mathbf{0}'_e$.*

Having found the first limitation in the cupping properties of $\mathcal{D}_e(\leq \mathbf{0}'_e)$ we consider a finer partition of its members arising from the difference hierarchy, defined by Ershov [8], [9]. For every $n \leq \omega$ we shall say that an enumeration degree is n-c.e. if it contains an n-c.e. set. Cooper [4] proves that the class of 2-c.e. enumeration degrees coincides with that of the Π^0_1 enumeration degrees, thus the first proper superclass of Π^0_1 enumeration degrees in this hierarchy consists of all 3-c.e. enumeration degrees. The last proper subclass of the Δ_2^0 enumeration degrees that we shall consider is that of the ω-c.e. degrees. For every $n \leq \omega$ the class of all n-c.e. sets has an n-c.e. enumeration, see [1], thus an immediate corollary from Theorem 1 is the following.

Corollary 1 *There exists a nonzero Δ_2^0 enumeration degree that cannot be cupped by any incomplete ω-c.e. enumeration degree.*

Soskova and Wu [18] prove that this Δ_2^0 enumeration degree cannot itself be an ω-c.e. enumeration degree as every nonzero ω-c.e. enumeration degree can even be cupped by an incomplete 3-c.e. enumeration degree. We face again the question of how much further we can limit our search for cupping partners when we restrict our attention to the smaller subclass of all n-c.e. enumeration degrees for some n, $3 \leq n \leq \omega$. In contrast to the Δ_2^0 enumeration degrees, there is even a 3-c.e. enumeration of a sequence of sets containing a representative of a cupping partner for every nonzero n-c.e. degree, namely the enumeration of all 3-c.e. sets. Can we limit this further to a finite set? Cooper, Seetapun and (independently) Li prove that there exists a single incomplete Δ_2^0 Turing degree that cups every nonzero c.e. degree. When we transfer this statement into the enumeration degrees we obtain a single incomplete Δ_2^0 enumeration degree that cups all nonzero Π_1^0 enumeration degrees, suggesting the possibility that a single incomplete degree from a larger class might be enough to cup all nonzero degrees from a smaller class in the considered hierarchy. Our next result shows that this is not the case as for every incomplete Σ_2^0 enumeration degree $\mathbf{a}$ there exists a nonzero member of the second class, a nonzero 3-c.e. enumeration degree $\mathbf{b}$, such that $\mathbf{b}$ is not cupped by $\mathbf{a}$. This provides a partial answer to the suggested question: If there is a finite set containing cupping partners for every nonzero n-c.e. enumeration degree, where $3 \leq n \leq \omega$, then it cannot be of cardinality 1.

Theorem 2 *Let $\mathbf{a}$ be an incomplete Σ_2^0 enumeration degree. There exists a nonzero 3-c.e. enumeration degree $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}_e'$.*

An extended abstract of Theorem 2 has been published in [17].

2 Preliminaries

We shall start by giving formal definitions of the notions used in the introduction and in the rest of this article.

Definition 1 A set A is *enumeration reducible* ($\leq_e$) to a set B if there is a c.e. set Φ such that:

$$n \in A \Leftrightarrow \exists u(\langle n, u\rangle \in \Phi \wedge D_u \subseteq B),$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an *enumeration operator* and its elements will be called *axioms*.

We can define a least upper bound operation in the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$. By setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and a jump operator $d_e(A)' = d_e(J_e(A))$. The enumeration jump of a set A, denoted by $J_e(A)$ is defined by Cooper [3] as $\overline{K_A} \oplus A$, where $K_A = \{\ n |\ n \in \Phi_n^A\}$.

The jump operator gives rise to the local structure of the enumeration degrees, consisting of all enumeration degrees $\mathbf{a} \leq \mathbf{0}'_e$. We note once again that Cooper [3] proves that these are exactly the Σ_2^0 enumeration degrees, where an enumeration degree is called Σ_2^0 if it contains a Σ_2^0 set, or equivalently if it consists entirely of Σ_2^0 sets.

Definition 2 1. For $n < \omega$ a set A is n-c.e. if there is a total computable function f such that for each x, $f(x, 0) = 0$, $|\{s \mid f(x, s) \neq f(x, s+1)\}| \leq n$ and $A(x) = \lim_s f(x, s)$.
2. A is ω-c.e. if there are two total computable functions $f(x, s)$ and $g(x)$ such that for all x, $f(x, 0) = 0$, $|\{s \mid f(x, s) \neq f(x, s + 1)\}| \leq g(x)$ and $A(x) = \lim_s f(x, s)$.

An enumeration degree which contains an n-c.e. set, where $n \leq \omega$, will be called an n-c.e. enumeration degree.

Next we make a note on the approximations used in the proofs of both theorems. We use the definition of a good approximation given by Lachlan and Shore [12].

Definition 3 Let $\{A[s]\}_{s<\omega}$ be a uniform computable sequence of finite sets. We say that $\{A[s]\}_{s<\omega}$ is a *good approximation* to the set A if:

1. $(\forall n)(\exists s)[\ A \upharpoonright n \subseteq A[s] \subseteq A\ ]$ and
2. $(\forall n)(\exists s)(\forall t > s)[\ A[t] \subseteq A \Rightarrow A \upharpoonright n \subseteq A[s]\ ]$.

Stages s at which $A[s] \subseteq A$ are called good stages.

That every Σ_2^0 set has a good Σ_2^0 approximation is proved by Jockusch [10]. In [12] Lachlan and Shore give a computable method for obtaining a good Δ_2^0 approximation from a given Δ_2^0 approximation to a set A. They prove furthermore the following proposition:

Proposition 1 (Lachlan, Shore) *If $\{A[s]\}_{s<\omega}$ is a good approximation to A, G the set of good stages and Φ is any enumeration operator then*

$$\lim_{s \in G} \Phi^A[s] = \Phi^A.$$

A useful notion when dealing with Σ_2^0 sets is the *age* of an element. This notion was used first by Nies and Sorbi [14] and was given its name by Kent [11].

Definition 4 Given a Σ_2^0 approximation $\{A[s]\}_{s<\omega}$ to a set A, a stage s, and an element $n \in A[s]$, we define $a(A, n, s)$, the age of n in A at stage s, to be the least s_n such that for all t, if $s_n \leq t \leq s$ then $n \in A[t]$. The age of a finite set $F \subset A[s]$ at stage s is $a(A, F, s) = \max\{\ a(A, n, s) |\ n \in F\}$.

An element n belongs to a Σ_2^0 set A if and only if its *age* relative to a fixed Σ_2^0 approximation reaches a finite limit value. We will denote this value by $a(A, n)$ and refer to it as the *limit age*. The *limit age* for a finite subset $F \subseteq A$ is defined, as one might expect, as $\max\{\, a(A, n)|\ n \in F\}$.

Finally we introduce one further notational convention. In what follows we will often need to work with a set C reducible to the least upper bound of two other sets, say A and B. To keep notation simple we will consider the set C as being enumerated relative to two sources and write $C = \Phi^{A,B}$, instead of $C = \Phi^{A \oplus B}$. Naturally we will assume that an axiom of the operator Φ has the structure $\langle n, D_A, D_B\rangle$ and that it is valid if an only if $D_A \subseteq A$ and $D_B \subseteq B$.

Further notation and terminology used in this article are based on that of [5] and [16].

3 The first limitation

We shall first give a proof of Theorem 1. Suppose we are given a sequence $\{A_i\}_{i<\omega}$ such that the enumeration $\mathbb{A} = \{\langle i, x\rangle|\ x \in A_i\}$ is Δ_2^0. We shall construct a Δ_2^0 set B whose enumeration degree is nonzero and such that $d_e(A_i)$ does not cup $d_e(B)$ for every incomplete member of the sequence $\{A_i\}_{i<\omega}$.

Fix a Δ_2^0 approximation $\{\mathbb{A}[s]\}_{s<\omega}$ of the set $\mathbb{A}$. For every i the sequence $\{\{x|\ \langle i, x\rangle \in \mathbb{A}[s]\}\}_{s<\omega}$ is a Δ_2^0 approximation to the set A_i. From this approximation, using Lachlan and Shore's algorithm, we effectively obtain a good Δ_2^0 approximation $\{A_i[s]\}_{s<\omega}$ to A_i for every i.

The requirements that the constructed set B needs to satisfy are:

1. The set B is not c.e. Let $\{W_e\}_{e<\omega}$ be the standard listing of all c.e. sets. For every natural number e we have a requirement:

$$\mathcal{N}_e : W_e \neq B.$$

2. The degree of the set B is not cupped by any incomplete member of the sequence. Let $\{\Theta_j\}_{j<\omega}$ be the standard listing of all enumeration operators. For every i and every j we will have a requirement:

$$\mathcal{P}_{i,j} : \Theta_j^{A_i,B} = \overline{K} \Rightarrow (\exists \Gamma_{i,j})[\Gamma_{i,j}^{A_i} = \overline{K}].$$

We shall use the priority method to construct the required set B. The requirements will be ordered linearly as follows:

$$\mathcal{P}_{0,0} < \mathcal{N}_0 < \mathcal{P}_{0,1} < \mathcal{N}_1 < \mathcal{P}_{1,0} \dots$$

Each particular requirement can be satisfied in more than one way. We connect to each such way an outcome. The choice of the correct way to satisfy a certain requirement depends on the outcomes of higher priority requirements. Therefore we represent the set of all possible sequences of outcomes as a *tree of outcomes*. Each node α on the tree is labelled by a requirement $\mathcal{R}$ and equipped with a finite set of instructions that determine its actions when activated. The node α will be referred to as an $\mathcal{R}$-strategy.

