Effective reducibilities and Degree spectra of Abstract structures

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[1]

Abstract

We introduce the basic notions and facts of the classical Computability theory. The accent is on the relative computability and some reducibilities among sets and functions, as Turing reducibility, many-one reducibility, c.e. reducibility and enumeration reducibility. We show how the structure of Turing degrees can be embedded isomorphically into the structure of the enumeration degrees. We illustrate some basic methods as forcing, genericity and prove some basic theorems as Fridberg's Jump Inversion Theorem for the Turing jump and the existence of quasi-minimal enumeration degree, the minimal pair theorem and the Selman's theorem for the enumeration degrees. We give an illustration how the Computability theory is applied in the Computable structure theory. We give an survey of some general properties of the degree spectra of a structure which show that they behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results are the analogs of Selman's Theorem, the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree. We consider the relationships between the spectra and the jump spectra. Our first result is that every jump spectrum is also a spectrum. The main result is a Jump inversion theorem for the degree spectra.

Contents

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1 Introduction

This is a short course on Computability theory and its applications in Computable structure theory. We start with Kleene's definition of the computable functions, Turing computable functions based on the notion of Turing machines and the Unlimited Register Machine - computable functions. It turns out that all these notions describe the same class of functions and this is by the Church-Turing thesis the class of all intuitively effective functions. The relativization of the notion of computable functions could be obtain in different ways, comparing the computability of two possibly incomputable sets of natural numbers and using a different information of them. We consider some basic properties of the Turing reducibility, many-one reducibility and c.e. reducibility. In the next section we introduce the properties of the Turing degrees and Turing jump, some properties of the generic sets and the forcing relation in order to represent some methods of this theory. We present the proof of the Fridberg's Jump Inversion Theorem. Next sections is devoted to the notion of enumeration operator, the properties of enumeration reducibility and the connections with the other reducibilities. We show how the Turing degrees are isomorphically embedded into the enumeration degrees. We prove some basic properties of the enumeration reducibility as the quasi-minimal degree, the minimal pair theorem and the Selman's theorem. In the last section we consider as an application of this theory the notion of degree spectrum of a countable structure which is a measure of complexity of the structure. We consider several examples of degree spectra and co-spectra and prove some general properties of the degree spectra which show that the degree spectra behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results we would like to mention are the analogs of Selmans Theorem, the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree. We introduce an analogue of a jump of a structure and prove a jump inversion theorem for degree spectra and that every jump spectrum is a spectrum of a structure.

We follow [5, 2] for the basic notions of Computability theory. The approach and the structure of the explanations come from of the course on Recursion theory in the Master program Logic and Algorithms, which was taught by the second author in the past several years at Sofia university. And the last section is based on [17, 18].

2 Relative computability

In this section we will introduce the class of computable functions. The goal is to fix he notations and definitions and to remind some basic facts and theorems, from the classical computability theory. All notions and theorems could be found in [5, 2].

2.1 Computable functions

After Gödel's formalization of the intuitive notion of a computable function, Stephen Kleene gave this model of computability its final form, and he developed the theory of computability (recursive function theory).

We will consider only partial functions on the set of the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Denote by $\mathcal{F}_n$ the set of all partial functions on n arguments on $\mathbb{N}$.

Definition 2.1.1 (Kleene)**.** A function f is *partial recursive* (p.r.) if it can be obtained from the basics $O = \lambda x.0$, $S = \lambda x.x + 1$, and $I^n_k = \lambda x_1, \dots, x_n.x_k$ by the operations superposition, primitive recursion and μ operation applied finitely many times, where

- the function $h \in \mathcal{F}_n$ is a superposition of $g_i \in \mathcal{F}_n$, $i = 1 \dots k$, and $f \in \mathcal{F}_k$ iff $\lambda x_1 \dots x_n.h(x_1, \dots, x_n) = f(g_1(x_1, \dots, x_n) \dots g_n(x_1, \dots, x_n))$,
- $h \in \mathcal{F}_{n+1}$ is a primitive recursion of $f \in \mathcal{F}_n$ and $g \in \mathcal{F}_{n+2}$ iff

  $h(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n)$ and

  $h(x_1, \dots, x_n, y + 1) = g(x_1, \dots, x_n, y, h(x_1, \dots, x_n, y))$

- $h \in \mathcal{F}_n$ is obtained by μ operation of $f \in \mathcal{F}_{n+1}$ iff

  $h(x_1, \ldots, x_n,) = \mu y[f(x_1, \ldots, x_n, y) = 0]$

  (i.e. $\forall z < y(\downarrow f(x_1, \ldots, x_n, z) > 0) \& f(x_1, \ldots, x_n, y) = 0$).

As usual a function in the natural numbers we call *primitive recursive* if it can be obtained from O, S and I_n^k by the operations superposition and primitive recursion, applied finitely many times (without using the μ operation). We call a p.r. function *computable (recursive)* if it is total.

Using the primitive recursive functions we can code different "data structures" in the natural numbers, i.e. the n-tuples of natural numbers, the finite sequences of natural numbers, the finite subsets of $\mathbb{N}$.

Let $\langle x, y \rangle = 2^x(2y+1) - 1$. It is easy to see that it is a bijection between $\mathbb{N}^2$ and $\mathbb{N}$. Moreover there are primitive recursive decoding functions $L(\langle x, y \rangle) = x$ and $R(\langle x, y \rangle) = y$. Let $\pi_1(x) = x$ and $\pi_{n+1}(x_1, \ldots, x_{n+1}) = \langle x_1, \pi_n(x_2, \ldots, x_{n+1}) \rangle$. Denote by $\langle x_1, \ldots, x_n \rangle$ the code $\langle n, \pi_n(x_1, \ldots, x_n) \rangle$ of the sequence $x_1, \ldots, x_n$.

The finite set $\{x_1, \ldots, x_n\}$ we denote by D_v, where $v = 2^{x_1} + \cdots + 2^{x_n}$.

Several models of computation arise at the beginning of 20th century. The first machine-based model is due to Alan Turing (1936). The basic hardware for any Turing machine consists of a tape, subdivided into cells, which is infinitely extendable in both directions, and a reading/writing head. Each Turing machine has a program, which uses finitely many internal states and a finite input alphabet. The computation begins at the start state with the first symbol of the input. The program decides at each step what action to be done: to replace the current symbol and to move the head to the left or to the right by one position or to stay at the same position. The natural numbers are represented by strings in a finite alphabet, e.g. binary represented. If the machine comes to a final state it stops and the result is on the tape, after the position of the head. A function f on the natural numbers is computable by a Turing machine M if for every input n the machine M stops whenever $\downarrow f(n)$ (is defined) and $M(n) = f(n)$.

As Cooper writes in [2] "It is a remarkable fact that computability exists independently of any language used to describe it. Any sufficiently general model of the computable functions gives the same class of functions." This prompts Alonzo Church and Alan Turing to conjecture the following in the 1930s.

Fact 2.1.2 (Church - Turing thesis)**.** *A function f is effectively computable iff f is Turing computable.*

What we mean by f being effectively computable is that there exists some description of an algorithm, in some language, which can be used to compute by finitely many steps any value $f(x)$ for which $\downarrow f(x)$.

In support of the Church-Turing thesis it happens that all sufficiently general known models of computability are equivalent, i.e. they compute the same class of functions. Other computational models characterizing the intuitive idea of effective computability are: general recursive functions of Gödel, Herbrand and Kleene(1936), λ-definable function of Church (1936), Post canonical systems (1943), Markov's algorithms (1951), Unlimited Register Machines by Shepherdson and Sturgis (1963).

Unlimited Register Machines (or URMs) are mathematical abstractions of real-life computers. URMs are more user-friendly than Turing machines. A URM has registers $R_1, R_2, \dots$ which store natural numbers $r_1, r_2, \dots$. A URM program is a finite list of instructions each having one of four basic types: $Z(n)$ which says that $r_n = 0$, $S(n) : r_n := r_n + 1$, $T(m, n) : r_n := r_m$ and the goto instruction $J(m, n, q)$: If $r_n = r_m$ then go to instruction with a label q else go to next instruction. Each URM computation using a given program starts with instruction number 1 on the list and carries out the rest in numerical order unless told to jump. A computation will halt if it runs out of instructions to obey. The result is obtained in R_1.

A function f is computable by a URM program P if P halts only on the elements of the domain of the function f and gives the same result as f. f is URM-computable if and only if there is a URM program which computes f.

Proposition 2.1.3. *A function is URM - computable iff it is Turing computable iff it is partial recursive.*

So we will identify p.r. functions with the Turing computable ones and with URM - computable ones.

Consider an effective coding of all URM programs. By φ_e we will denote the function, computable by the program with code e. Let $\{\varphi_e^{(n)}\}_{e\in\omega}$ be the standard listings of the URM computable functions on n arguments.

Theorem 2.1.4 (Kleene's normal form theorem)**.** *There exists a primitive recursive function T_n such that*

1. $\downarrow \varphi_e^{(n)}(\bar{x}) \iff \exists z[T_n(e, \bar{x}, z) = 0]$;
2. $\varphi_e^{(n)}(\bar{x}) = L(\mu z[T_n(e, \bar{x}, z) = 0])$.

Theorem 2.1.5 (S_n^m-theorem)**.** *There is a primitive recursive function S_n^m :*

$$\varphi_a^{(m+n)}(\bar{x}, \bar{y}) \simeq \varphi_{S_n^m(a,\bar{x})}^{(n)}(\bar{y}),$$

for every $a, \bar{x}, \bar{y}$.

Theorem 2.1.6 (Universal function theorem)**.** *There is a partial recursive function $U_n \in \mathcal{F}_{n+1}$*

$$\varphi_a^{(n)}(\bar{x}) \simeq U_n(a, \bar{x})$$

for every $a, \bar{x}$.

The *characteristic function* of a subset A of $\mathbb{N}$ we denote by

$$c_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A. \end{cases}$$

The set A is *computable* iff c_A is computable.

The *semi-characteristic function* of A we denote by:

$$\chi_A(x) \simeq \begin{cases} 1 & , x \in A \\ \neg\downarrow & , x \notin A. \end{cases}$$

The set A is *computable enumerable* (c.e.) if χ_A is p.r. The c.e. sets are the domains of the p.r. functions. The set A is c.e if there is an effective process for enumerating all the members of A, or more formally: if $A = \emptyset$ or there is a computable function f such that $A = \{f(0), f(1), ...\} = range(f)$. The computable sets are closed under union, intersection and complement, and the c.e. sets are closed under union and intersection. Every computable set is c.e..

The connection between computable and c.e sets is given by Post's theorem: the set A is computable iff A and $\bar{A}$ are c.e.. The Kleene set $K = \{x \mid x \in W_x\}$ is c.e. (by the Universal function theorem K is the domain of $\lambda x.U_1(x, x)$), but not computable since its complement is not c.e..