The set of all possible outcomes for each requirement will be linearly ordered ($<_L$, defined below) and the nodes of the tree of strategies will be ordered by the induced lexicographical ordering $\leq$. The construction is in stages. Every c.e. set W_e and every enumeration operator Θ_j will be approximated by its standard Σ^0_1 approximation. The set $\overline{K}$ is approximated by a sequence of co-finite sets, obtained from the standard Σ^0_1 approximation to the set K, namely for every s, $\overline{K}[s] = \overline{K[s]}$. At each stage s we construct a set $B[s]$ approximating B and a string $\delta[s]$ of length s in the tree of strategies. The initial segments $\delta \subseteq \delta[s]$ are the nodes of the tree visited during stage s of the construction; they are the strategies that might act to satisfy their requirements. If $\delta \subset \delta[s]$ then s is a δ-true stage. The intent is that there will be a true path, a leftmost path of nodes visited infinitely often, such that all nodes along the true path are able to satisfy their requirements.

3.1 Basic strategies

We shall describe the basic strategies with the context of the tree in mind.

Consider a $\mathcal{P}$-strategy α working on the requirement $\mathcal{P}_{i,j}$. We shall denote Θ_j by Θ_α, A_i by A_α and $\Gamma_{i,j}$ by Γ_α. The basic goal of α is to construct the operator Γ_α so that $\Gamma_\alpha^{A_\alpha} = \overline{K}$. It shall have two outcomes $i <_L w$.

The strategy will perform cycles k of increasing length, examining each element $n < k$ on each cycle. The cycles do not necessarily correspond to the stages at which α is active. In fact α can take any number of stages to complete one of its cycles. When examining a particular element n, the strategy α shall try to rectify the operator Γ_α at this element n, using information from the current approximation of the set $\Theta_\alpha^{A_\alpha, B}$. The strategy will act differently depending on whether or not the element is in the current approximation of the set $\overline{K}$.

If $n \in \overline{K}$ then the strategy will try to find an axiom to enumerate in Γ_α which is valid at almost all stages s. Candidates for such an axiom come from the axioms currently enumerated in Θ_α. The strategy α shall select the axiom $\langle n, A_n, B_n \rangle$ that has been valid the longest (i.e. of least age) including at all stages since the strategy last examined the element n during the previous cycle. If there is such an axiom then α will record it as its current guess in a special parameter $Ax_\alpha(n)$ and enumerate a corresponding axiom $\langle n, A_n \rangle$ in the operator Γ_α. Then during the same stage it will move on to examine the next element in the cycle. If the current guess recorded in $Ax_\alpha(n)$ has not been valid at some stage since the last time n was examined or if there is no appropriate axiom valid long enough in the current approximation of the operator Θ_α then the strategy shall indicate that it has been unsuccessful to rectify Γ_α at n via the outcome i, ending its actions for this particular stage and cancelling the value of $Ax_\alpha(n)$. When α is active again it will move on to examine the next element in the cycle. Thus if α has outcome i infinitely often in relation to a particular element n, this yields that Θ_α is unable to supply

α with an axiom for n that is valid at all but finitely many stages and hence $n \notin \Theta_\alpha^{A_\alpha,B}$.

If $n \notin \overline{K}$, then to rectify Γ_α the strategy should ensure that all previously enumerated axioms for n in Γ_α are invalid. It is enough to ensure that there are infinitely many stages s at which $n \notin \Gamma_\alpha^{A_\alpha}$. Thus the strategy first searches for such a stage looking back at all stages since the last time that n was examined. If such a stage is found, α assumes that the operator will be rectified eventually and moves on to the next element, without any further actions related to n. If α is not able to spot a stage at which $n \notin \Gamma_\alpha^{A_\alpha}$, then it shall enumerate the element n back in the set $\Theta_\alpha^{A_\alpha,B}$ by enumerating back in B the B-part, B_n, of each axiom that is used for n in Γ_α and we shall say that α is restraining these elements in B. The strategy will indicate that it has been unsuccessful to rectify Γ_α at n via the outcome w. It shall next concentrate its attention on this element at further stages not moving on to the next element in the cycle, until it observes a stage at which the operator Γ_α is rectified. Thus if α has outcome w at all but finitely many stages then the strategy is never able to rectify Γ_α at some element n. From the properties of a good approximation we can deduce that in this case $n \in \Theta^{A_\alpha,B} \setminus \overline{K}$.

To sum up we have three possibilities for the outcomes of a $\mathcal{P}$-strategy α:

1. The strategy α has outcome w at all but finitely many stages. Then α performs finitely many cycles, reaching an element $n \in \Theta_\alpha^{A_\alpha,B} \setminus \overline{K}$.
2. The strategy α has outcome i related to a particular element n at infinitely many stages. Then α performs infinitely many cycles and for this element n we have that $n \in \overline{K} \setminus \Theta_\alpha^{A_\alpha,B}$.
3. The strategy α has outcome i at infinitely many stages, but for every element n the outcome i is related to n only at finitely many stages. In this case the construction of Γ_α is successful, i.e. $\Gamma_\alpha^{A_\alpha} = \overline{K}$.

Now consider an $\mathcal{N}$-strategy β working on the requirement $\mathcal{N}_e$. We shall denote W_e by W_β. This strategy attempts to prove that $B \neq W_\beta$. First it selects a fresh witness x_β, one that has not appeared in the construction so far. While $x_\beta \notin W_\beta$ the strategy keeps x_β in B and indicates this via a rightmost outcome w. If x_β enters W_β then every time the strategy β is activated it extracts x_β from B and indicates this by a leftmost outcome d.

3.2 Interactions between strategies

The strategies are designed so that they do not interfere with each other. Every $\mathcal{N}$-strategy β is responsible for its unique witness x_β, which will never be extracted unless β decides to extract it. If it is extracted then β will extract it every time it is activated. A $\mathcal{P}$-strategy is most of the time only an observer, it does not modify the approximation to B except in one case, when it restrains elements in B. In this case it has outcome w and we ensure that all strategies extending this outcome are in initial state. Thus lower priority strategies will not injure this restraint. Higher priority strategies initialize α if they injure this restraint.

The only risk we face is that the set B turns out to be properly Σ^0_2 as an element x is extracted and enumerated back in B infinitely often. Consider a $\mathcal{P}$-strategy α and an $\mathcal{N}$-strategy $\beta \supseteq \alpha\hat{\ }i$. Assume that both strategies are visited infinitely often. The strategy β has a witness x_β which is extracted from B. The strategy α has used in the definition of its operator Γ_α an axiom $\langle n, A_n, B_n\rangle$ such that $x_\beta \in B_n$. At stage s the strategy α examines the element n, which is not in $\overline{K}[s]$ and enumerates the witness x_β back in the set B. Then at the next β-true stage x_β is extracted once again from B. If we were dealing with a Σ^0_2 set A_α it would be possible that an element in the finite set A_n is extracted and enumerated back in in the approximation of A_α infinitely many times and as a consequence the element x_β would be extracted and enumerated back in B infinitely often. In this construction however we are given a Δ^0_2-approximation to A_α. As a result the approximation to A_α will eventually settle down on the elements of the finite set A_n and hence the described situation will not happen.

It is still possible however that after α enumerates x_β in B and before the strategy β is activated again, α moves on to a new element n', enumerates a new axiom for it , say $\langle n', A_{n'}, B_{n'}\rangle$, and again $x_\beta \in B_{n'}$. Then β is activated and extracts x_β. Now we are faced with a new situation, which could be repeated infinitely often, regardless of the Δ^0_2-ness of A_α. If it does then the number x_β will again be extracted infinitely often from B.

To avoid this risk we shall require that a $\mathcal{P}$-strategy α always restores the set B in its initial state after it has observed a rectifying stage. In this way after α is done with the element n it extracts the witness x_β, preempting β's actions, before it enumerates a new axiom for n'. Thus the new axiom used for n' will not contain the witness x_β in its B-part. This action makes the $\mathcal{P}$-strategies a bit more aggressive as now they will extract numbers from B as well. This turns out not to provoke further conflicts and is dealt with in detail in Section 3.5.

3.3 Parameters and the tree of strategies

A $\mathcal{P}$-strategy α will have a parameter Γ_α, the enumeration operator that it will construct. At initialization Γ_α is set to the empty set. It will have also parameters k_α denoting the current cycle of the strategy and $n_\alpha \leq k_\alpha$ denoting the current element of the cycle that α is working with. At initialization the values of the parameters are set to $k_\alpha = 0$ and $n_\alpha = 0$. For every n it will have a parameter $Ax_\alpha(n)$ denoting the axiom in Θ_α for n which is currently assumed to be permanently valid. Finally it will keep track in a set Out_α of all witnesses that the strategy has currently re-enumerated back in B, initially empty. At initialization α will give up any restraints.

An $\mathcal{N}$-strategy β shall have a parameter x_β, which will be undefined if β is initialized. At initialization β will give up any restraints.

The tree of strategies is a computable function T with domain $D(T) \subset \{w, d, i\}^{<\omega}$ and range $R(T)$ the set of all requirements with the following inductive definition:

1. $T(\varnothing) = P_{0,0}$.
2. Let α be a $\mathcal{P}$-node in the domain of T. Then $\alpha\hat{\ }o$, where $o \in \{i, w\}$, is also in the domain of T and $T(\alpha\hat{\ }o) = \mathcal{N}_{|\alpha|/2}$, the least $\mathcal{N}$-requirement in the priority listing which is not yet assigned to any node.
3. Let β be an $\mathcal{N}$-node in the domain of T. Then $\beta\hat{\ }o$, where $o \in \{d, w\}$, is in the domain of T and $T(\beta\hat{\ }o) = \mathcal{P}_{i,j}$, where $\mathcal{P}_{i,j}$ is the least $\mathcal{P}$-requirement in the priority listing which is not yet assigned to any node.

3.4 Construction

The set B shall be approximated by a sequence of cofinite sets with $B[0] = \mathbb{N}$ and every set $B[s]$ obtained from $B[s-1]$ by allowing the active strategies at stage s to enumerate or extract numbers from it. A traditional Δ^0_2 approximation $\{\widehat{B}[s]\}_{s<\omega}$ to the set B can be obtained by setting $\widehat{B}[s] = B[s] \upharpoonright s$.