Denote by $W_e^{(n)} = \mathrm{dom}(\varphi_e^{(n)})$ the computably enumerable (c.e.) set which is the domain of $\varphi_e^{(n)}$.

The normal form of the c.e sets is: $W_e^{(n)} = \{\bar{x} \mid \exists z[T_n(e, \bar{x}, z) = 0]\}$.

2.2 Turing reducibility

Let ϕ be a partial function on $\mathbb{N}$, which we will refer to as an oracle.

Definition 2.2.1. A function is computable in ϕ if it can be obtained from the basic functions O, S, I_k^n and ϕ by superposition, primitive recursion and μ-operation, applied finitely many times.

We can relativize the notion of a URM program: a URM program with oracle ϕ is a URM program with an additional command $O(n)$, which sends a query to the oracle about its value on argument: the number, which is contained in the nth register. If the oracle $\phi(r_n)$ is defined, then the result $\phi(r_n)$ is obtained in the nth register, otherwise the computation does not halt. A function f is URM computable with oracle ϕ if there is a URM program with oracle ϕ which computes f. It turns out that a function f is URM computable with oracle ϕ iff f is computable in ϕ.

The Normal form theorem for the functions computable in ϕ, the S_n^m theorem, the Universal function theorem, and all the properties of the computable functions and c.e. sets are relativized naturally.

Definition 2.2.2. A function f is Turing reducible to the function ϕ ($f \leq_T \phi$) if f is computable in ϕ.

Let $A \subseteq \mathbb{N}$ be given. We say that a function f *is computable in* A, if f is computable in c_A.

We will denote the functions that are computable in A by $\{\varphi_e^A\}_{e\in\omega}$. We use also the notation $\{e\}^A$ for φ_e^A. The computable in A functions coincide with the A - Turing computable ones, i.e. computable with a Turing machine with oracle A.

For sets $A, B \subseteq \mathbb{N}$:

Definition 2.2.3. B is Turing reducible to A ($B \leq_T A$) if the characteristic function $c_B \leq_T c_A$.

The intuition behind this definition is as follows: we want B to be computable from A if we can give a yes or no answer to every membership question of the form Is $n \in B$? using additionally finitely many answers to similar questions about the set A Is $m_1 \in A$?, Is $m_2 \in A$?, …, Is $m_k \in A$?. The immediate properties are:

1. If $A \subseteq \mathbb{N}$ is computable, then $(\forall B \subseteq \mathbb{N})(A \leq_T B)$, i.e. $c_A \leq_T B$ for an arbitrary oracle B.
2. $A \leq_T \mathbb{N} \Rightarrow A$ is computable.
3. $\overline{A} \leq_T A$.

2.3 Many-one reducibility

A stronger reducibility is the many-one reducibility (m-reducibility), which gives a very natural way of comparing the computability of different possibly incomputable sets of natural numbers A and B.

Definition 2.3.1. The set A is *many-one reducible* (m-reducible) to B if

$$A \leq_m B \iff (\exists \text{ total computable function } h)(\forall x)(x \in A \iff h(x) \in B).$$

It is clear that if B is computable (c.e) and $A \leq_m B$ then A is computable (c.e.).

Proposition 2.3.2. *A is c.e. iff $A \leq_m K$.*

Proof. Let A be c.e. Define:

$$g(x,y) \simeq \begin{cases} 0 & , x \in A \\ \neg\downarrow & , x \notin A. \end{cases}$$

Since g is p.r., by the S^m_n theorem, there is a primitive recursive function h, such that $\varphi_{h(x)}(y) \simeq g(x,y)$. Then $x \in A \iff \downarrow \varphi_{h(x)}(h(x)) \iff h(x) \in K$. For the other direction it is enough to mention that K is c.e. □

Proposition 2.3.3. *$A \leq_m B, B \leq_m C \Rightarrow A \leq_m C$.*

Proof. Let $x \in A \iff h(x) \in B$ and $x \in B \iff g(x) \in C$, where g and h are computable. Then $x \in A \iff g(h(x)) \in C$, i.e. $A \leq_m C$. □

Proposition 2.3.4. *If $A \leq_m B$, then $A \leq_T B$. Morover $\bar{K} \leq_T K$, $\bar{K} \not\leq_m K$.*

Proof. Let $A \leq_m B$ by h. Using h we construct a program which computes $h(x)$ and asks the oracle B if $h(x) \in B$. If we assume that $\bar{K} \leq_m K$, then $\bar{K}$ will be c.e. which we know is a contradiction. □

2.4 c.e.-reducibility

Definition 2.4.1. A is *computable enumerable* (c.e.) in$B$:

$$A \leq_{c.e.} B \iff A = \mathrm{dom}(\{a\}^B)$$

for some program with code a.

It follows from the definition:

1. If A is c.e., then $(\forall B)(A \leq_{c.e.} B)$.
2. If $A \leq_{c.e.} \mathbb{N}$, then A is c.e.

Proposition 2.4.2. *$A \leq_m B, B \leq_{c.e.} C \Rightarrow A \leq_{c.e.} C$.*

Proof. Let $A \leq_m B$ and h be a computable function: $x \in A \iff h(x) \in B$ and e: $B = \mathrm{dom}(\varphi_e^C)$. Consider $g(x) \simeq \varphi_e^C(h(x))$. Then $x \in A \iff h(x) \in B \iff \downarrow g(x)$. Hence $A = dom(g)$ but $g \leq_T C$, then $A \leq_{c.e.} C$. □

Proposition 2.4.3. *$A \leq_T B \Rightarrow A \leq_{c.e.} B$.*

Proof. Let $c_A = \{a\}^B$. Construct a new program:

1. execute $\{a\}^B(x)$ with output y
2. if $y = 1$ stop
3. if $y = 0$ infinite loop.

□

Definition 2.4.4. $W_a^B = \mathrm{dom}(\{a\}^B)$, $K_B = \{a \mid a \in W_a^B\}$.

It is clear that: $K_B \leq_{c.e.} B$ and $\overline{K_B} \not\leq_{c.e.} B$. We could see that $K_B \not\leq_T B$ from the next properties.

Proposition 2.4.5. *$A \leq_{c.e.} B, B \leq_T C \Rightarrow A \leq_{c.e.} C$.*

Proof. Let $A = \mathrm{dom}(\{a\}^B)$, $B = \{b\}^C$. We can translate carefully the program with code a to another substituting everywhere the call $O(n)$ by *calling* the program b. □

Using the same technique we can prove the transitivity of $\leq_T$.

Proposition 2.4.6. *$A \leq_T B, B \leq_T C \Rightarrow A \leq_T C$.*

Notice that from $A \leq_{c.e.} B, B \leq_{c.e.} C$ it does not follow that $A \leq_{c.e.} C$. Since $\overline{K} \leq_{c.e.} K$ and $K \leq_{c.e.} \emptyset$, (K is c.e.), if we assume the transitivity of $\leq_{c.e.}$ then $\overline{K} \leq_{c.e.} \emptyset \leq_T \mathbb{N}$. Thus $\overline{K}$ will be c.e., a contradiction.

Proposition 2.4.7 (Post)**.** *$A \leq_T B \iff A \leq_{c.e.} B$ & $\overline{A} \leq_{c.e.} B$.*

Proof. Let $A \leq_T B$. Since $\overline{A} \leq_T B$ then $A \leq_{c.e.} B$, and $\overline{A} \leq_{c.e.} B$.

Let $A \leq_{c.e.} B$ and $\overline{A} \leq_{c.e.} B$. Then there are programs P and Q, such that $\{P\}^B = \chi_A$, $\{Q\}^B = \chi_{\overline{A}}$. Using them we construct a program PQ, which computes P and Q parallel, step by step, and gives in output 1 if P halts, and 0 if Q halts.

$$(\forall x)(\downarrow \{P\}^B(x) \vee \downarrow \{Q\}^B(x)) \Rightarrow (\forall x) \downarrow \{PQ\}^B(x)).$$

So $A \leq_T B$. □

3 Turing degrees and Turing jump

3.1 Turing degrees

Definition 3.1.1. $A \equiv_T B \iff (A \leq_T B \,\&\, B \leq_T A)$.

The relation $\equiv_T$ is an equivalence relation.

Definition 3.1.2. *The Turing degree of the set* A is the equivalence class containing A:

$$d_T(A) = \{B \mid B \equiv_T A\}.$$

Definition 3.1.3. $d_T(A) \leq d_T(B) \iff A \leq_T B$

Let D_T be the set of all Turing degrees. The structure $(D_T, \leq)$ is a partial order.

Definition 3.1.4 (The operation join)**.** $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

Proposition 3.1.5. *$d_T(A \oplus B)$ is the least upper bound of $d_T(A)$ and $d_T(B)$.*

Proof.

1. $x \in A \iff 2x \in A \oplus B \Rightarrow A \leq_m A \oplus B \Rightarrow A \leq_T A \oplus B$, so $A \oplus B$ is a upper bound of A and B.

2. Let $c_A = \{a\}^C, c_B = \{b\}^C$. Then $c_{A \oplus B}$ is computable in C since

$$c_{A\oplus B}(x) \simeq \begin{cases} \varphi_a^C([x/2]) & \text{, if } x \text{ is odd,} \\ \varphi_b^C([x/2]) & \text{, if } x \text{ is even.} \end{cases}$$

□

The least Turing degree is $\mathbf{0}_T = d_T(\emptyset) = d_T(R)$, where R is an arbitrary computable set.

So we have that the structure $D_T = (D_T, \leq, \oplus, \mathbf{0}_T)$ is an upper semi-lattice.

3.2 The Turing jump

Definition 3.2.1. The Turing jump of a set A is the set:
$A' = K_A = \{x \mid x \in \mathrm{dom}(\varphi_x^A)\}$.

The Turing jump has the following properties.

Proposition 3.2.2. *1.* $K_A \leq_{c.e.} A$.

2. $B \leq_{c.e.} A \Rightarrow B \leq_m K_A$.

Hint: Let $B \leq_{c.e.} A$, *Consider:*

$$g(x,y) \simeq \begin{cases} 0 & , x \in B \\ \neg\downarrow & , x \notin B. \end{cases}$$

By the S_n^m *theorem there is a computable h:* $\varphi_{h(x)}^A(y) \simeq g(x,y)$.

Then $x \in B \iff \downarrow \varphi_{h(x)}^A(h(x)) \iff h(x) \in K_A$.

3. $A <_T K_A$, *since* $\overline{K^A} \not\leq_{c.e.} A$.

Here $A <_T K_A$ means that $A \leq_T K_A$ & $A \not\equiv_T K_A$.

Proposition 3.2.3. $A \leq_T B \iff A' \leq_m B'$.

Proof. ($\Rightarrow$) Let $A \leq_T B$. We have $A' \leq_{c.e.} A$ and then $A' \leq_{c.e.} B$. Thus $A' \leq_m B'$ (by 2.).

($\Leftarrow$) Let $A' \leq_m B'$. We have $A \leq_{c.e.} A \Rightarrow A \leq_m A' \leq_m B'$ and $\overline{A} \leq_{c.e.} A \Rightarrow \overline{A} \leq_m A' \leq_m B'$. Then $A \leq_m B', \overline{A} \leq_m B'$, But (by 1.) $B' \leq_{c.e.} B$ and then $A \leq_{c.e.} B$, $\overline{A} \leq_{c.e.} B$. By Post theorem. $A \leq_T B$. □

Corollary 3.2.4 (Monotonicity of the jump)**.** $A \leq_T B \Rightarrow A' \leq_T B'$.

Definition 3.2.5. $(d_T(A))' = d_T(A')$.

Since $A <_T K_A$, then $d_T(A) < d_T(A')$.

3.3 Genericity

Definition 3.3.1. Every finite mapping $\tau : [0; n-1] \longrightarrow \mathbb{N}$ we call *a finite part.* We denote by $|\tau| = n$ the length of the interval, where τ is defined. For any $a \in \mathbb{N}$ and $\tau : [0; n-1] \longrightarrow \mathbb{N}$, let $\lambda x.(\tau * a)(x)$ be the finite part:

$$(\tau * a)(x) \simeq (\tau * n \to a)(x) \simeq \begin{cases} \tau(x) & \text{if } 0 \leq x < n, \\ a & \text{if } x = n. \end{cases}$$

If A is a set, we write $\tau \subseteq A$ instead of $\tau \subseteq c_A$, i.e. τ is a subfunction of c_A.

We denote the finite parts with the Greek letters: $\alpha, \beta, \delta, \tau, \rho\dots$. We say that $\alpha \subseteq \beta$ if $(\forall x)$ $(\downarrow \alpha(x) \Rightarrow \downarrow \beta(x)$ & $\alpha(x) = \beta(x))$.

Definition 3.3.2. The set A is *generic*, if for every c.e. set S of finite parts:

$$(\exists\alpha \subseteq A)\underbrace{(\alpha \in S \vee (\forall\beta \supseteq \alpha)(\beta \notin S))}_{\alpha \text{ decides } S}.$$

The set of finite parts S is called *dense in* A, if $(\forall\alpha \subseteq A)(\exists\beta \in S)(\alpha \subseteq \beta)$. It is easy to see that A is generic, if whenever S is dense in A, then A meets S, i.e. $(\exists\alpha \subseteq A)(\alpha \in S)$.

Let $\mathcal{S}$ be the set of all finite parts and $S_e = W_e \cap \mathcal{S}$, $e \in \mathbb{N}$. There is a total computable function h, such that $S_e = W_{h(e)}$ for every e.

We will show how to construct a generic set.

The construction of a generic set: We construct by steps finite parts α_n, which will approximate c_A, $\alpha_n \subseteq \alpha_{n+1} \subseteq A$.

- We start with $\alpha_0 = \emptyset$.
- For α_{n+1} we ask if there is an extension of α_n in S_n. If there is, set α_{n+1} to be the least one. If there is not then we let $\alpha_{n+1} = \alpha_n$.

The construction assures that A is generic and one can see that $c_A = \bigcup_n \alpha_n$ is a total function.

Proposition 3.3.3. *If A is generic then A is not a finite set.*

Proof. Assume that A is finite. There exists an upper bound n, such that $x \in A \Rightarrow x < n$. Let $S = \{\alpha \mid (\exists m > n)(\alpha(m) \simeq 1)\}$. S is c.e. Since A is generic then $(\exists\alpha \subseteq A)(\alpha \in S \vee (\forall\beta \supseteq \alpha)(\beta \notin S))$. Since A is finite $\alpha \notin S$. Then $(\forall\beta \supseteq \alpha)(\forall m > n)(\beta(m) \not\simeq 1)$, which is impossible. Hence A is infinite. □

Proposition 3.3.4. *If A is generic then every c.e. $V \subseteq A$ is finite.*

Proof. Let $S = \{\alpha \mid (\exists x)(\alpha(x) \simeq 0 \,\&\, x \in V)\}$. S is c.e. Since A is generic then $\exists\alpha \subseteq A$ such that $\alpha \in S \vee (\forall\beta \supseteq \alpha)(\beta \notin S)$. Clearly $\alpha \notin S$ $(V \subseteq A)$. Then $(\forall\beta \supseteq \alpha)(\forall x)(\beta(x) \simeq 0 \Rightarrow x \notin V)$. Let $n \geq |\alpha|$, then for every $\beta \supseteq \alpha$, with $|\beta| > n$ and $\beta(n) = 0$ we have $n \notin V$. Thus V is finite. □

As corollary we have that if A is generic since $A \subseteq A$ and A is infinite, then

Corollary 3.3.5. *If A is generic then A is not c.e.*

Proposition 3.3.6. *Let A be generic. If $V \leq_T A$ is c.e., then V is computable.*

Proof. We know $\overline{V} \leq_T V \leq_T A$, hence there is an a, such that $\overline{V} = dom(\{a\}^A)$. Let $S = \{\alpha \mid (\exists x \in V)(\downarrow \{a\}^\alpha(x))\}$. Since S is c.e. and A is generic there is $\alpha \subseteq A$, such that $\alpha \in S \vee (\forall\beta \supseteq \alpha)(\beta \notin S)$. If $\alpha \in S$, then $(\exists x \in V)(\downarrow \{a\}^A(x))$. Then $x \in \overline{V}$, a contradiction.

Then $(\forall\beta \supseteq \alpha)(\forall x \in V)(\neg \downarrow \{a\}^\beta(x))$. If $x \in \overline{V}$ then $\downarrow \{a\}^A(x)$. By the compactness of the computation there is $\beta \subseteq A$, $\downarrow \{a\}^\beta(x)$. We can suppose that $\beta \supseteq \alpha$. Hence

$$x \in \overline{V} \iff (\exists\beta \supseteq \alpha)(\downarrow \{a\}^\beta(x)),$$

i.e. $\overline{V}$ is c.e. But V is c.e., therefore V is computable. □

Definition 3.3.7. The set A *models* the formula $F_e(x)$:

$$A \models F_e(x) \iff \downarrow \{e\}^A(x) \iff x \in W_e^A.$$

Definition 3.3.8. The finite part α *forces* formula $F_e(x)$:

$$\alpha \Vdash F_e(x) \iff \downarrow \{e\}^\alpha(x).$$

Here are some properties of this relations.

1. $\alpha \subseteq A \,\&\, \alpha \Vdash F_e(x) \Rightarrow A \models F_e(x)$.
2. $\alpha \subseteq \beta \,\&\, \alpha \Vdash F_e(x) \Rightarrow \beta \Vdash F_e(x)$.
3. $A \models F_e(x) \Leftrightarrow (\exists \alpha \subseteq A)(\alpha \Vdash F_e(x))$.

Lemma 3.3.9. *The set* $\{(\alpha, e, x) \mid \alpha \Vdash F_e(x)\}$ *is c.e.*

Definition 3.3.10. $A \models \neg F_e(x) \iff A \not\models F_e(x) \iff \neg \downarrow \{e\}^A(x)$.

Definition 3.3.11. $\alpha \Vdash \neg F_e(x) \iff (\forall \beta \supseteq \alpha)(\beta \not\Vdash F_e(x))$.

Theorem 3.3.12. *Let A be a generic set. Then*

$$A \models \neg F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x)).$$

Proof. ($\Leftarrow$) Let $\alpha \subseteq A \,\&\, \alpha \Vdash \neg F_e(x)$. Suppose that $A \models F_e(x)$. Then $(\exists \beta \subseteq A)(\beta \Vdash F_e(x))$. Let $\gamma = \alpha \cup \beta$. Then $\gamma \supseteq \beta \Rightarrow \gamma \Vdash F_e(x)$, but $\gamma \supseteq \alpha, \alpha \Vdash \neg F_e(x) \Rightarrow \gamma \not\Vdash F_e(x)$ - a contradiction.

($\Rightarrow$) Let $A \models \neg F_e(x)$. We search for $\alpha \subseteq A$, $\alpha \Vdash \neg F_e(x)$, i.e. no extension of α could forces $F_e(x)$. But A is generic. Suppose that $(\forall \alpha \subseteq A)(\alpha \not\Vdash \neg F_e(x))$. Hence $(\forall \alpha \subseteq A)(\exists \beta \supseteq \alpha)(\beta \Vdash F_e(x))$. Set $S_{e,x} = \{\beta \mid \beta \Vdash F_e(x)\}$. $S_{e,x}$ is c.e. and dense in A, then there is $\alpha \subseteq A, \alpha \in S_{e,x}$, i.e. $\alpha \Vdash F_e(x)$. Then $A \models F_e(x)$, a contradiction. So $(\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x))$. ◻

Corollary 3.3.13 (Truth lemma)**.** *If A is generic, then*

$$A \models (\neg)F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash (\neg)F_e(x)).$$

Notice that $\{(\alpha, e, x) \mid \alpha \Vdash \neg F_e(x)\} \leq_T \emptyset'$.

Corollary 3.3.14. *For every generic A we have $A' \equiv_T A \oplus \emptyset'$.*

Proof.

1. A' is a upper bound of $\emptyset'$ and A. Hence $\emptyset' \oplus A \leq_T A'$.
2. $A' = K_A = \{x \mid x \in W_x^A\} \leq_{c.e.} A$. Then there is e, such that $x \in K_A \iff \downarrow \{e\}^A(x) \iff A \models F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash F_e(x))$. Thus $K_A \leq_{c.e.} A \oplus \emptyset'$. A is generic then $x \in \overline{K_A} \iff \neg \downarrow \{e\}^A(x) \iff A \not\models F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x))$. And $\overline{K_A} \leq_{c.e.} A \oplus \emptyset'$. Thus $K_A = A' \leq_T A \oplus \emptyset'$ by Post theorem.

◻

3.4 Jump Inversion Theorem

Theorem 3.4.1 (Fridberg's Jump Inversion Theorem). *Let $\emptyset' \leq_T B$. There exists a generic A, such that $A' \equiv_T B$.*

Proof. We will construct the set A by steps, so that $A \leq_T B$ and A will be generic. Then $A' \equiv_T A \oplus \emptyset' \Rightarrow A' \leq_T B$. For the other direction we will code B in $A \oplus \emptyset'$. On each step n we will define a finite part α_n of c_A.