At stage 0 all nodes are initialized. Suppose we have constructed $\delta[t]$ for $t < s$. We construct $\delta[s](n)$ with an inductive definition. We always start at the root of the tree: $\delta[s](0) = \emptyset$. Suppose that we have constructed $\delta[s] \upharpoonright n$. If $n = s$, we end this stage and move on to $s+1$, initializing all nodes $\sigma > \delta[s]$. Otherwise we visit the strategy $\delta[s] \upharpoonright n$ and let it determine its outcome o. We define $\delta[s](n+1) = o$. We have two cases depending on the type of strategy associated with $\delta[s] \upharpoonright n$:

I. $\delta[s] \upharpoonright n = \alpha$ is a $\mathcal{P}$-node:
Let s^- be the previous α-true stage, if α has not been initialized since, and $s^- = s$ otherwise. The strategy α will inherit the values of its parameters from stage s^- and during its actions it can change their values several times. Thus we will omit the subscript indicating the stage when we discuss α's parameters.
If the current element n_α does not need further actions we shall move on to the next element in the cycle. As this is a subroutine which is frequently performed in the construction, we define it here once and for all and refer to it with the phrase "reset the parameters".

Definition 5 (Resetting the parameters)
Denote the current values of n_α by n and of k_α by k. We *reset the parameters* by performing the following actions:
- Initialize the strategies extending $\alpha\hat{\ }w$.
- If Out_α is not empty then extract Out_α from B and set $Out_\alpha = \emptyset$.
- Remove any restraint imposed by α.
- If $n < k$ then set $n_\alpha := n + 1$.
- If $n = k$ then set $k_\alpha := k + 1$, $n_\alpha := 0$ and end this sub-stage with outcome i.

At stage s we perform the following actions:

1. Let $k = k_\alpha$ and $n = n_\alpha$. Let s_n^- be the previous stage when n was examined, if α has not been initialized since, $s_n^- = s$ otherwise.
2. If $n \in \overline{K}[s]$ and $n \in \Gamma_\alpha^{A_\alpha}[t]$ for all stages t with $s_n^- < t \leq s$ then *reset the parameters* and go to step 1.
3. If $n \in \overline{K}[s]$, but $n \notin \Gamma_\alpha^{A_\alpha}[t]$ at some stage t with $s_n^- < t \leq s$ then:
   a. If $Ax_\alpha(n)$ is not defined then define it as the axiom that has been valid longest including at all stages $s_n^- < t \leq s$ and move on to step c. If there is no such axiom then let the outcome be i and *reset the parameters.*
   b. If $Ax_\alpha(n)$ is defined but was not valid at some stage t with $s_n^- < t \leq s$, then cancel its value (make it undefined) and let the outcome be i, *reset the parameters.*
   c. If $Ax_\alpha(n) = \langle n, A_n, B_n \rangle$ is defined and has been valid at all stages t with $s_n^- < t \leq s$ then enumerate in Γ_α the axiom $\langle n, A_n \rangle$. *Reset the parameters* and go back to step 1.
4. If $n \notin \overline{K}[s]$ and $n \notin \Gamma_\alpha^{A_\alpha}[t]$ at some stage t, with $s_n^- < t \leq s$, *reset the parameters* and go back to step 1.
5. Suppose $n \notin \overline{K}[s]$ but $n \in \Gamma_\alpha^{A_\alpha}[t]$ at all t such that $s_n^- < t \leq s$. For each axiom $\langle n, A_n \rangle \in \Gamma_\alpha[s]$, consider the corresponding B-part B_n of the axiom $\langle n, A_n, B_n \rangle \in \Theta_\alpha$. If $B_n \nsubseteq B[s]$ then enumerate all elements from B_n that are not in $B[s]$ back in the set B. Out of these elements enumerate in the set Out_α the ones that are currently restrained out of B. Restrain the elements of B_n in B. Let the outcome be w. Note that we will not reset the parameters at this point, thus the construction will keep going through this step while there is no change in A_α. If later on there is a change in A_α then the strategy will move on to the next element in the cycle but only after it has restored the set B to its original state by extracting Out_α from B.

II. $\delta[s] \upharpoonright n = \beta$ is an $\mathcal{N}$-node:
Let s^- be the previous β-true stage if β has not been initialized since. The strategy β inherits the values of its parameters from stage s^- and goes to the step indicated at stage s^-. Otherwise $s^- = s$ and the strategy starts from step 1.

1. Define x_β as a fresh number - one that has not appeared in the construction so far. Go to the next step.
2. If $x_\beta \notin W_\beta[s]$ then let the outcome be w, return to this step at the next β-true stage. Otherwise go to the next step.
3. If $x_\beta \in B[s]$, then extract x_β from $B[s]$ and restrain it out of B. Let the outcome be d, come back to this step at the next β-true stage.

3.5 Proof

We define the true path h to be the leftmost infinite path in the tree of strategies of nodes visited at infinitely many stages, i.e.:

1. $(\forall n)(\exists^\infty s)[h \upharpoonright n \subseteq \delta[s]]$;
2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \not<_L h \upharpoonright n]$.

The true path exists as the tree is finitely branching. We shall prove that the strategies along the true path do not get initialized infinitely often.

Proposition 2 *For all n there exists a stage $s_i(n)$ such that $h \upharpoonright n$ does not get initialized at stages $t \geq s_i(n)$.*

Proof We prove this proposition by induction on n. The case $n = 0$ is trivial as $h \upharpoonright 0$ does not get initialized at any stage $t > 0$, thus $s_i(0) = 1$.

Suppose that we have proved the statement for n. Then $h \upharpoonright (n+1)$ does not get initialized at any stage $t \geq \max(s_i(n), s_l(n+1))$ unless it is initialized by $h \upharpoonright n$. The only case when this is possible is when $h \upharpoonright n$ is a $\mathcal{P}$-strategy and $h \upharpoonright (n+1) = (h \upharpoonright n)\hat{\ }w$. It follows from the construction that $h \upharpoonright n$ performs only finitely many cycles, as the actions on *resetting the parameters* ensure that every time the strategy starts a new cycle it has outcome i. Thus after a certain stage $s_i(n+1)$ the strategy $h \upharpoonright n$ will not reset its parameters and hence will not initialize strategies below outcome w. □

The next lemma shows that the only elements that are ever extracted from the set B are the witnesses that are extracted by an $\mathcal{N}$-strategy.

Proposition 3 *1. Let x_β be a witness of an $\mathcal{N}$-strategy β. If β does not extract x_β at any stage, then $x_\beta \in B[s]$ for all s.*
2. If x is not a witness to an $\mathcal{N}$-strategy, then $x \in B[s]$ for all s.

Proof 1. It follows that the witness x_β will never be restrained out of B. From the choice of a witness in step $II.1$ of the construction it follows that x_β is not a witness to any other $\mathcal{N}$-strategy. On the other hand it cannot be extracted by a $\mathcal{P}$-strategy α as in order to be extracted by α it must first enter the set Out_α and elements in this set are necessarily restrained out of B.
2. Part 2. is proved by a similar argument as Part 1. □

This is all we need to prove that the $\mathcal{N}$-requirements are satisfied.

Lemma 1 *The set B is not c.e.*

Proof For every i there is a strategy β along the true path working with $\mathcal{N}_i$. This strategy is visited infinitely often and not initialized at any stage $t \geq s_i(|\beta|)$, where $s_i(|\beta|)$ is defined in Proposition 2. Let $x = x_\beta[s_i(|\beta|)]$ be $\beta's$ permanent witness at stages $t \geq s_i|\beta|$. If $\beta\hat{\ }w \subset h$ then x is never enumerated in W_i and Proposition 3 yields $x \in B$, hence $x \in B \setminus W_i$. If $\beta\hat{\ }d \subset h$ then there is a stage s_x such that $x \in W_i[t]$ at all $t \geq s_x$. At every β-true stage $t \geq s_x$ the strategy β ensures $x \notin B[t]$, hence $x \in W_i \setminus B$. □

We shall turn our attention to the $\mathcal{P}$-strategies. Before we can prove that they are successful we will show that the restraints that they impose on B are respected.

Lemma 2 *Let α be a $\mathcal{P}$-strategy. If α restrains an element n in B at stage s then $n \in B[t]$ at all stages $t > s$ until α removes the restraint.*

Proof Suppose for a contradiction that a strategy γ extracts n from B at stage $s_1 \geq s$ strictly before α has removed the restraint. And let s_1 be the least such stage and γ be the least such strategy. We have to consider different cases depending on the type and priority of the strategy γ.

1. $\gamma <_L \alpha$. Then γ is visited at stage s_1 and hence α is initialized at stage s_1 and removes its restraints.
2. $\alpha <_L \gamma$. Then γ is initialized at stage s. If γ is an $\mathcal{N}$-strategy then γ chooses its witness after stage s hence bigger than n. If γ is a $\mathcal{P}$-strategy then γ will extract only elements from B that enter Out_γ at a stage t such that $s < t < s_1$. Elements that enter the set $Out_\gamma[t]$ are not in $B[t]$. By our choice of stage s_1 as the least stage greater than s at which $n \notin B$, we have that $n \in B[t]$ and hence does not enter Out_γ.
3. $\alpha \subset \gamma$. The only strategies that are accessible while α is restraining elements in B are the strategies extending outcome w. By the actions in *Resetting the parameters* these strategies are in initial state at stage s. Thus the argument in 2 is valid for these strategies as well.
4. $\gamma \subset \alpha$ and γ is an $\mathcal{N}$-strategy. Then n is the witness of γ. If $\gamma\hat{\ }w \subseteq \alpha$ then at stage s_1 the strategy γ has outcome d and initializes α, forcing it to drop any restraints. If $\gamma\hat{\ }d \subseteq \alpha$ then the element n is extracted by γ at every α-true stage since the last initialization of α. Thus no axiom $\langle m, A_m, B_m\rangle$ with $n \in B_m$ is valid at an α-true stage after the last initialization of α and hence no such axiom will be used by α in the construction of Γ_α. This contradicts the fact that α restrains n at stage s.
5. $\gamma \subset \alpha$ and γ is a $\mathcal{P}$-strategy. Then $n \in Out_\gamma[s_1]$. Suppose n enters the set Out_γ at stage $t < s_1$. Then $n \notin B[t]$ and by the choice of s_1 it must be that $t \leq s$. Then at stage s the element n is in Out_γ and by the construction γ has outcome w at stage s, as whenever it has outcome i the set Out_γ is empty. As α is visited at stage s, $\gamma\hat{\ }w \subseteq \alpha$. By the actions of *Resetting the parameters* when the set Out_γ is extracted from B, γ initializes α at stage s_1. □

We are ready to prove that every $\mathcal{P}$-requirement is satisfied.