Let $\alpha_0 = \emptyset$. Let α_n be constructed. We ask: "Is it true that: $(\exists \beta \supseteq \alpha_n)(\beta \in S_n)$?". Since the set $V = \{(\alpha, n) \mid (\exists \beta \supseteq \alpha)(\beta \in S_n)\}$ is c.e., then $V \leq_T K = \emptyset'$. If **yes**, set α_n^* will be the minimal such β, if **no**, then $\alpha_n^* = \alpha_n$. Thus assures that A is generic. Set $\alpha_{n+1} = \alpha_n^* * c_B(n)$.

1. $A \leq_T B$. Since $|\alpha_{n+1}| \geq n$, $n \in A \iff \alpha_{n+1}(n) = 1$. And $\alpha_{n+1} \leq_T B \oplus \emptyset' \leq_T B$.
2. A is generic, since α_n^* assures genericity with respect to S_n.
3. $B \leq_T A \oplus \emptyset'$. We have $k \in B \iff \alpha_{k+1}(|\alpha_k^*|) = 1$. We can construct B repeating the construction, changing $c_B(n)$ with $c_A(|\alpha_n^*|)$. So, using oracle A and $\emptyset'$ we have $B \leq_T A \oplus \emptyset'$.

Thus A is generic and $A' \equiv_T B$. □

Corollary 3.4.2. *There exists a generic $A \not\equiv_T \emptyset$ such that $A' \equiv_T \emptyset'$.*

Corollary 3.4.3. *There exists a generic A such that $\emptyset \lneq_T A \lneq_T A' \equiv_T \emptyset'$.*

A more strong result is obtained by Jockush and Shore: for every c.e. non-computable $V \subseteq \mathbb{N}$ there is a generic set A such that $A \leq_T V$.

4 Enumeration reducibility

In this section we will consider a positive reducibility between sets : enumeration reducibility. Intuitively a set A is enumeration reducible to the set B if we can computably enumerate the members of A from an enumeration of the members of B. To explain this notion more formally we start with the definition of enumeration operator.

4.1 Enumeration operator

Definition 4.1.1. An operator $\Gamma : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ is an *enumeration operator* (e-operator) if:

1. Γ is compact : $x \in \Gamma(A) \iff (\exists D \subseteq A)(x \in \Gamma(D) \,\&\, D$ - finite$)$ for all x,
2. Γ is effective : there is a computable function h, such that $\Gamma(W_a) = W_{h(a)}$.

Definition 4.1.2. A set A is *enumeration reducible* to B:

$$A \leq_e B \iff (\exists \Gamma \text{ - e-operator})(A = \Gamma(B)).$$

Proposition 4.1.3. $A \leq_e B, B \leq_e C \Rightarrow A \leq_e C$.

Proposition 4.1.4. Γ *is an e-operator* $\iff$ *there exists a c.e.* W*, such that:*

$$\Gamma(A) = \{x \mid (\exists v)(\langle x, v\rangle \in W \,\&\, D_v \subseteq A)\}$$

Proof. ($\Leftarrow$) Let $x \in \Gamma(A) \iff (\exists v)(\langle x, v\rangle \in W \,\&\, D_v \subseteq A)$.

1. (compact) $x \in \Gamma(A) \Rightarrow (\exists v)(\langle v, x\rangle \in W \,\&\, D_v \subseteq A) \Rightarrow x \in \Gamma(D)$.
2. (monotone) Let $A \subseteq B$. Then $x \in \Gamma(A) \Rightarrow (\exists v)(\langle v, x\rangle \in W \,\&\, D_v \subseteq A \subseteq B) \Rightarrow x \in \Gamma(B)$.
3. (effective) $x \in \Gamma(W_a) \iff (\exists v)(\langle v, x\rangle \in W \,\&\, D_v \subseteq W_a)$. Consider:

$$R = \{(a, x) \mid \underbrace{(\exists v)(\langle v, x\rangle \in W \,\&(\forall y \in D_v)(y \in W_a))}_{\text{c.e. condition}}\}.$$

Let $R = W_e$ and $h(a) = S^1_1(e, a)$.

$x \in \Gamma(W_a) \iff (a, x) \in R \iff x \in W_{h(a)}$.

($\Rightarrow$) Let Γ be compact and effective and h is computable function, such that $\Gamma(W_a) = W_{h(a)}$. Consider also a computable function $\lambda : D_v = W_{\lambda(v)}$. Then

$$\begin{aligned}
x \in \Gamma(A) &\iff (\exists D \text{ - finite})(D \subseteq A \,\&\, x \in \Gamma(D)) \\
&\iff (\exists v)(D_v \subseteq A \,\&\, x \in \Gamma(D_v)) \\
&\iff (\exists v)(D_v \subseteq A \,\&\, x \in \Gamma(W_{\lambda(v)})) \\
\text{(effective)} &\iff (\exists v)(D_v \subseteq A \,\&\, x \in W_{h(\lambda(v))}) \\
&\iff (\exists v)(D_v \subseteq A \,\&\langle x, v\rangle \in W), \\
&\text{where } W = \{\langle x, v\rangle \mid x \in W_{h(\lambda(v))}\}.
\end{aligned}$$

□

Since the e-operator Γ is completely determined by the c.e. set W from the last Proposition we will use the following notation $W(A) = \Gamma(A) = \{x \mid (\exists v)(\langle x, v\rangle \in W \,\&\, D_v \subseteq A)\}$. For the finite set $D = D_v$ and $x \in \mathbb{N}$ we will use the notation $\langle x, D\rangle$ instead of $\langle x, v\rangle$, i.e.
$W(A) = \{x \mid (\exists D)(\langle x, D\rangle \in W \,\&\, D \subseteq A)\}$.

Here are several examples which shows some basic properties of the enumeration reducibility.

1. $A \leq_e A$ via the c.e. set $W = \{\, \langle x, \{x\}\rangle \mid x \in \mathbb{N}\,\}$.
2. If A is c.e. then $A \leq_e B$ via the c.e. set $W = \{\, \langle x, \emptyset\rangle \mid x \in A\,\}$.
3. If f is computable function for $A \leq_m B$, i.e. $A = f^{-1}(B)$, then $A \leq_e B$ via the c.e. set $W = \{\, \langle x, \{f(x)\}\rangle \mid x \in \mathbb{N}\,\}$.

Denote by $\langle\varphi\rangle$ the graph of the partial function φ, i.e $\langle\varphi\rangle = \{\langle x, y\rangle \mid \varphi(x) \simeq y\}$.
Let φ and ψ are partial functions.

Definition 4.1.5. $\varphi \leq_e \psi \iff \langle\varphi\rangle \leq_e \langle\psi\rangle$.

If $A \subseteq \mathbb{N}$ we will write also $A \leq_e \varphi$ and $\varphi \leq_e A$ instead of $A \leq_e \langle\varphi\rangle$ and $\langle\varphi\rangle \leq_e A$.

Proposition 4.1.6. $\varphi \leq_T \psi \Rightarrow \varphi \leq_e \psi$.

Proof. Let $\varphi \leq_T \psi$ and $\varphi = \{e\}^\psi$. We are looking for a c.e. set W such that

$$\langle x, y\rangle \in W(\langle\psi\rangle) = \langle\varphi\rangle \iff (\exists v)(\langle\langle x, y\rangle, v\rangle \in W \,\&\, D_v \subseteq \langle\psi\rangle),$$

Consider the c.e set $W = \{\langle\langle x, y\rangle, v\rangle \mid \{e\}^{\theta_v}(x) \simeq y\}$ where $\theta_v(x) \simeq \mu y[\langle x, y\rangle \in D_v]$.

$$\begin{aligned}\langle x, y\rangle \in W(\langle\psi\rangle) &\iff (\exists v)(\{e\}^{\theta_v}(x) \simeq y \,\&\, D_v \subseteq \langle\psi\rangle)\\ &\iff \{e\}^\psi(x) \simeq y \iff \varphi(x) \simeq y\\ &\iff \langle x, y\rangle \in \langle\varphi\rangle .\end{aligned}$$

Thus $\varphi \leq_e \psi$. □

The enumeration reducibility is weaker than Turing reducibility in the following sense. We will prove that $c_K \leq_e \chi_{\overline{K}}$, but $c_K \not\leq_T \chi_{\overline{K}}$.

For $c_K \leq_e \chi_{\overline{K}}$ consider the c.e. set:

$$\begin{aligned}W =&\{\langle\langle x, 0\rangle, v\rangle \mid x \in \mathbb{N} \,\&\, D_v = \{\langle x, 1\rangle\}\}\cup\\ &\{\langle\langle x, 1\rangle, v\rangle \mid x \in K \,\&\, D_v = \emptyset\}.\end{aligned}$$

Then W defines an e-operator and $W(\chi_{\overline{K}}) = c_K$.

Suppose that $c_K \leq_T \chi_{\overline{K}}$. Then there is an e, such that $\{e\}^{\chi_{\overline{K}}} = c_K$. Note that c_K is a total function. But $\downarrow \chi_{\overline{K}}(x) \simeq y \iff y = 1$. We could change in the program e all oracle questions $O(n)$ by $Z(n), S(n)$, i.e. instead of asking the oracle we write 1. Then we can compute c_K by this new URM program (without using the oracle) which is a contradiction, since K is not computable.

It turns out that if the function ψ is total then $\varphi \leq_e \psi$ and $\varphi \leq_T \psi$ are equivalent.

Definition 4.1.7. Let $A \subseteq \mathbb{N}$, $g : \mathbb{N} \longrightarrow \mathbb{N}$. The function g *uniformizes* the set A, if $\langle g\rangle \subseteq A$ and for every x $(\exists y)(\langle x, y\rangle \in A) \Rightarrow \downarrow g(x)$.

Proposition 4.1.8. *If ψ is total and $\varphi \leq_e \psi$, then $\varphi \leq_T \psi$.*

Proof. $\varphi \leq_e \psi \Rightarrow (\exists W)(W(\langle\psi\rangle) = \langle\varphi\rangle)$. Since W is c.e., then there is a computable function f, which enumerates it, i.e. $f(i) = \langle\langle x_i, y_i\rangle, v_i\rangle$. Consider the function g defined as:

$$g(x) \simeq \begin{cases} y_i & \text{, if } x = x_i, (\forall \langle u, v\rangle \in D_{v_i})(\psi(u) \simeq v);\\ \neg\downarrow & \text{, otherwise.}\end{cases}$$

It is clear that $g \leq_T \psi$, thus there is an e, such that $\{e\}^\psi = \varphi$ and e does not depend of ψ. Then $\Delta(\psi) = \{e\}^\psi$ uniformizes $W(\langle\psi\rangle) = \langle\varphi\rangle$. So $\Delta(\psi) = \varphi$, since φ is a function. □

4.2 Enumeration degrees

Definition 4.2.1. $A \equiv_e B \iff A \leq_e B \,\&\, B \leq_e A$.

The relation $\equiv_e$ is an equivalence relation, and the equivalence classes we call enumeration degrees.

Definition 4.2.2. *Enumeration degree* of A is the equivalence class of A with respect to the relation $\equiv_e$:

$$d_e(A) = \{B \mid B \equiv_e A\}.$$

Definition 4.2.3. Define the order between the e-degrees $d_e(A) \leq d_e(B) \iff A \leq_e B$

Denote by D_e the set of all enumeration degrees. The structure $(D_e, \leq)$ is a partially ordered set. The operation $\oplus$ gives the least upper bound of two e-degrees.

Proposition 4.2.4. *$d_e(A \oplus B)$ is the least upper bound of $d_e(A)$ and $d_e(B)$.*

Proof.

1. $x \in A \iff 2x \in A \oplus B$ then $A \leq_m A \oplus B \Rightarrow A \leq_T A \oplus B \Rightarrow A \leq_e A \oplus B$. And $x \in B \iff 2x+1 \in A \oplus B$ then $B \leq_e A \oplus B$, i.e. $A \oplus B$ is an upper bound of A and B.

2. Let $A \leq_e C, B \leq_e C$, i.e. $A = W_1(C), B = W_2(C)$. Denote by $W = \{\langle x, v\rangle \mid (\exists v_1)(\exists v_2)(\langle x, v_1\rangle \in W_1 \,\&\langle x, v_2\rangle \in W_2 \,\&\, D_v = D_{v_1} \oplus D_{v_2})\}$. W is c.e. and $W(C) = A \oplus B$, i.e. $C \leq_e A \oplus B$.

□

Denote by $\mathbf{0}_e = \{W \mid W \text{ is c.e.}\}$, $\mathbf{0}_e \leq \mathbf{a}$ for an arbitrary $\mathbf{a}$. The structure $D_e = (D_e, \mathbf{0}_e, \oplus, \leq)$ is an upper semi-lattice.

Denote by $A^+ = A \oplus \overline{A}$. Then A is total, if $A \equiv_e A^+$.

Proposition 4.2.5. *1. $A^{++} \equiv_e A^+$, i.e.. A^+ is total.*
2. $A^+ \equiv_e \langle c_A\rangle$.
3. A is total $\iff A \equiv_e \langle c_A\rangle$.
4. If f is a total function, then $\langle f\rangle$ is a total set via the c.e. set

$$W = \{\, \langle\langle x, y\rangle, \{\langle x, z\rangle\}\rangle \mid y \neq z \,\}\,.$$

Every computable set is total. But K is not total since $\overline{K} \not\leq_e K$. For example $A \oplus \overline{A}$ is total for any A and if f is a total function then $\langle f\rangle$ is total.

It turns out that we can express the c.e. reducibility and the Turing reducibility in terms of the enumeration reducibility. By Proposition 4.1.8 we have:

Corollary 4.2.6. *$A \leq_{c.e.} B \iff \langle \chi_A\rangle \leq_T \langle c_B\rangle \iff A \leq_e B \oplus \overline{B}$.*
$A \leq_T B \iff \langle c_A\rangle \leq_T \langle c_B\rangle \iff \langle c_A\rangle \leq_e \langle c_B\rangle \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

Definition 4.2.7. $\mathbf{a} \in D_e$ is *total*, if there is a total $A \in \mathbf{a}$.

Notice that $\mathbf{0}_e$ is a total e-degree, but $K \in \mathbf{0}_e$ is not total.

The total degrees in D_e form an upper semi-lattice isomorphic to D_T. Denote by $Tot = \{\mathbf{a} \mid \mathbf{a} \text{ is total e-degree}\}$.

Definition 4.2.8 (Rogers, Myhill)**.** The mapping $\kappa : D_T \to D_e$ is defined as $\kappa(d_T(A)) = d_e(A^+)$ for any set A.

The mapping κ is an isomorphic embedding of D_T into D_e.

- $A \equiv_T B \iff A^+ \equiv_e B^+$ (correctness);
- range(κ) = Tot. Indeed if $A \in \mathbf{a} \in Tot$, then since $A \equiv_e A^+$ we have $A^+ \in a \Rightarrow \kappa(d_T(A)) = \mathbf{a}$ and thus $\mathbf{a} \in$range(κ). But if $\mathbf{a} \in$ range(κ), then $\mathbf{a} = d_e(A^+) \Rightarrow \mathbf{a} \in Tot$.
- Let $\kappa(d_T(A)) = \kappa(d_T(B)) \Rightarrow A^+ \equiv_e B^+ \Rightarrow A \equiv_T B \Rightarrow d_T(A) = d_T(B)$. (injective)
- $A \leq_T B \Rightarrow \kappa(d_T(A)) = A^+ \leq_e B^+ = \kappa(d_T(B))$. (isomorphic embedding)

So, κ is an isomorphism between D_T in the total degrees in D_e. To see that it is strong embedding of D_T into D_e: range(κ) $= Tot \subsetneq D_e$, we have to show that there are non-total degrees.

4.3 Quasi-minimal degree

In the proof we will use forcing.

Definition 4.3.1. Modeling and forcing relations.

$$A \models_e F_a(x,y) \iff \langle x,y\rangle \in W_a(A),$$
$$\alpha \Vdash_e F_a(x,y) \iff \langle x,y\rangle \in W_a(\alpha^+), \text{ where } \alpha^+ = \{x \mid \alpha(x) \simeq 1\}.$$

We have the following properties of these relations:
1. $\alpha \subseteq A \,\&\, \alpha \Vdash_e F_a(x,y) \Rightarrow A \models_e F_a(x,y)$ (monotonicity).
2. $\alpha \subseteq \beta \,\&\, \alpha \Vdash_e F_a(x,y) \Rightarrow \beta \Vdash_e F_a(x,y)$, since $\alpha^+ \subseteq \beta^+$.
3.$A \models_e F_a(x,y) \Rightarrow (\exists \alpha \subseteq A)(\alpha \Vdash_e F_a(x,y))$ (compactness).

Proposition 4.3.2. *Let A be a generic set and $\varphi \leq_e A$. There exists a computable function ψ, such that $\varphi \subseteq \psi$.*

Proof. Let A be a generic set and $\varphi \leq_e A$, $\langle\varphi\rangle = W_a(A)$. Then $\langle x,y\rangle \in \langle\varphi\rangle \iff A \models_e F_a(x,y)$. Consider the c.e. set (the relation $\Vdash_e$ is c.e.):
$S = \{\alpha \mid (\exists x)(\exists y_1)(\exists y_2)(\alpha \Vdash_e F_a(x,y_1) \,\&\, \alpha \Vdash_e F_a(x,y_2) \,\&\, y_1 \neq y_2)\}$.

There is $\alpha \subseteq A$, $\alpha \in S$ or $(\forall \beta \supseteq \alpha)(\beta \notin S)$. But $\alpha \notin S$. Let $\psi(x) \simeq y \iff (\exists \beta \supseteq \alpha)(\beta \Vdash_e F_a(x,y))$. ψ is a function and ψ is computable since $\langle\psi\rangle$ is c.e. Moreover if $\varphi(x) \simeq y \Rightarrow A \models_e F_a(x,y) \Rightarrow (\exists \beta \supseteq \alpha)(\beta \Vdash_e F_a(x,y)) \Rightarrow \psi(x) \simeq y$.

Thus $\varphi \subseteq \psi$. ∎

Corollary 4.3.3. *For every generic set A, the e-degree $d_e(A)$ is non-total.*

Corollary 4.3.4. *If A is generic, X total and $X \leq_e A$, then $X \leq_e \emptyset$.*

Definition 4.3.5. A is *quasi-minimal* if $A \not\leq_e \emptyset$ and if $X \leq_e A$ and X is a total set then $X \leq_e \emptyset$.

From the previous Corollary:

Proposition 4.3.6. *Each generic set is quasi-minimal.*

4.4 Selman's theorem

We will describe the technique of the regular enumerations in order to prove some more results as Selman;s theorem and Minimal pair theorem.

Definition 4.4.1. Let $B \subseteq \mathbb{N}$. The total function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a *regular enumeration* of B, if $f(2\mathbb{N}+1) = B$.

In other words we code the elements of the set B on the odd position of f. If $B = \emptyset$, we consider enumerations of $\mathbb{N} = \overline{\emptyset} \equiv_e \emptyset$.

If f is a regular enumeration of B then $\chi_B(x) \simeq c_1(\mu n[x = f(2n+1)])$, where $c_1 = \lambda x.1$, So $\chi_B \leq_T f$ and $B \leq_e f$.

Definition 4.4.2. *B-regular finite part* is a function $\tau : [0; 2q+1] \longrightarrow \mathbb{N}$, such that $2x+1 \in dom(\tau) \Rightarrow \tau(2x+1) \in B$.

If τ is a B-regular finite part, then there exists a regular enumeration f of B such that $f \supseteq \tau$.

Definition 4.4.3. The modeling and forcing relation for regular enumerations of B and B-regular finite parts.

$$f \models F_e(x) \iff x \in W_e(\langle f \rangle),$$
$$\tau \Vdash F_e(x) \iff x \in W_e(\langle \tau \rangle).$$

The set $S_\tau = \{\langle e, x\rangle \mid \tau \Vdash F_e(x)\}$ is c.e.

We have the usual properties of monotonicity and compactness:

1. $\tau \subseteq f \,\&\, \tau \Vdash F_e(x) \Rightarrow f \models F_e(x)$.
2. $\tau \subseteq \rho \,\&\, \tau \Vdash F_e(x) \Rightarrow \rho \Vdash F_e(x)$.
3. $f \models F_e(x) \Rightarrow (\exists \tau \subseteq f)(\tau \Vdash F_e(x))$.

Proposition 4.4.4. *Let $A \not\leq_e B$. There exists a regular enumeration f of B, such that $A \not\leq_e f$.*

Proof. We construct a sequence of B-regular finite parts $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_q \subseteq \ldots$

Let $\tau_0(0) = 0, \tau_0(1) = z_0 \in B$. If τ_q is constructed:

1. $q = 2e$. Let $z_0 = \mu z[z \in B \,\&\, z \notin \tau_q(2\mathbb{N}+1)]$. Set $\tau_{q+1} = \tau_q * 0 * z_0$.

2. $q = 2e + 1$.

$$C = \{x \mid (\exists \rho \supseteq \tau_q)(\rho \text{ is a B-regular finite part} \,\&\, \rho(|\tau_q|) = x \,\&\, \rho \Vdash F_e(|\tau_q|))\}.$$

Since $C \leq_e B$, then $C \neq A$. There are two cases:

2 (a). $(\exists x)(x \in C \,\&\, x \notin A)$. Then τ_{q+1} is the minimal ρ satisfying C.

2 (b). $(\exists x)(x \notin C \,\&\, x \in A)$. Then $\tau_{q+1} = \tau_q * x * z_0$ for some $z_0 \in B$.