Lemma 3 *For every i if $A_i \not\equiv_e \overline{K}$ then $A_i \oplus B \not\equiv_e \overline{K}$.*

Proof Suppose that A_i is incomplete and for each j consider the strategy $\alpha \subset h$ along the true path labelled by the requirement $\mathcal{P}_{i,j}$. Then $\Theta_j = \Theta_\alpha$ and $A_i = A_\alpha$. We will prove that $\Theta_\alpha^{A_\alpha,B} \neq \overline{K}$. By Proposition 2 after stage $s_i(|\alpha|)$ the strategy α is not initialized. Let $\Gamma_\alpha = \bigcup_{t>s_i(|\alpha|)} \Gamma_\alpha[t]$. Then by assumption $\Gamma_\alpha^{A_\alpha} \neq \overline{K}$.

Suppose there is an $m \in \Gamma^{A_\alpha} \setminus \overline{K}$. Then there is a valid axiom $\langle m, A_m\rangle$ in Γ_α for m. Let $s > s_i(|\alpha|)$ be the stage at which this axiom is enumerated in Γ_α. As $A_m \subseteq A_\alpha$, A_α is a Δ_2^0 set and $\overline{K}$ is a Π_1 set, there is a stage $s_1 > s$

such that $(\forall t \geq s_1)[A_m \subseteq A_\alpha[t] \wedge m \notin \overline{K}[t]]$. If after stage s_1 the strategy α considers m then by $I.5$ of the construction α will never again move on to a different element and will have outcome w forever. Thus α will perform finitely many cycles.

If α performs finitely many cycles then let n be the last element it considers and let s_2 be the least stage such that $n_\alpha[t] = n$ for all $t \geq s_2$. Then again by $I.5$ of the construction $n \notin \overline{K}[t]$ and $n \in \Gamma_\alpha^{A_\alpha}[t]$ at all $t \geq s_2$ or else $I.4$ of the construction would be valid at an α-true stage and α would move on to the next element. The good approximation that we have chosen for A_α and Proposition 1 guarantee that in this case $n \in \Gamma_\alpha^{A_\alpha}$ and hence there is a valid axiom $\langle n, A_n\rangle$ in Γ_α. By the actions that α performs at stage s_2 under $I.5$ each axiom for n including $\langle n, A_n, B_n\rangle$ is restored, i.e. $B_n \subseteq B[s_2]$ and α restrains B_n in B at all stages $t \geq s_2$. By Lemma 2 $B_n \subseteq B$. Thus $n \in \Theta_\alpha^{A_\alpha, B}$.

Suppose now that $\Gamma_\alpha^{A_\alpha} \subseteq \overline{K}$ and that α performs infinitely many cycles. Let n be the least element such that $n \in \overline{K} \setminus \Gamma_\alpha^{A_\alpha}$. We will prove that in this case $n \notin \Theta_\alpha^{A_\alpha, B}$. Suppose not. Then there is a valid axiom in Θ_α. Consider the oldest valid axiom $\langle n, A_n, B_n\rangle$ in Θ_α, i.e. the one with least limit age $a(A_\alpha \oplus B, A_n \oplus B_n)$.

By assumption the strategy will perform infinitely many cycles and hence at infinitely many stages it will examine n. As $n \notin \Gamma_\alpha^{A_\alpha}$ and we have chosen a good approximation to A_α there will be infinitely many stages at which $n \notin \Gamma_\alpha^{A_\alpha}[t]$. Let s_0 be the first stage at which α examines n and at which the oldest valid axiom $\langle n, A_n, B_n\rangle$ has reached its limit age, i.e. $a(A_\alpha \oplus B, A_n \oplus B_n, t) = a(A_\alpha \oplus B, A_n \oplus B_n)$ at all $t \geq s_0$ and all other axioms for n enumerated in $\Theta[s_0]$ have greater age.

Let t_0 be the least stage after s_0 such that $n \notin \Gamma_\alpha^{A_\alpha}[t_0]$. Consider the least stage $s_1 > t_0$ at which n is again considered by α. Then step $I.3$ of the construction will be executed. If $Ax_\alpha(n)$ is currently undefined then α will select $\langle n, A_n, B_n\rangle$ as the new value of $Ax_\alpha(n)$ and enumerate it in Γ_α. If $Ax_\alpha(n)$ does have a value then it will be cancelled as the corresponding axiom is already enumerated in Γ_α and was not valid at stage t_0. Let t_1 be the next stage at which $n \notin \Gamma_\alpha^{A_\alpha}[t_1]$ and s_2 be the next stage at which α considers n. Finally $I.3.a$ and $I.3.c$ will be executed and the axiom $\langle n, A_n\rangle$ will be enumerated in Γ_α. By assumption this axiom is valid at all stages $t > s_0$ hence $n \in \Gamma_\alpha^{A_\alpha}$ and we have reached the desired contradiction. □

Finally to complete the proof we need to show that the constructed set B is in fact a Δ_2^0 set. We will do this in two steps.

Lemma 4 *Suppose α is a strategy visited at stages s_1 and s_2. Suppose x is a witness of a higher priority strategy $\beta < \alpha$. If $B(x)[s_1] = 1$ and $B(x)[s_2] = 0$ then α is initialized at a stage t, with $s_1 < t \leq s_2$.*

Proof First note that in order to have $B(x)[s_2] = 0$, β must extract the witness x at stage $s_x \leq s_2$ by Proposition 3. If $s_1 < s_x$ then β is visited at stage s_x and has outcome d. Then if $\beta <_L \alpha$ or $\beta\hat{\ }w \subseteq \alpha$, the strategy α is initialized at stage s_x. If $\beta\hat{\ }d \subseteq \alpha$ then as at stage s_1 the witness $x \in B[s_1]$, β must have a different witness at stage s_1 and must have been initialized together with all its successors including α at a stage t such that $s_1 < t \leq s_x \leq s_2$.

Suppose $s_x < s_1$. As all strategies of lower priority than β are in initial state at stage s_x, the element x must be enumerated back in B before or at stage s_1 by a $\mathcal{P}$-strategy γ of higher priority than β. By construction at this stage γ executes step $I.5$ of the construction with outcome w and restrains x in B. This restraint is still valid at stage s_1 hence $\alpha \geq \gamma\hat{}w$. By Lemma 2 γ gives up its restraint at a stage $t \leq s_2$ as otherwise $x \in B[s_2]$. By construction when γ gives up its restraint, it is either initialized (together with all nodes of lower priority than γ) or else it itself initializes all strategies of lower priority than $\gamma\hat{}w$, hence α is initialized at stage t. □

Lemma 5 *The set B is Δ^0_2.*

Proof We will prove that for every number n the value of $B(n)$ changes only finitely often. By Proposition 3 this is true for numbers that are not witnesses to any $\mathcal{N}$-strategy and for numbers that are witnesses to an $\mathcal{N}$-strategy and are never extracted by it.

Suppose that n is the witness x_β to the $\mathcal{N}$-strategy β extracted for the first time at stage s_x. No other $\mathcal{N}$-strategy will affect $B(x_\beta)$ as the sets of witnesses to each $\mathcal{N}$-strategy are disjoint. If β is initialized at stage $s > s_x$ then β gives up its restraint on x_β and will not extract x_β at any further stage. Furthermore x_β cannot enter the set $Out_\alpha[t]$ for any $t > s$ and any $\mathcal{P}$-strategy α. At stage s the element x_β can belong to finitely many sets Out_α for finitely many strategies α. Each such strategy can extract the element x_β only once, when emptying the set Out_α. Altogether $B(x_\beta)$ changes finitely often.

Suppose that β is not initialized after stage s_x. Then the element x_β has a permanent restraint out of B and no strategy $\alpha <_L \beta$ is visited after stage s_x.

First we note that $\mathcal{P}$-strategies of lower priority than β will not change the value of $B(x_\beta)$ as in order to do this they must be visited at a stage s_1 at which $B(x_\beta)[s_1] = 1$ to include an axiom that uses x_β and then again at a stage s_2 at which $B(x_\beta)[s_2] = 0$ to enumerate x_β back in B, without being initialized in between and by Lemma 4 this is not possible.

Thus we only need to prove that the finitely many $\mathcal{P}$-strategies $\alpha \subset \beta$ do not change the value of the $B(x_\beta)$ infinitely often. Assume for a contradiction that this is not true and let $\alpha \subset \beta$ be the largest strategy that changes the value of x_β infinitely often. It follows that α is visited infinitely often, not initialized and performs infinitely many cycles. Let $s > s_x$ be a stage after which no lower priority $\mathcal{P}$-strategy ever changes the value of $B(x_\beta)$. Let $s_0 > s$ be the least stage at which $B(x_\beta) = 0$.

At any stage $t > s_0$ if α is visited and chooses a new axiom to enumerate in Γ_α then $B(x_\beta)[t] = 0$. Indeed higher priority strategies $\alpha'\hat{}i \subset \alpha$ always have empty $Out_{\alpha'}$ when they have outcome i. Higher priority strategies $\alpha''\hat{}w$ do not enumerate any further elements after stage s_0 or else α and hence β are initialized. If α enumerates the element x_β in B at stage t_0 it executes step $I.5$ and does not define new axioms in Γ_α. The element x_β is permanently restrained out of B and hence enters the set $Out_\alpha[t_0]$. If α chooses a new axiom at stage $t > t_0$ then its set Out_α is empty and $Out_\alpha[t_0]$ is extracted from B, hence $x_\beta \notin B[t]$.