Let $f = \bigcup_q \tau_q$. It is clear that f is a regular enumeration of B.

Suppose that $A \leq_e f$, i.e. $A = W_e(\langle f \rangle)$. Then $f^{-1}(A) = \{x \mid f(x) \in A\} \leq_e f$ and there is e, such that $n \in f^{-1}(A) \iff f \models F_e(n)$.

Consider the step $q = 2e + 1$. Let $n = |\tau_q| = 2q + 2$.

Case1. $n \in f^{-1}(A) \Rightarrow f(n) \in A \Rightarrow (\exists \rho \supseteq \tau_q)(\rho \Vdash F_e(n) \,\&\, \rho(n) = f(n))$. Then $f(n) \in C \cap A$, a contradiction.

Case 2. $n \notin f^{-1}(A) \Rightarrow f(n) \notin A \Rightarrow (\forall \rho \supseteq \tau_q)(\rho(n) = f(n) \Rightarrow \rho \nVdash F_e(n))$, then $f(n) \notin C$ - a contradiction.

So $A \not\leq_e f$. ◻

Theorem 4.4.5 (Selman)**.** *[15] $A \leq_e B \iff (\forall X$ - total$)(B \leq_e X \Rightarrow A \leq_e X)$.*

Proof. ($\Rightarrow$) From the transitivity of $\leq_e$.

($\Leftarrow$) Suppose that $A \not\leq_e B$. Then by Proposition 4.4.4 there is a B-regular enumeration f, such that $A \not\leq_e \langle f \rangle$. But $\langle f \rangle$ is total and $B \leq_e \langle f \rangle$, then $A \leq_e \langle f \rangle$ - a contradiction. ◻

Corollary 4.4.6. *If $a, b \in D_e$, then*

$$a \leq b \iff (\forall x \text{ - } total)(b \leq x \Rightarrow a \leq x).$$

The last Corollary shows that the set Tot of all total degrees in D_e is a base of the automorphisms in D_e, i.e. each automorphism $\varkappa : D_e \to D_e$, for which $\varkappa(\mathbf{a}) = \mathbf{a}$ for any $\mathbf{a} \in Tot$ is the identity on D_e.

4.5 Minimal pair theorem

Definition 4.5.1. The sets F and G form a *minimal pair for B*, if

1. $B \lneq_e F, B \lneq_e G$;
2. $A \lneq_e F, A \lneq_e G \Rightarrow A \leq_e B$,

i.e. B is the greatest lower bound for F and G.

Definition 4.5.2. We call f a *generic regular enumeration* of B, if f is a regular enumeration of B and for every set of B-regular finite parts and $S \leq_e B$ it holds $(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S))$.

Proposition 4.5.3. *Let $B \subseteq \mathbb{N}$, and $\{A_n\}$ be a sequence of sets, such that $(\forall n)(A_n \not\leq_e B)$. Then there exists a generic regular enumeration f of B, such that $(\forall n)(A_n \not\leq_e f)$.*

Proof. The proof is similar to Proposition 4.4.4.

Denote by $R_B = \{\tau \mid \tau \text{ is B-regular finite part}\}$. We construct a monotone increasing sequence of B-regular finite parts $\tau_q : [0; 2q+1] \longrightarrow \mathbb{N}$.

Let $\tau_0(0) = \tau_0(1) = b_0 \in B$. Suppose that we have constructed τ_q.

1. $q = 3e$. Set $\tau_{q+1} = \tau_q * 0 * b$, where b is the first non-enumerated element of B.

2. $q = 3e+1$. We assure the genericity of f. Consider $S_e = W_e(B) \cap R_B$. If there is $\tau \supseteq \tau_q$ and $\tau \in S_e$, let τ_{q+1} be the least one. If not $\tau_{q+1} = \tau_q$.

3. $q = 3e+2$. Let $e = \langle n, k\rangle$. We will assure that $f^{-1}(A_n) \neq W_k(\langle f\rangle)$. From here $f^{-1}(A_n) \not\leq_e f$ and $A_n \not\leq_e f$. Define

$$n_q = |\tau_q|$$
$$C_q = \{x \mid (\exists \tau \supseteq \tau_q)(\underbrace{\tau \text{ is } B\text{-regular finite part}}_{\leq_e B} \,\&\, \underbrace{\tau(n_q) \simeq x}_{\text{c.e.}} \,\&\, \underbrace{\tau \Vdash F_k(n_q)}_{\text{c.e.}}\}.$$

Then $C_q \leq_e B$ and thus $C_q \neq A_n$.

3.(a) $(\exists x)(x \in C_q \,\&\, x \notin A_n)$. We get the minimal such x and set τ_{q+1} to be the minimal such τ.

3.(b) $(\exists x)(x \notin C_q \,\&\, x \in A_n)$. Then set $\tau_{q+1} = \tau_q * x * b$, for some $b \in B$.

Finally we define f as follows: $f(n) \simeq x \iff (\exists q)(\tau_q(n) \simeq x)$. By the construction f is a generic regular enumeration of B.

We will show that $f^{-1}(A_n) \not\leq_e f$.

Suppose that $f^{-1}(A_n) \equiv_e W_k(\langle f\rangle)$ for some n and k. Consider the step $q = 3\langle n, k\rangle + 2$. We know $f(n_q) \simeq x$ is a witness for $A_n \neq C_q$.

1. $x \in C_q \,\&\, x \notin A_n$. Then

$$f \models F_k(n_q) \Rightarrow n_q \in f^{-1}(A_n) \Rightarrow f(n_q) = x \in A_n \text{ - a contradiction.}$$

2. $x \notin C_q \,\&\, x \in A_n$. Then

$$n_q \in f^{-1}(A_n) \Rightarrow (\exists \tau \supseteq \tau_q)(\tau(n_q) \simeq x \,\&\, \tau \Vdash F_k(n_q)) \Rightarrow x \in C_q \text{ - a contradiction.}$$

Thus $f^{-1}(A_n) \not\leq_e f \Rightarrow A_n \not\leq_e f$. □

Lemma 4.5.4. *If f is a generic regular enumeration of B, then $f \not\leq_e B$.*

Proof. Suppose that $f \leq_e B$. Consider

$$S = \{\tau \text{ - } B\text{-regular finite part} \mid (\exists x)(\downarrow \tau(x) \,\&\, \tau(x) \not\simeq f(x))\}.$$

$S \leq_e B \oplus \langle f\rangle$, but $\langle f\rangle \leq_e B \Rightarrow S \leq_e B$. By genericity of f we have

$$(\exists \tau \subseteq f)(\underbrace{\tau \in S}_{\tau \not\subseteq f} \vee \underbrace{(\forall \rho \supseteq \tau)(\rho \notin S)}_{\text{not true } f \not\supseteq \rho \supseteq \tau}).$$

In both cases we have a contradiction. □

Theorem 4.5.5 (Minimal pair theorem)**.** *For any $B \subseteq \mathbb{N}$ there is a minimal pair F and G for B.*

Proof. Let f be an arbitrary generic regular enumeration of B. Let $\{A_n\}_n$ be a sequence of those sets which are enumeration reducible to f and which are not enumeration reducible to B. By Proposition 4.5.3 we can construct a generic regular enumeration g of B, such that $(\forall n)(A_n \not\leq_e g)$. Set $F = \langle f \rangle, G = \langle g \rangle$. By Lemma 4.5.4 since $\langle f \rangle, \langle g \rangle$ are the graphs of generic regular enumerations of B we have that $B \lneq_e F, B \lneq_e G$. Suppose that $A \leq_e F, A \leq_e G$. Then $A \notin \{A_n\}$, otherwise $A \not\leq_e G$. Since $A \notin \{A_n\}_n$, then $A \leq_e B$. □

Definition 4.5.6. (Cooper, McEvoy) For any set $A \subseteq \mathbb{N}$ denote by $E_A = \{\langle i, x \rangle | x \in \Psi_i(A)\}$. The set $J_e(A) = E_A^+$ is called *the enumeration jump of A*.

The enumeration jump J_e is monotone and agrees with the Turing jump $J_T(A) = K_A$ in the following sense:

Theorem 4.5.7. *For any $A \subseteq \mathbb{N}$, $J_T(A)^+ \equiv_e J_e(A^+)$.*

Corollary 4.5.8. *Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding κ.*

5 Degree Spectra and Co-spectra

We will illustrate some results of the Computable Structure Theory based on the enumeration and Turing reducibilities. We start with the notion of degree spectrum of a structure, which in some sense is a measure of the complexity of a countable structure.

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a denumerable structure. *An enumeration of* $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Definition 5.0.9. (Richter) *The Turing Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an one to one enumeration of } \mathfrak{A})\}.$$

The notion of a degree spectrum of a countable structure is introduced by Richter [14] and further studied by Ash, Downey, Jockush and Knight in [10, 1, 6]. If **a** is the least element of $DS_T(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$.

5.1 Enumeration Degree Spectra

We will follow [17] to represent a notion of degree spectra of structure based on the enumeration reducibilities.

Definition 5.1.1. *The Enumeration Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$.

Proposition 5.1.2. *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one to one enumeration* f *of* $\mathfrak{A}$*.*

This Proposition shows that if we are interested of the least element of the degree spectrum it is not important that we consider all enumerations of the structure not only one to one. The benefit of considering arbitrary enumerations is that in this way we ensure that the degree spectrum is closed upwards with respect to the total e-degrees.

Proposition 5.1.3. *If* $\mathbf{a} \in DS(\mathfrak{A})$*,* $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$*.*

Definition 5.1.4. The structure $\mathfrak{A}$ is called *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition 5.1.5. *If* $\mathfrak{A}$ *is a total structure then* $DS(\mathfrak{A}) = \kappa(DS_T(\mathfrak{A}))$*.*

Given a structure $\mathfrak{A} = (\mathbb{N}, R_1, \dots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, R_1^c, \dots, R_k^c)$.

Proposition 5.1.6. *The following properties hold:*

1. $\kappa(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$*.*
2. *If the structure* $\mathfrak{A}$ *is total then* $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$*.*

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. And $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.

We are interested to answer to the following question: Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ such that $DS(\mathfrak{A}) = DS(\mathfrak{B})$?

Definition 5.1.7. Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Example 5.1.8. *Fix* $\mathbf{a} \in \mathcal{D}_e$ *and set* $\mathcal{A}_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{a} \leq_e \mathbf{b}\}$*. Then* $co(\mathcal{A}_{\mathbf{a}}) = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{b} \leq_e \mathbf{a}\}$*.*

Definition 5.1.9. The set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$. is called *the co-spectrum* of $\mathfrak{A}$. If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

Definition 5.1.10. A set A of natural numbers is *admissible in* $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $A \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ iff $\mathbf{a} = d_e(A)$ for some admissible in $\mathfrak{A}$ set A.