Thus α will use only finitely many axioms whose B-part contains x_β in the definition of Γ_α. These are axioms for finitely many numbers, only part of which are not elements of the set $\overline{K}$. For each such element $n \notin \overline{K}$ there will be finitely many axioms $Ax(n)$ enumerated in Γ_α. Let s_n be a stage after which the approximation of the Δ^0_2 set A_α does not change on the A-parts of the axioms $Ax(n)$. After stage s_n the value of $\Gamma^{A_\alpha}_\alpha(n)$ does not change. If $\Gamma^{A_\alpha}_\alpha(n) = 1$ then when α examines n after stage s_n it will restrain x_β in B forever and never move on to a different element contrary to the assumption that α performs infinitely many cycles. If $\Gamma^{A_\alpha}_\alpha(n) = 0$ then whenever α examines n at stages after s_n, step $I.4$ of the construction will be valid and α will not enumerate x_β back in B. This proves that our assumption is wrong and hence $B(x_\beta)$ changes its value only at finitely many stages. □

4 The second limitation

In this section we give a proof of Theorem 2. Given an incomplete Σ^0_2 enumeration degree **a** we will construct a nonzero 3-c.e. enumeration degree **b** which is not cupped by **a**.

Let A be a representative of the given Σ^0_2 enumeration degree. Let $\{A[s]\}_{s<\omega}$ be a good Σ^0_2 approximation to A. We shall construct two 3-c.e. sets X and Y, so that ultimately the degree of one of them will have the requested properties. The sets will actually be constructed as co-d.c.e., i.e. as complements of 2-c.e. sets.

Consider the following requirements:

1. Let $\{\Theta_i\}_{i<\omega}$ and $\{\Psi_i\}_{i<\omega}$ be standard listings of all enumeration operators. For every i we will have a pair of requirements:

$$\mathcal{P}^0_i : \Theta^{A,X}_i \neq \overline{K} \quad \text{and} \quad \mathcal{P}^1_i : \Psi^{A,Y}_i \neq \overline{K}.$$

2. Let $\{W_e\}_{e<\omega}$ be the standard listing of all c.e. sets. For every natural number e we have a requirement:

$$\mathcal{N}_e : W_e \neq X \wedge W_e \neq Y.$$

We shall construct the sets X and Y so that for all e the requirement $\mathcal{N}_e$ is satisfied, thus both X and Y have nonzero enumeration degree, and if $\mathcal{P}^j_i$ is not satisfied for some i then for all i' the requirement $\mathcal{P}^{1-j}_{i'}$ is satisfied, thus the degree of at least one of the sets $A \oplus X$ or $A \oplus Y$ is incomplete. The construction resembles the one used in Section 3, it is carried out in stages and uses a tree of strategies.

4.1 Basic strategies

We shall again describe the basic strategies with the context of the tree in mind. The tree of strategies shall be designed so that each node shall be assigned either an $\mathcal{N}$-requirement or a pair of a $\mathcal{P}^0$- and a $\mathcal{P}^1$-requirement.

A $\mathcal{P}$-strategy α is associated with a pair of requirements, $\mathcal{P}^0_\alpha$ and $\mathcal{P}^1_\alpha$. It will attempt at proving that at least one of them is satisfied. To do this the strategy constructs an enumeration operator Γ_α, threatening to prove that $A \geq_e \overline{K}$. The outcomes of the strategy will be divided into two groups, *finitary*, i.e. requiring a finite number of actions, and *infinitary* outcomes, requiring an infinite number of actions. There will be infinitely many infinitary outcomes - two for each number n arranged from left to right by the order of the natural numbers: $\langle X, 0\rangle <_L \langle Y, 0\rangle < \langle X, 1\rangle \ldots$ Then there will be two finitary rightmost outcomes $\langle X, w\rangle <_L \langle Y, w\rangle$. Thus all the outcomes of a $\mathcal{P}$-node are arranged as follows:

$$\langle X, 0\rangle <_L \langle Y, 0\rangle <_L \ldots <_L \langle X, n\rangle <_L \langle Y, n\rangle \ldots <_L \langle X, w\rangle <_L \langle Y, w\rangle.$$

For each outcome the first element of the pair indicates which requirement has been satisfied. The next $\mathcal{P}$-strategy below outcomes $\langle X, -\rangle$ shall be associated with a new $\mathcal{P}^0$-requirement and the same $\mathcal{P}^1$-requirement. Similarly the next $\mathcal{P}$-strategy below outcomes $\langle Y, -\rangle$ will be associated with the same $\mathcal{P}^0$-requirement and a different $\mathcal{P}^1$-requirement. Thus if $\mathcal{P}^j_i$ never gets satisfied for some i then all $\mathcal{P}^{1-j}_{i'}$ must be.

Similarly to the $\mathcal{P}$-strategy described in Section 3 the strategy α performs cycles of increasing length. On the k-th cycle it examines all elements $n = 0, 1, \ldots, k$ in turn. While it examines an element n the strategy can choose to end its actions for the particular stage by selecting an outcome or move on to the next element in the cycle, possibly even starting a new cycle. Suppose α is examining the element n. If the element n currently belongs to $\overline{K}$ then the only possible outcomes that it can choose for this element are the infinitary $\langle X, n\rangle$ or $\langle Y, n\rangle$. If the element n is in both sets $\Theta^{A,X}_\alpha$ and $\Psi^{A,Y}_\alpha$ and has been there at all stages since α last looked at n then it will enumerate an axiom for n in Γ_α which comprises the A-parts of the two axioms for n in Θ_α and in Ψ_α that have been valid the longest, i.e. have least age, and move on to the next element. Otherwise α will select the appropriate outcome corresponding to the set that has failed to provide a valid axiom and end its actions for this stage. When α is active again, it will start working with the next element of the cycle.

If the element n has left the approximation of $\overline{K}$ then for each axiom in Γ_α for this element the strategy shall select and restore one of the axioms in either Θ_α or Ψ_α by enumerating the corresponding X-part back in X or Y-part back in Y and have the corresponding finitary outcome $\langle X, w\rangle$ or $\langle Y, w\rangle$. The precise method for this selection will be described in the next section. Note that for α's success this selection is irrelevant. The strategy shall then wait until it has observed a change in A that rectifies the operator Γ_α, i.e. it will not move on to the next element in the cycle until (if ever) this happens and it will keep having the same finitary outcome.

As A is incomplete the strategy will eventually include in its cycles an element n such that $\Gamma_\alpha^A(n) \neq \overline{K}(n)$. If there is an element n such that $n \in \Gamma_\alpha^A \backslash \overline{K}$ then $n \in \Gamma_\alpha^A[s] \setminus \overline{K}[s]$ at all but finitely many stages s. Thus eventually $\Gamma_\alpha^A(n)$ will not be rectified by any change in A and α will have a finitary outcome proving the successful diagonalization. Otherwise α will have infinitely many cycles and each element n will be examined infinitely often. Consider the least n such that $n \in \overline{K} \backslash \Gamma_\alpha^A$. By the properties of a good approximation we have that at infinitely many stages s, in fact at all good stages, $n \notin \Gamma_\alpha^A[s]$. Thus infinitely often α will discover that at least one of the operators Θ_α or Ψ_α has failed to provide it with an axiom that is permanently valid, i.e. infinitely often α will have proof that $\Theta_\alpha^{A,X}(n) = 0$ or $\Psi_\alpha^{A,Y}(n) = 0$ and have outcome $\langle X, n\rangle$ or $\langle Y, n\rangle$ respectively.

An $\mathcal{N}$-strategy β working on W_β would like to prove that $W_\beta \neq X$ and $W_\beta \neq Y$. The obvious strategy for β would be the one described in Section 3. It will select a witness x_β and wait until $x_\beta \in W_\beta$. The sets X and Y will initially be approximated by $\mathbb{N}$, then during the construction the strategies extract or enumerate back elements in the sets. Thus if x_β never enters W_β the strategy will be successful and will have outcome w. If the element does enter W_β then the strategy will extract x_β from both sets X and Y, have outcome d, where $d <_L w$, and again will have proved a difference. This strategy is unfortunately incomplete as we shall see in the next section.

4.2 Elaborating the $\mathcal{N}$-strategy to avoid conflicts

The naive $\mathcal{N}$-strategy described in the previous section is in conflict with the need of higher priority $\mathcal{P}$-strategies to restore axioms by enumerating elements back in one of the sets X or Y, constructed as 3-c.e. sets. Therefore the strategy for an $\mathcal{N}$-node β will have to be more elaborate. This conflict justifies the introduction of nonuniformity.

The elaborated strategy will start off as the original strategy: select a witness x_β as a fresh number and wait until $x_\beta \in W_\beta$. If this never happens then the requirement will be satisfied with outcome w. Otherwise it will extract x_β from both sets X and Y. Suppose a higher priority $\mathcal{P}$-strategy α wants to restore an axiom that includes x_β in its X- or Y-part. As was noted previously, the strategy α can make a choice between enumerating elements back in X or enumerating elements back in Y. In this case β shall permanently restrain x_β out of X and only allow α to enumerate it back in Y, i.e. if α has the choice between enumerating the finite set X_θ back in X or Y_ψ back in Y, and $x_\beta \in X_\theta$, then α will select to enumerate Y_ψ back in Y. The strategy β shall then initialize all lower priority strategies, choose a new witness y_β that has not been used in any axiom so far. From this point on any axiom that appears in the construction shall necessarily have $x_\beta \notin X$, thus x_β and y_β cannot appear in the same axiom. The strategy β will wait again with outcome w until y_β

enters W_β and then extract it from Y with outcome d. Should a higher priority α require that an axiom be restored which involves y_β then β will only give permission to enumerate back in X.

This will resolve the central conflict between strategies. Note that as the only actions that the $\mathcal{P}$-strategies ever take is enumerating certain elements back in the sets X and Y, the $\mathcal{P}$-strategies are not in conflict with each other. Possible conflicts between $\mathcal{N}$-strategies are resolved via initialization. Whenever a higher priority $\mathcal{N}$-strategy β decides to extract a number n from X or Y all strategies below outcome w are initialized and all strategies below outcome d are in initial state. Thus lower priority strategies will operate at further stages under the assumption that n is extracted, the axioms used by lower $\mathcal{P}$-strategies will not include this element and the witnesses used by $\mathcal{N}$-strategies will be chosen as big numbers that do not appear in any axiom seen so far, thus cannot appear in an axiom that includes the element n.

4.3 Parameters and the tree of strategies

A $\mathcal{P}$-strategy α will have a parameter Γ_α, the enumeration operator that it will construct when visited. At initialization Γ_α is set to the empty set. The strategy will also have parameters k_α denoting the current cycle of the strategy and $n_\alpha \leq k_\alpha$ denoting the current element of the cycle that α is working with. On initialization the values of the parameters are set to $k_\alpha = 0$ and $n_\alpha = 0$. Furthermore for each element $n < \omega$ the strategy α shall have one more parameter $D_\alpha(n)$, a list of all pairs of X- and Y-parts of axioms from Θ_α and Ψ_α respectively, for which the A-parts are used in axioms for n in Γ_α. Initially the values of all such lists will be $\emptyset$. Finally it will have two parameters $Ax^\theta_\alpha(n)$ and $Ax^\psi_\alpha(n)$ denoting axioms in Θ_α and Ψ_α respectively which will be candidates for the construction of a new axiom in Γ_α, initially undefined.

An $\mathcal{N}$-strategy β shall have parameters x_β, y_β, initially undefined. Furthermore on initialization β will give up any restraint it has imposed so far.

Let $O_\mathcal{P}$ denote the set of all possible outcomes of a $\mathcal{P}$-strategy and $O_\mathcal{N} = \{d, w\}$. Let $O = O_\mathcal{P} \cup O_\mathcal{N}$ be the collection of all possible outcomes and $\mathcal{R}$ the collection of all requirements. The tree of strategies is a computable function T with domain a downwards closed subset of $O^{<\omega}$ and range a subset of $\mathcal{R}^2 \cup \mathcal{R}$ with the following inductive definition:

1. $T(\emptyset) = \langle \mathcal{P}^0_0, \mathcal{P}^1_0 \rangle$.
2. Let α be in the domain of T and α be a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node. Then $\alpha\hat{\ }o$, where $o \in O_\mathcal{P}$, is also in the domain of T and $T(\alpha\hat{\ }o) = \mathcal{N}_{|\alpha|/2}$.
3. Let β be an $\mathcal{N}$-node in the domain of T. Then $\beta = \alpha\hat{\ }o$, where α is a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node for some i and j. Then $\beta\hat{\ }o'$, where $o' \in O_\mathcal{N}$, is in the domain of T. If $o = \langle X, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_{i+1}, \mathcal{P}^1_j \rangle$. If $o = \langle Y, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_i, \mathcal{P}^1_{j+1} \rangle$.

4.4 Construction

At stage 0 all nodes are initialized and $X[0] = Y[0] = \mathbb{N}$, $\delta[0] = \emptyset$.

Suppose we have constructed $\delta[t]$, $X[t]$ and $Y[t]$ for $t < s$. The sets $X[s]$ and $Y[s]$ shall be obtained by allowing the strategies visited at stage s to modify the approximations $X[s-1], Y[s-1]$ obtained at the previous stage. We construct $\delta[s](n)$ with an inductive definition. Define $\delta[s](0) = \emptyset$. Suppose that we have constructed $\delta[s] \upharpoonright n$. If $n = s$, we end this stage and move on to $s+1$. Otherwise we visit the strategy $\delta[s] \upharpoonright n$ and let it determine its outcome o. Then $\delta[s](n+1) = o$. We have two cases depending on the type of the node $\delta[s] \upharpoonright n$.

I. If $\delta[s] \upharpoonright n = \alpha$ is a $\mathcal{P}$-node, we perform the following actions:
Let s^- be the previous α-true stage if α has not been initialized since and $s^- = s$ otherwise. The strategy α will inherit the values of its parameters from stage s^- and during its actions it can change their values several times. The actions that α makes when moving on to a new element in the cycle will be defined in advance as was done in Section 3.4.

Definition 6 (Resetting the parameters) Denote the current values of n_α by n and of k_α by k. We *reset the parameters* by changing the values of the parameters as follows:

- If $n < k$ then set $n_\alpha := n + 1$.
- If $n = k$ then set $k_\alpha := k + 1$, $n_\alpha := 0$.
- Initialize the strategies extending $\alpha\hat{\ }\langle X, w\rangle$ and $\alpha\hat{\ }\langle Y, w\rangle$.

At stage s we perform the following actions:

1. Let $k = k_\alpha$ and $n = n_\alpha$. Let s_n^- be the previous stage when n was examined, if α has not been initialized since, $s_n^- = s$ otherwise.
2. If $n \in \overline{K}[s]$ and $n \in \Gamma_\alpha^A[t]$ for all stages t with $s_n^- < t \leq s$ then *reset the parameters* and go to step 1.
3. If $n \in \overline{K}[s]$, but $n \notin \Gamma_\alpha^A[t]$ at some stage t with $s_n^- < t \leq s$ then:
   - a.X If $Ax_\alpha^\theta(n)$ is not defined, then define it as the axiom $\langle n, A_\theta, X_\theta\rangle$ with least age $a(A[s] \oplus X[s], A_\theta \oplus X_\theta, s) \leq s_n^-$ and move on to step $a.Y$. If there is no such axiom then let the outcome be $\langle X, n\rangle$ and *reset the parameters.*
   - b.X If $Ax_\alpha^\theta(n)$ is defined but was not valid at some stage t with $s_n^- < t \leq s$ then cancel its value (make it undefined) and let the outcome be $\langle X, n\rangle$, *reset the parameters*. Otherwise go to step $a.Y$.
   - a.Y If $Ax_\alpha^\psi(n)$ is not defined, then define it as the axiom $\langle n, A_\psi, Y_\psi\rangle$ with least age $a(A[s] \oplus Y[s], A_\psi \oplus Y_\psi, s) \leq s_n^-$ and move on to step c. If there is no such axiom then let the outcome be $\langle Y, n\rangle$ and *reset the parameters.*
   - b.Y If $Ax_\alpha^\theta(n)$ is defined but was not valid at some stage t with $s_n^- < t \leq s$ then cancel its value (make it undefined) and let the outcome be $\langle Y, n\rangle$, *reset the parameters*. Otherwise go to step c.

c. If both $Ax^{\theta}_{\alpha}(n) = \langle n, A_\theta, X_\theta\rangle$ and $Ax^{\psi}_{\alpha}(n) = \langle n, A_\psi, Y_\psi\rangle$ are defined and have been valid at all stages t with $s_n^- < t \leq s$ then enumerate in Γ_α the axiom $\langle n, A_\theta \cup A_\psi\rangle$. Enumerate $\langle X_\theta, Y_\psi\rangle$ in $D_\alpha(n)$. *Reset the parameters* and go back to step 1.

4. If $n \notin \overline{K}[s]$ and $n \notin \Gamma^A_\alpha[t]$ at some stage t, with $s_n^- < t \leq s$, *reset the parameters* and go back to step 1.
5. Suppose $n \notin \overline{K}[s]$ but $n \in \Gamma^A_\alpha[t]$ at all t such that $s_n^- < t \leq s$. For every pair $\langle X_\theta, Y_\psi\rangle \in D_\alpha(n)$ find the highest priority $\mathcal{N}$-strategy $\beta \supset \alpha$ that has permanently restrained an element $x \in X_\theta$ out of X or $y \in Y_\psi$ out of Y. If there is such a strategy β and it has a permanent restraint on X, enumerate Y_ψ in $Y[s]$; if it has a permanent restraint on Y, enumerate X_θ back in $X[s]$. Otherwise if there is no such strategy enumerate Y_ψ back in $Y[s]$. Choose the axiom $\langle n, A_\theta \cup A_\psi\rangle$ in Γ^A_α with least age $a(A[s], A_\theta \cup A_\psi, s)$. Let X_θ and Y_ψ be the corresponding X and Y parts of the axioms $\langle n, A_\theta, X_\theta\rangle \in \Theta_\alpha$ and $\langle n, A_\psi, Y_\psi\rangle \in \Psi_\alpha$.

   a. If $X_\theta \subseteq X[s]$ then this will ensure that $n \in \Theta^{A,X}_\alpha[s]$. Let the outcome be $\langle X, w\rangle$.

   b. If $X_\theta \nsubseteq X[s]$ then $Y_\psi \subseteq Y[s]$ and this will ensure that $n \in \Psi^{A,Y}_\alpha[s]$. Let the outcome be $\langle Y, w\rangle$.

II. If $\delta[s] \restriction n = \beta$ is an $\mathcal{N}$-node, we perform the following actions:
Let s^- be the previous β-true stage if β has not been initialized since, go to the step indicated at stage s^-. Otherwise $s^- = s$ and go to step 1.

1. Define x_β as a fresh number, one that has not appeared in the construction so far and is bigger than s. Go to the next step.
2. If $x_\beta \notin W_\beta[s]$ then let the outcome be w, return to this step at the next β-true stage. Otherwise go to the next step.
3. Extract x_β from $X[s]$ and $Y[s]$. Restrain permanently x_β out of X. Let the outcome be d, go to the next step at the next β-true stage.
4. If $x_\beta \in Y[s]$ then define y_β as a fresh number, initialize all strategies of lower priority than β and go to the next step. Otherwise the outcome is d, return to this step at the next β-true stage.
5. If $y_\beta \notin W_\beta$ then let the outcome be w. Return to this step at the next β-true stage. Otherwise go to the next step.
6. If y_β is not yet restrained then restrain y_β permanently out of Y and extract y_β from $Y[s]$. Let the outcome be d, return to this step at the next β-true stage.

4.5 Proof

The tree is infinitely branching and therefore there is a risk that there might not be an infinite path in the tree that is visited infinitely often. However we shall start the proof by establishing some basic facts about the relationship between strategies.

For technical convenience we shall define one more notation. Let α be a $\mathcal{P}$-strategy. To every axiom $Ax = \langle n, A_\theta \cup A_\psi\rangle \in \Gamma_\alpha$ we shall associate a corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$ so that $\langle n, A_\theta, X_\theta \rangle \in \Theta_\alpha$ and $\langle n, A_\psi, Y_\psi \rangle \in \Psi_\alpha$ are the corresponding axioms used to construct Ax.