We will give a characterization of the admissible in $\mathfrak{A}$ sets in terms of the structure. Thus we shall obtain some information about the elements of $CS(\mathfrak{A})$.

We will use the following forcing relation. For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in W_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

As usual an enumeration f is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ such that $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

Definition 5.1.11. A set A of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist finite part δ and natural number e such that

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Theorem 5.1.12. *Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathfrak{A})$. Then the following conditions are equivalent:*

1. *A is admissible in $\mathfrak{A}$.*
2. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$.*
3. *A is forcing definable.*

If the structure $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true. For example, consider the linear ordering $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$. It is easy to see by a direct analysis of the forcing definable on $\mathfrak{A}$ sets that the co-degree of A is $\mathbf{0}_e$. The first results about degrees of structures are obtained by Richter [14]. Richter showed an example of a linear ordering which has no degree. Let $\mathfrak{A} = (\mathbb{N}; <)$ be a linear ordering. She proved that $DS(\mathfrak{A})$ contains a minimal pair of degrees and hence $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. Clearly $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. Therefore if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

Knight [10] defined the so called *jump degree* of a structure. The *jump spectrum* $DS_1(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all jumps of the elements of $DS(\mathfrak{A})$. The co-set of the jump spectrum $DS_1(\mathfrak{A})$ is denoted by $CS_1(\mathfrak{A})$. The least elements of $DS_1(\mathfrak{A})$ and $CS_1(\mathfrak{A})$ are called a jump degree and a co-jump degree respectively.

Knight proved that for a linear ordering $\mathfrak{A}$, $CS_1(\mathfrak{A})$ consists of all Σ^0_2 sets. The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$. So if a linear ordering has a jump degree it should be $\mathbf{0}'_e$. In the last section we will present more results about degrees and co-degrees of sructures

There are examples of structures with more sophisticating properties. Independently Slaman [16] and Wehner [20] proved that:

Example 5.1.13. *(Slaman 1998, Wehner 1998) There exists an $\mathfrak{A}$ such that*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

Obviously if a structure A has a co-degree, then $CS(\mathfrak{A})$ is a principal ideal. Building on results of Coles, Downey and Slaman [4] we shall show that every principle ideal of enumeration degrees can be represented as $CS(G)$ from some subgroup G of the additive group of the rational numbers $Q = (Q; +, =, \neq)$. Let G be a nontrivial torsion free Abelian group of rank 1, i.e. G is a subgroup of Q. Let $a \neq 0 \in G$. For every prime number p let $h_p(a)$ be the greatest k such that $(\exists x \in G)(p^k.x = a)$, and $h_p(a) = \infty$ if $p^k | a$ in G for all k. Let $\{p_i\}_i$ be the sequence of the prime numbers. Consider

$$S_a = \{\langle i, j\rangle \mid j \leq h_{p_i}(a)\}.$$

For $a, b \neq 0 \in G$, it is easy to see that $S_a \equiv_e S_b$.

Set $\mathbf{s}_G = d_e(S_a)$. Then $DS(G) = \{\mathbf{b} : \mathbf{b}$ is total and $\mathbf{s}_G \leq_e \mathbf{b}\}$. The co-degree of G is $\mathbf{s}_G$. The group G has a degree iff $\mathbf{s}_G$ is total. $\mathbf{s}'_G$ is the jump degree of G.

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, such that $\mathbf{s}_G = \mathbf{d}$. Hence every principal ideal of enumeration degrees is $CS(G)$ for some G.

Similar results on algebraic fields are obtained by W. Calvert, V. Harizanov and A. Shlapentokh (2007) [3] and A. Frolov, I. Kalimullin and R. Miller (2009) [7].

In [17] the representation of an arbitrary countable ideal I of enumeration degrees as a co-spectrum of a structure is obtained. Without loss of generality we may assume that there exists a sequence $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k, \ldots$ of elements of the ideal I such that $a \in I \iff (\exists k)(a \leq \mathbf{b}_k)$. For every k fix a set $B_k \in \mathbf{b}_k$. Set $\mathfrak{A} = (\mathbb{N}; f; \sigma, =, \neq)$, where

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \;\&\; i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \ldots, d_e(B_n), \ldots)$

Definition 5.1.14. Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are in $\mathcal{A}$.

There are some general properties of upwards closed sets of enumeration degrees. Let $\mathcal{A}$ be an upwards closed set of degrees. First property is an analougue of Selman's theorem [15] for enumeration degrees.

Proposition 5.1.15. *(Selman's Theorem) Let $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \;\&\; \mathbf{a}$ is total$\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_t)$.*

The elements of an upwards closed set $\mathcal{A}$ with arbitrary high jumps determine completely the co-set of $\mathcal{A}$.

Proposition 5.1.16. *Let $\mathbf{b}$ be an arbitrary enumeration degree and $n > 0$. Set $\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \;\&\; \mathbf{b} \leq_e \mathbf{a}^{(n)}\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n})$.*

We will consider some specific properties of the degree spectra which are not true for an arbitrary upwards closed set of enumeration degrees.

Theorem 5.1.17. *Let* $\mathfrak{A}$ *be a structure,* $1 \leq n$ *and* $\mathbf{c} \in DS_n(\mathfrak{A})$*. Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

The Theorem shows that the elements of the degree spectrum $DS(\mathfrak{A})$ with low jumps also determine its co-set $CS(\mathfrak{A})$. Here is an example of an upwards closed set for which the last Theorem is not true.

Consider two sets A and B of natural numbers such that $B \not\leq_e A$ and $A \not\leq_e B'$. It is enough to take an arbitrary non c.e. set and construct and construct a set A as a B' generic and $B \not\leq A$.

Let $\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}$. Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \&\ \mathbf{a}' = d_e(B)'\}$. Clearly if $\mathbf{a} \geq d_e(A)$, then $\mathbf{a} \notin \mathcal{A}$. Then $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$. $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

5.2 The Minimal pair theorem for Degree spectra

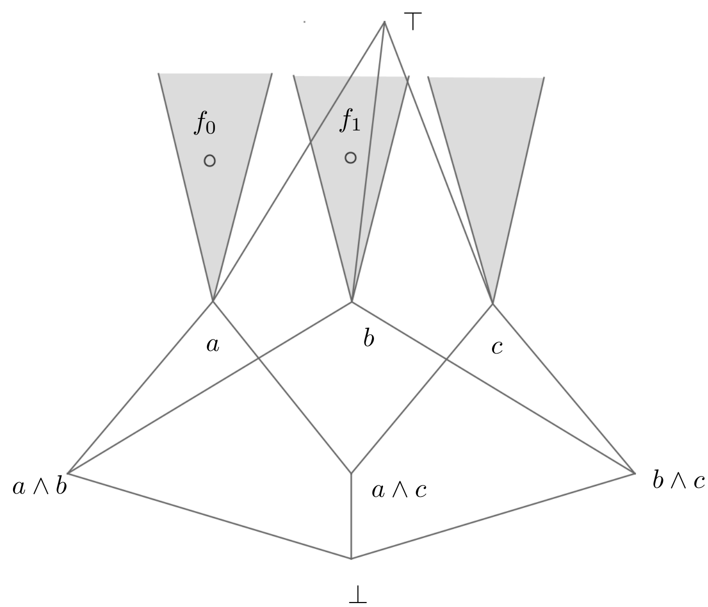
An analog of the Minimal pair theorem for the enumeration degrees is the following result:

Theorem 5.2.1. *Let* $\mathbf{c} \in DS(\mathfrak{A})$*. There exist* $\mathbf{f}$,$\mathbf{g} \in DS(\mathfrak{A})$ *such that* $\mathbf{f}$,$\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}''$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ there exists a structure $\mathfrak{A}_{\mathbf{b}}$ s. t. $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. Hence

Corollary 5.2.2 (Rozinas)**.** *For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which form a minimal pair over* $\mathbf{b}$.

Not every upwards closed set of enumeration degrees has a minimal pair. Indeed consider the finite lattice L consisting of the elements $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{c}, \mathbf{b} \wedge \mathbf{c}, \top, \bot$ such that $\top$ and $\bot$ are the greatest and the least element of L. Since every finite lattice can be embedded in the semi-lattice of the Turing degrees, the lattice Lcan be embedded in $(D_T, \leq)$ and hence it can be embedded in $(D_e, \leq)$. So we may assume that L is a substructure of $(D_e, \leq)$. Let $\mathcal{A} = \{\mathbf{d} \in D_e : (\mathbf{d} \geq \mathbf{a}) \vee (\mathbf{d} \geq \mathbf{b}) \vee (\mathbf{d} \geq \mathbf{c})\}$. Clearly $\mathcal{A}$ is an upwards closed set of enumeration degrees. Assume that there exist $\mathbf{f_0}, \mathbf{f_1} \in \mathcal{A}$ such that $co(\{\mathbf{f_0}, \mathbf{f_1}\}) = co(\mathcal{A})$. Let $\mathbf{x_0}, \mathbf{x_1} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be such that $\mathbf{f_0} \geq \mathbf{x_0}$ and $\mathbf{f_1} \geq \mathbf{x_1}$. Let $\mathbf{x_2} = \min\{\mathbf{x_0}, \mathbf{x_1}\}$. Clearly $\mathbf{x_2} \in co(\{\mathbf{f_0}, \mathbf{f_1}\})$ but $\mathbf{x_2} \notin co(\mathcal{A})$. A contradiction.



5.3 The Quasi-minimal degree

Now we present the third property of $DS(\mathfrak{A})$ which shows the existence of enumeration degrees which are quasi minimal with respect to $CS(A)$.

Definition 5.3.1. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

1. $\mathbf{q} \notin co(\mathcal{A})$.
2. If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
3. If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Theorem 5.3.2. *If $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.*

Theorem 5.3.3. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary 5.3.4 (Slaman and Sorbi)**.** *Let I be a countable ideal of enumeration degrees. There exist an enumeration degree $\mathbf{q}$ such that*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

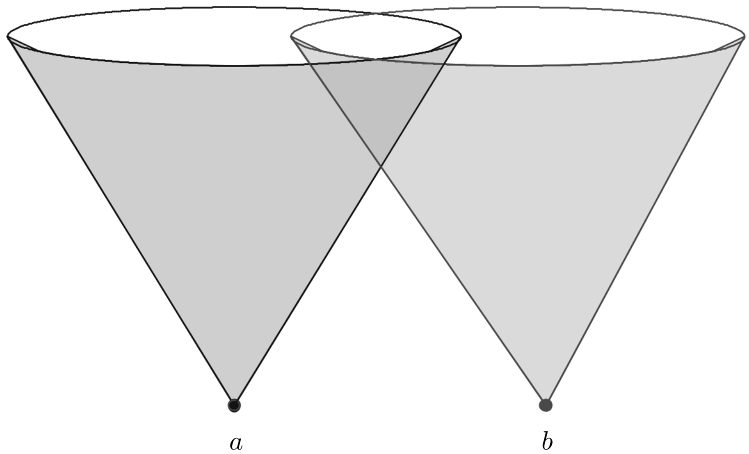
Definition 5.3.5. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem 5.3.6. *Let $\mathcal{A}$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.*

Corollary 5.3.7. *A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.*

Now we can construct an upwards closed set $\mathcal{A}$ of degrees which does not possess a quasi-minimal degree. Indeed let $\mathbf{a}$ and $\mathbf{b}$ be two incomparable total degrees. Let $\mathcal{A} = \{\mathbf{c} : \mathbf{c} \geq \mathbf{a} \vee \mathbf{c} \geq \mathbf{b}\}$. Clearly $\mathcal{A}$ has a countable base of total degrees, but it has not a least element. So, $\mathcal{A}$ has no quasi-minimal degree.