Lemma 6 *Let β be an $\mathcal{N}$-strategy, initialized for the last time at stage s_i. If β has a witness x_β that is extracted by β at stage $s_x > s_i$ then $x_\beta \notin X[t]$ at all $t \geq s_x$. If β has a witness y_β that is extracted from Y at stage $s_y > s_x$ then $y_\beta \notin Y[t]$ at all $t \geq s_y$.*

Proof There are only finitely many $\mathcal{N}$-strategies of higher priority than β that are ever visited in the construction as after stage s_i no strategy to the left of β is visited. Every higher priority strategy $\beta' < \beta$ that is ever visited is not initialized after stage s_i, as otherwise β would be initialized after stage s_i contrary to our assumption. We can inductively assume that the statement is valid for every higher priority strategy β'.

Suppose β chooses the witness x_β at stage $s_1 > s_i$. We can furthermore prove the following:

Claim: Any witness which is permanently extracted by a higher priority strategy β' is extracted before or at stage s_1.

Indeed, suppose that β' permanently extracts a new witness at stage $s_2 > s_1$. Then at stage s_2 the strategy β' has outcome d. Thus if $\beta >_L \beta'$ or $\beta \supseteq \beta' \hat{} w$ then β would be initialized at stage s_2 contrary to assumption. This leaves us with the only possibility that $\beta \supseteq \beta' \hat{} d$. Then at stage s_1, as β was visited, β' was visited and had outcome d. As β' is not initialized after stage s_1 and permanently extracts a new witness at stage s_2 it must be the case that β' permanently extracts a witness $y_{\beta'}$ from Y and $x_{\beta'}$ was already extracted before or at stage s_1. It follows that between stages s_1 and s_2, β' has selected this new witness $y_{\beta'}$ passing through $II.4$ of the construction and initializing all lower priority strategies including β. This leads again to a contradiction with the assumption that β is not initialized after stage s_1 and hence the claim is correct.

Thus at stage s_1 all witnesses of higher priority strategies that are ever permanently restrained out of either set X or Y are already permanently restrained out of X or Y. At stage s_1 the strategy β selects x_β as a fresh number. And at stage s_x the witness x_β is permanently restrained out of X.

Now we will prove again inductively but this time on the stage t, that $x_\beta \notin X[t]$ at all stages $t \geq s_x$.

So suppose this is true for $t < s_3$ and that at stage $s_3 > s_x$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi \rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$. We have the following cases to consider:

1. Suppose $\alpha > \beta$. If $\alpha >_L \beta \hat{} d$ then α is initialized at stage s_x. If $\alpha \subset \beta \hat{} d$, then α was initialized at stage s_i and was not accessible before stage s_x. Thus in both cases the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α at stage t with $s_x \leq t < s_3$, at which both $\langle n, A_\theta, X_\theta \rangle$ and $\langle n, A_\psi, Y_\psi \rangle$ were valid i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By induction $x_\beta \notin X[t]$ hence $x_\beta \notin X_\theta$ and thus α does not enumerate x_β back in X.

2. Suppose $\alpha < \beta$. If $\alpha <_L \beta$ then β would be initialized at stage s_3, hence $\alpha \subset \beta$. Suppose the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α at stage t. If $t \leq s_1$ then by the choice of x_β as a fresh number at stage s_1 we have that $x_\beta \notin X_\theta$. If $t > s_1$ then both $\langle n, A_\theta, X_\theta \rangle$ and $\langle n, A_\psi, Y_\psi \rangle$ were valid at stage t i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By $I.5$ of the construction α will consider all $\mathcal{N}$-strategies that extend it and select the one with highest priority that has permanently restrained an element out of either set X or Y.
   Consider any $\beta' < \beta$. By our *Claim* any witness $x_{\beta'}$ or $y_{\beta'}$ of β' that is ever permanently restrained out of X or Y is already restrained out at stage s_1 and by induction at all stages $s \geq s_1$ including at stage t. Thus X_θ does not contain $x_{\beta'}$ and Y_ψ does not contain $y_{\beta'}$. As this is true for an arbitrary strategy β' of higher priority than β that is ever visited, if $x_\beta \in X_\theta$ then β will be the strategy selected by α and α will choose to enumerate Y_ψ back in Y. Thus again α does not enumerate x_β back in X.

To prove the second part of the lemma suppose y_β is selected at stage s_4 and extracted at stage s_y. Because $s_1 < s_4$ and all strategies of lower priority than β are initialized at stage s_4 the interactions between β and other strategies are dealt with in the same way as in the case when we were considering x_β. The only thing left for us to establish is that β does not come into conflict with itself. So suppose that at stage $s_5 > s_y$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi \rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$. We will prove that if $x_\beta \in X_\theta$ then $y_\beta \notin Y_\psi$. Let t be the stage at which the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α. If $t < s_4$ then $y_\beta \notin Y_\psi$ by the choice of y_β at stage s_4 as a fresh number. If $t \geq s_4 > s_x$ then we have already proved that $x_\beta \notin X[t]$. The axiom $\langle n, A_\theta, X_\theta \rangle$ was valid at stage t, thus $X_\theta \subseteq X[t]$, and hence $x_\beta \notin X_\theta$.

This completes the induction step and the proof of the lemma. $\square$

Lemma 7 *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs finitely many cycles then:*

1. *There is a stage $s_n \geq s_i$ after which the value of n_α does not change.*
2. *At all α-true stages $t > s_n$, α has either outcome $\langle X, w \rangle$ or outcome $\langle Y, w \rangle$.*
3. *There is a stage $s_d \geq s_n$ such that at all α-true stages $t > s_d$, α has a fixed outcome o.*
4. *If $o = \langle X, w \rangle$ then $\Theta_\alpha^{A,X} \neq \overline{K}$ and if $o = \langle Y, w \rangle$ then $\Psi_\alpha^{A,Y} \neq \overline{K}$.*

Proof It follows from the construction and the definition of the actions *resetting the parameters* that if the value of n_α changes infinitely often, then there will be infinitely many cycles. Thus part (1) of the lemma is true. Let s_n be the stage after which the value of n_α does not change. The only case when the value of the parameter $n_\alpha = n$ is not reset is when $n \notin \overline{K}$ and $n \in \Gamma_\alpha^A[t]$ at all stages t since the last time n was examined at stage s_n^-, thus α will have only outcomes $\langle X, w \rangle$ or $\langle Y, w \rangle$ at all stages after s_n and part (2) is true. It follows from step $I.5$ of the construction and the fact that n_α does not change any longer that at all stage $t > s_n$, $n \in \Gamma_\alpha^A[t]$. By the properties of a good approximation and under these circumstances $n \in \Gamma_\alpha^A$. Then there will be an axiom $\langle n, A_\theta \cup A_\psi \rangle \in \Gamma_\alpha$ which is valid at all but finitely many stages. Select the axiom with least limit age. This axiom has corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$. The strategy α will eventually be able to spot this precise axiom, after possibly finitely many wrong guesses. So after a stage $s_d \geq s_n$ the strategy α will consider this axiom to select its outcome.

At stage s_n either $X_\theta \subseteq X[s_n]$ or $Y_\psi \subseteq Y[s_n]$. As we initialize all strategies below outcomes $\langle X, w \rangle$ and $\langle Y, w \rangle$ whenever we reset the parameters, we can be sure that $\mathcal{N}$-strategies visited at stages $t > s_n$ of lower priority than α will not extract any elements of $X_\theta \cup Y_\psi$ from X or Y. Higher priority $\mathcal{N}$-strategies will not extract any elements at all, otherwise α would be initialized. Thus if $X_\theta \subseteq X[s_n]$ then for all stages $t \geq s_n$ we have $X_\theta \subseteq X[t]$ and similarly if $Y_\psi \subseteq Y[s_n]$ then for all stages $t \geq s_n$ we have $Y_\psi \subseteq Y[t]$.

Suppose $X_\theta \subseteq X[s_n]$. Then at stages $t \geq s_d$ the strategy α will always have outcome $\langle X, w \rangle$. The axiom $\langle n, A_\theta, X_\theta \rangle \in \Theta_\alpha$ will be valid at all stages $t \geq s_d$, thus $n \in \Theta^{A,X}$, and $n \notin \overline{K}$.

If $X_\theta \nsubseteq X[s_n]$ then there is a strategy $\beta \supset \alpha$ which is permanently restraining some element $x \in X_\theta$ out of X at stage s_n. Then $\beta <_L \alpha\hat{\ }\langle X, w \rangle$ as strategies extending $\alpha\hat{\ }\langle X, w \rangle$ or to the right of it are in initial state at stage s_n and do not have any restraints. This strategy β will not be initialized at stages $t \geq s_n$ according to part (2) of this lemma and the choice of $s_n > s_i$. By Lemma 6 $x \notin X[t]$ at all $t \geq s_n$. Hence case $I.5.b$ of the construction is valid at all $t \geq s_d$. Thus α will have outcome $\langle Y, w \rangle$ at all stages $t \geq s_d$ and $n \in \Psi^{A,Y}$. This proves parts (3) and (4) of the lemma. □

Proposition 4 *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If v is an element such that $\Gamma_\alpha^A(v) = \overline{K}(v)$ then there is a stage s_v after which the outcomes $\langle X, v \rangle$ and $\langle Y, v \rangle$ are not accessible any longer.*

Proof If α has finitely many cycles then by Lemma 7 there will be a stage s_n after which $\langle X, v \rangle$ and $\langle Y, v \rangle$ are not accessible. Suppose there are infinitely many cycles.

If $v \notin \overline{K}$ then there is a stage s_v at which v exits $\overline{K}$. Then after stage s_v the outcomes $\langle X, v \rangle$ and $\langle Y, v \rangle$ are not accessible.