5.4 Jump spectra

Definition 5.4.1. The nth jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$ then $\mathbf{a}$ is called n-th jump degree of $\mathfrak{A}$.

Here we consider the relationships between the spectra and the jump spectra. Since the degree spectra are upward closed sets of e-degrees we have the following property.

Proposition 5.4.2. *For every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subseteq DS(\mathfrak{A})$.*

Our first result is that every jump spectrum is also a spectrum of a structure, i.e. we show that for every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$. The structure $\mathfrak{B}$ is constructed in two stages. First, we define the least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ which we call *Moschovakis' extension* of $\mathfrak{A}$. Roughly speaking $\mathfrak{A}^*$ is an extension of $\mathfrak{A}$ with additional coding machinery. Using this coding machinery we define the set $K_{\mathfrak{A}}$ which is an analogue of Kleene's set K. Finally we set $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$.

Let $\bar{0} \notin \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function such that none of the elements of $\mathbb{N}_0$ is a pair and N^* be the least set containing $\mathbb{N}_0$ and closed under $\langle ., . \rangle$.

Definition 5.4.3. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle.,.\rangle}, G_L, G_R, =).$$

Proposition 5.4.4. $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x\rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$. Set $\mathfrak{A}^*_K = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$. The structure $\mathfrak{A}^*_K$ is total. And $DS_1(\mathfrak{A}) = DS(\mathfrak{A}^*_K)$.

Theorem 5.4.5. *[18] For every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$.*

Montalban (2009)[13] presented a different approach adding to the structure a complete set of computable Π^c_1 relations and the received structure he called the jump of the structure. It turns out that the both approaches lead to a structure whose degree spectrum is the jump spectrum of the initial structure.

5.5 The Jump Inversion Theorem

The main result in this subsection sounds like a Jump inversion theorem. Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

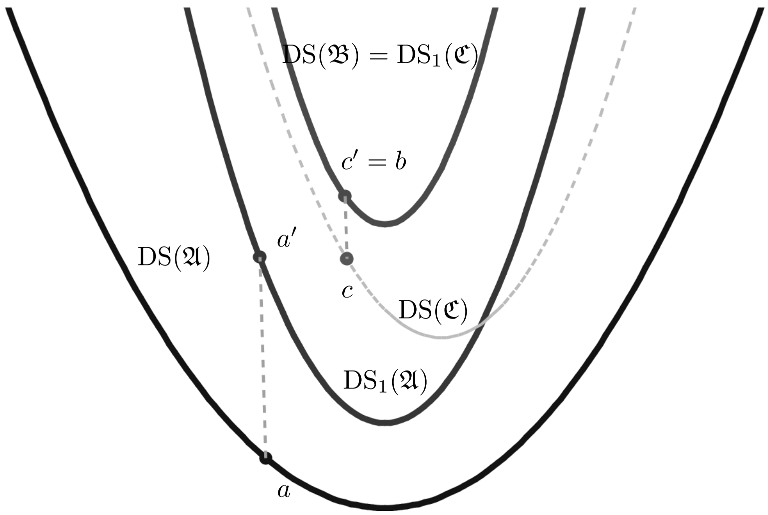
Theorem 5.5.1. *[18] There exists a structure $\mathfrak{C}$ such that $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

The structure $\mathfrak{C}$ is constructed as a Markers extension [12] of $\mathfrak{A}$, an idea influenced by the results of Goncharov and Khoussainov [8] .

Corollary 5.5.2. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ such that $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary 5.5.3. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}'$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C}')$.*

This Corollary gives a positive answer of the question above: If the degree spectrum of a structure consists of total degrees greater than or equal to $\mathbf{0}'$, then there is a total structure with the same degree spectrum.



We can generalize the Jump inversion theorem by induction using the results from Theorem 5.4.5 and Theorem 5.5.1.

Theorem 5.5.4. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary 5.5.5. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Stukachev (2009)[19] proved a similar result for Σ reducibility using Marker's extensions.

5.6 Applications

Recall that a nth jump degree is called the minimal element (if it exists) of the nth jump spectra. Notice that if a structure $\mathfrak{A}$ possesses a nth jump degree then it possesses $(n+k)$th jump degrees for all k. The definition of the nth jump degree can be naturally generalized for all recursive ordinals α. In [6] Downey and Knight proved with a fairly complicated construction that for every recursive ordinal α there exists a linear order $\mathfrak{A}$ such that $\mathfrak{A}$ has αth jump degree equal to $\mathbf{0}^\alpha$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathfrak{A}$.

As an application of Theorem 5.5.4 we will present a construction of a total structure $\mathfrak{C}$ such that $\mathfrak{C}$ has a $(n+1)$-st jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.

Suppose that we have a structure $\mathfrak{B}$ satisfying the following conditions:
1. $DS(\mathfrak{B})$ has no least element.
2. $\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.
3. All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.

Let $\mathfrak{A} = (N; =)$ be a structure such that $DS(\mathfrak{A})$ is equal to the set of all Turing degrees. Clearly $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. By Theorem 5.5.4, there exists a structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$. Therefore $\mathfrak{C}$ does not have a nth jump degree and hence it has no kth jump degree for $k \leq n$. On the other hand $DS_{n+1}(\mathfrak{C}) = DS_1(\mathfrak{B})$ and hence the $(n+1)$th jump degree of $\mathfrak{C}$ is $\mathbf{0}^{(n+1)}$.

Now we could provide an example of a structure satisfying the conditions 1. 3. Consider a set B satisfying the following conditions:

(a) B is quasi-minimal above $\mathbf{0}^{(n)}$.

(b) $B' \equiv_e \mathbf{0}^{(n+1)}$.

Let G be a subgroup of the additive group of the rationales such that $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$. So the structure G satisfies the conditions 1.- 3.

Our second application is a generalization of results of Slaman [16] and Wehner [20]. They give an example of a structure with degree spectrum consisting of all noncomputable Turing degrees.

The relativized result is the following: Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} | \mathbf{0}^{(n)} <_e \mathbf{a}\}$.

It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$. We use the Wehner's construction using a special family of sets. First we relativize this construction. Let $B \subseteq \mathbb{N}$. There is a family $\mathcal{F}$ of sets, which has no c.e. in B enumeration, and for every set $X >_T B$ there is a enumeration of the family $\mathcal{F}$, computable in X. Following an idea of Kalimullin [9] we consider the following family of sets $\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F$ - finite set, $F \neq W_n^B\}$. We construct the structure $\mathfrak{B}$, such that $DS(\mathfrak{B}) = \{\mathbf{x} \mid \mathbf{x} >_T \mathbf{b}\}$, using the family $\mathcal{F}$ in the same way as is done in [20]. Let $\mathfrak{A} = (N; =)$. It is clear that $\mathbf{b} \in DS_n(\mathfrak{A})$ for each $\mathbf{x} \geq \mathbf{0}^{(n)}$. Thus $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. By the Jump inversion Theorem 5.5.4 there exists a structure $\mathfrak{C}$, such that $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.

References

[1] Ash, C. J., Jockush, C., Knight, J. F. : Jumps of orderings. Trans. Amer. Math. Soc. **319** (1990) 573–599

[2] Cooper S. B. : *Computability Theory*, Chapman Hall/CRC Mathematical series (2003).

[3] Calvert, W., Harizanov V., Shlapentokh A. : Turing degrees of isomorphism types of algebraic objects. Journal of the London Math. Soc. **73** (2007) 273-286.

[4] Coles, R., Downey, R. and Slaman, T. :Every set has a least jump enumeration. Journal London Math. Soc, **62** (2000) 641–649

[5] Cutland, N. : *An introduction to recursive function theory*, Cambridge University Press (1980)

[6] Downey, R. G., Knight, J. F. : Orderings with αth jump degree $\mathbf{0}^{(\alpha)}$. Proc. Amer. Math. Soc. **114** (1992) 545–552

[7] Frolov, A. Kalimullin, I. Sh. Miller R. : Spectra of Algebraic Fields and Subfields. In Proceedings of CiE'2009, (2009) 232-241

[8] Goncharov, S., Khoussainov, B. : Complexity of categorical theories with computable models. Algebra and Logic, **43**, No. 6 (2004) 365–373

[9] Kallimulin, I. : Some notes on degree spectra of the structures. LNCS **4497** (2007) 389–397

[10] Knight, J. F. : Degrees coded in jumps of orderings. J. Symbolic Logic **51** (1986) 1034–1042.

[11] McEvoy, K. : Jumps of quasi-minimal enumeration degrees. J. Symbolic Logic, **50** (1985) 839–848

[12] Marker, D. : Non Σ_n-axiomatizable almost strongly minimal theories. J. Symbolic Logic **54** No. 3,(1989) 921–927

[13] Montalban, A. : Notes on the jump of a structure. Mathematical Theory and Computational Practice, (2009), 372-378.

[14] Richter, L. J. : Degrees of structures. J. Symbolic Logic **46** (1981) 723–731.

[15] Selman, A.L. : Arithmetical reducibilities I. Z. Math. Logik Grundlag. Math **17** (1971) 335–350

[16] Slaman T. A. : Relative to any nonrecursive set. Proc. Amer. Math. Soc. **126** (1998) 2117–2122

[17] Soskov, I. N. : Degree spectra and co-spectra of structures. Ann. Univ. Sofia **96** (2004) 45–68

[18] Soskova, A., Soskov, I.: A Jump Inversion Theorem for the Degree Spectra, Journal of Logic and Computation, **19**, (2009) 199–215

[19] Stukachev, A. : A jump inversion theorem for the semilattices of Sigma-degrees. Siberian Advances in Mathematics, **20**, N1 (2010), 68 - 74.

[20] Wehner St. : Enumerations, Countable Structures and Turing Degrees. Proc. of Amer. Math. Soc. **126** (1998) 2131–2139