If $v \in \Gamma_\alpha^A$ then there is an axiom in Γ_α that is valid at all but finitely many stages, say at all stages $t \geq s'_v$. If α is on its k-th cycle during stage s'_v then let s_v be the beginning of the $(k+2)$-nd cycle. Then after stage s_v, whenever α considers v, part $I.2$ of the construction holds and hence α will never have outcome $\langle X, v \rangle$ or $\langle Y, v \rangle$. □

Lemma 8 *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs infinitely many cycles, then there is leftmost outcome $o <_L \langle X, w \rangle$ that α has at infinitely many stages and*

1. *If $o = \langle X, u \rangle$ then $\Theta_\alpha^{A,X}(u) \neq \overline{K}(u)$.*

2. If $o = \langle Y, u\rangle$ then $\Psi_\alpha^{A,Y}(u) \neq \overline{K}(u)$.

Proof The set A is not complete by assumption, hence $\Gamma_\alpha^A \neq \overline{K}$. Let u be the least difference between the sets. By Proposition 4 for every $v < u$ the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not visited at stages $t > s_v$. Let s_0 be a stage bigger than $\max\{\ s_v|\ v < u\}$. As α has infinitely many cycles there will be infinitely many stages $t > s_0$ at which $n_\alpha[t] = u$. If $u \notin \overline{K}$ and $u \in \Gamma_\alpha^A$ then there is a stage $s_1 > s$ such that at all stages $t > s_1$ we have $u \in \Gamma_\alpha^A[t]$ and $u \notin \overline{K}[t]$ and when α considers u at the first stage after s_1, it will never move on to the next element, and α would have finitely many cycles. Hence $u \in \overline{K}$ and $u \notin \Gamma_\alpha^A$.

1. If $u \notin \Theta_\alpha^{A,X}$ then all axioms for u in Θ_α are invalid at infinitely many stages. Let t be any stage greater than or equal to s_0. We will prove that there is a stage $t' \geq t$ at which α has outcome $\langle X, u\rangle$. As $u \notin \Gamma_\alpha^A$ and $\{A[s]\}_{s<\omega}$ is a good approximation to A there are infinitely many stages s at which $u \notin \Gamma_\alpha^A[s]$ and hence part $I.3$ of the construction holds at infinitely many stages at which we consider u. Let $t_1 \geq t$ be such that $n_\alpha[t_1] = u$ and part $I.3$ of the construction is true. If $Ax_\alpha^\theta(u)$ is not defined and we are not able to define it as there is no appropriate axiom in Θ_α valid for long enough then α will have outcome $\langle X, u\rangle$ at stage t_1, hence $t' = t_1$ proves the claim. Otherwise $Ax_\alpha^\theta(u)$ is defined at stage t_1 and by assumption there are infinitely many stages $t \geq t_1$ at which it is invalid. Let $t_2 > t_1$ be the next stage when $Ax_\alpha^\theta(u)$ is invalid and let $t' \geq t_2$ be the first stage after t_2 at which again $n_\alpha[t'] = u$ and part $I.3$ of the construction is true. By $I.3.b.X$ of the construction α will have outcome $\langle X, u\rangle$ at stage t'.
2. Now assume that $u \in \Theta_\alpha^{A,X}$. Then there is an axiom $\langle u, A_\theta, X_\theta\rangle \in \Theta_\alpha$ valid at all but finitely many stages. Select the axiom, say Ax, with least limit age. Then $Ax_\alpha^\theta(u)$ will have permanent value Ax after a certain stage s_1. It follows that $u \notin \Psi_\alpha^{A,Y}$ as otherwise we would be able to find an axiom in $\Psi_\alpha^{A,Y}$ valid at all but finitely many stages, and construct an axiom in Γ_α valid at all but finitely many stages. Now a similar argument as the one used in part (1) of this lemma proves that α will have outcome $\langle Y, u\rangle$ at infinitely many stages. □

As an immediate corollary from Lemmas 7 and 8 we obtain the existence of the true path:

Corollary 2 *There exists an infinite path through the tree of strategies with the following properties:*

1. $(\forall n)(\exists^\infty s)[h \restriction n \subseteq \delta[s]]$*;*
2. $(\forall n)(\exists s_l(n))(\forall t > s_l(n))[\delta[t] \not<_L h \restriction n]$*;*
3. $(\forall n)(\exists s_i(n))(\forall t > s_i(n))[h \restriction n$ *is not initialized at stage* $t]$.

Proof We will define the true path by induction on n and prove that it has the properties needed. The case $n = 0$ is trivial: $h \restriction 0 = \emptyset$ is visited at every stage of the construction and is never initialized, $s_l(0) = s_i(0) = 0$. Suppose we have constructed $h \restriction n$ with the required properties. We shall define $h \restriction (n + 1)$.

If $h \upharpoonright n = \beta$ is an $\mathcal{N}$-strategy then let $o \in \{d, w\}$ be the leftmost outcome that β has at infinitely many stages. The design of the strategy ensures that there is a stage $s_o > s_i(n)$ such that $\beta\hat{\ }o \subseteq \delta[t]$ at all $t \geq s_o$. If β does not define y_β after stage $s_i(n)$ then s_o is the first stage after $s_i(n)$ at which β has outcome o. If β defines y_β at stage s_y then s_o is the first stage after s_y at which β has outcome o. We define $h \upharpoonright (n+1) = \beta\hat{\ }o$ and $s_i(n+1) = s_o$.

Suppose $h \upharpoonright n = \alpha$ is a $\mathcal{P}$-strategy. If α performs finitely many cycles then by Lemma 7 there is a stage $s_o > s_i(n)$ after which α does not reset the parameters and has the same fixed outcome o. We define $h \upharpoonright (n+1) = \alpha\hat{\ }o$ and $s_i(n+1) = s_o$.

If α performs infinitely many cycles then by Lemma 8 there is a leftmost outcome $o <_L \langle X, w\rangle$ that α has at infinitely many stages. Let $s_o > s_i(n)$ be a stage such that at stages $t > s_o$ the strategy α does not have outcomes $o' <_L o$. Then $h \upharpoonright (n+1) = \alpha\hat{\ }o$ and $s_i(n+1) = s_o$. □

Corollary 3 *X and Y are not c.e.*

Proof For every requirement $\mathcal{N}_e$ there is an $\mathcal{N}_e$-strategy β along the true path, visited infinitely often and not initialized at any stage $t > s_i$. Let x_β and y_β be the final values of β's witnesses. If $\beta\hat{\ }w \subset h$ then there is an element $u \in \{x_\beta, y_\beta\}$ that never enters W_e. The way each $\mathcal{N}_e$-strategy chooses its witnesses ensures that only β can extract u from either of the sets X or Y. The construction and the definition of the true path ensure that β does not extract u from X and Y at any stage. Hence $u \in X \cap Y$ and $u \notin W_e$.

If $\beta\hat{\ }d \subset h$ then $x_\beta \in W_e$ and there is a β-true stage s_x at which β extracts x_β from X and Y. By Lemma 6 $x_\beta \notin X[t]$ at all stages $t \geq s_x$. If at any stage $t \geq s_x$ we have that $x_\beta \in Y[t]$ then β selects y_β at its next true stage. As the true outcome is d, $y_\beta \in W_e[t']$ at some stage $t' \geq t$. Then at the next β-true stage $s_y \geq t'$ the strategy β will permanently restrain y_β out of Y and by Lemma 6, we have that $y_\beta \notin Y$. □

Corollary 4 *$A \oplus X \not\equiv_e \overline{K}$ or $A \oplus Y \not\equiv_e \overline{K}$.*

Proof Consider the $\mathcal{P}$-nodes on the true path. From the definition of the tree it follows that either for every $\mathcal{P}^0_e$-requirement there is a node on the tree α which is associated with $\mathcal{P}^0_e$ or there is a fixed requirement $\mathcal{P}^0_e$ associated with all but finitely many nodes. In the latter case there is a node on the true path for every $\mathcal{P}^1_e$-requirement.

Suppose there is a node on the tree for each $\mathcal{P}^0_e$-requirement. We can show that $A \oplus X \not\equiv_e \overline{K}$. Assume for a contradiction $\Theta^{A,X}_e = \overline{K}$ and let $\alpha \subset h$ be the last node associated with $\mathcal{P}^0_e$. Then α has true outcome $\langle X, u\rangle$ for some $u \in \omega \cup \{w\}$. It follows from Lemma 7 and Lemma 8 that $\Theta^{A,X}_e \neq \overline{K}$.

The case when there is a node for every $\mathcal{P}^1_e$-requirement yields by a similar argument that $A \oplus Y \not\equiv_e \overline{K}$. □

Lemma 9 *The sets X and Y are 3-c.e.*

Proof We can easily obtain a 3-c.e. approximation of each of the sets X and Y from the one constructed. Define $\widehat{X}[s] = X[s] \upharpoonright s$ and $\widehat{Y}[s] = Y[s] \upharpoonright s$.

It follows from the construction that elements extracted from X and Y are necessarily witnesses of $\mathcal{N}$-strategies. Suppose therefore that n is the witness x_β for an $\mathcal{N}$-strategy β. Then n appears in the defined approximations $\{\widehat{X}[s]\}_{s<\omega}$ and $\{\widehat{Y}[s]\}_{s<\omega}$ at stage $n+1$. If β never extracts x_β then we are done - as no other strategy can extract it. If β extracts x_β then it does so only once at stage s_x when it goes through $II.3$ and moves on to $II.4$ at the next stage. In order for β to return to step $II.3$ of the construction, β has to be initialized and will select new witnesses. Thus after its extraction at stage s_x from both $\widehat{X}[s_x]$ and $\widehat{Y}[s_x]$, the number x_β can only be enumerated back in either set and hence $|\{\ s|\ \widehat{X}[s-1](x_\beta) \neq \widehat{X}[s](x_\beta)\}| \leq 3$ and $|\{\ s|\ \widehat{Y}[s-1](x_\beta) \neq \widehat{Y}[s](x_\beta)\}| \leq 3$.

If n is the witness y_β then it will never be extracted from X. If it is ever extracted from Y it is extracted only once by β at the first stage it reaches step $II.6$. After that y_β is already restrained by β and whenever β executes step $II.6$ it will ignore the first sentence of the instruction and just have outcome $o = d$. Thus again $|\{\ s|\ \widehat{Y}[s-1](y_\beta) \neq \widehat{Y}[s](y_\beta)\}| \leq 3$.

This concludes the proof of the lemma and of the theorem. □

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