ENUMERATION DEGREE SPECTRA OF ABSTRACT STRUCTURES

IVAN N. SOSKOV, ALEXNDRA A. SOSKOVA

The degree spectrum $DS(\mathfrak{A})$ of a countable structure $\mathfrak{A}$ we define to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. In this paper we consider the connections between degree spectra and their co-spectra. We present variants of Selman's theorem, the minimal pair theorem and quasi-minimal degree theorem for degree spectra. A structure $\mathfrak{A}$ is called total if all presentations of $\mathfrak{A}$ are total sets. For every total structure $\mathfrak{A}$ the set $DS(\mathfrak{A})$ contains only total degrees. We prove that if $DS(\mathfrak{A})$ consists of total degrees above $\mathbf{0}'$, then there exists a total structure $\mathfrak{B}$ such that $DS(\mathfrak{B}) = DS(\mathfrak{A})$. We prove a generalized Jump inversion theorem for degree spectra. As an application we receive structures with interesting degree spectra.

1. Preliminaries

Definition 1.1. *(Friedberg and Rogers, 1959) We say that* $\Psi : 2^\omega \to 2^\omega$ *is an* enumeration operator *iff for some c.e. set* W_i *and for each* $B \subseteq \omega$

$$\Psi(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \ \& D \subseteq \mathfrak{B}]\}$$

Here $\{W_i\}_{i\in\omega}, \{D_i\}_{i\in\omega}$ are the standard listings of computably enumerable sets and the finite sets of numbers.

For any sets A and B define A is *enumeration reducible to* B, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator Ψ. Let $A^+ = A \oplus (\omega \setminus A)$. The connection with the Turing reducibility is shown by $A \leq_T B$ iff $A^+ \leq_e B^+$.

Let $E_A = \{\langle i, x\rangle | x \in \Psi_i(A)\}$. The set $J_e(A) = E_A^+$ is called the enumeration jump of A [1, 3]. The enumeration jump J_e is monotone and agrees with the Turing jump J_T in the following sense: $J_T(A)^+ \equiv_e J_e(A^+)$.

Let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$ and $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

A set A is called *total* iff $A \equiv_e A^+$. The Rogers embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ is defined by $\iota(d_T(A)) = d_e(A^+)$. The enumeration degrees in the range of ι are called *total*.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding ι.

2. Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a denumerable structure. Enumeration of $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

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Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \dots, x_a\rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Definition 2.1. *(Richter* [5]*) The* Turing Degree Spectrum of $\mathfrak{A}$ *is the set*

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \textit{ is an one to one enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_T(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$

Definition 2.2. [7] The e-Degree Spectrum of $\mathfrak{A}$ *is the set*

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \textit{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$

The e-degree spectrum is closed upwards: if $\mathbf{a} \in DS(\mathfrak{A})$,$\mathbf{b}$ is a total e-degree and $\mathbf{a} \leq_e \mathbf{b}$ then $\mathbf{b} \in DS(\mathfrak{A})$.

Definition 2.3. *The structure* $\mathfrak{A}$ *is called* total *if for every enumeration* f *of* $\mathfrak{A}$ *the set* $f^{-1}(\mathfrak{A})$ *is total.*

If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.

Given a structure $\mathfrak{A} = (\mathbb{N}, R_1, \dots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, R_1^c, \dots, R_k^c)$. The following are true:

(1) $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
(2) If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Example. Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.

The question here is: if $DS(\mathfrak{A})$ consists of total degrees do there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$? We will give a positive answer when all elements of $DS(\mathfrak{A})$ are total above $\mathbf{0}'$.

Definition 2.4. *Let $\mathcal{A}$ be a nonempty set of enumeration degrees the* co-set of $\mathcal{A}$ *is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely*

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Example. Fix $\mathbf{a} \in \mathcal{D}_e$ and set $\mathcal{A}_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{a} \leq_e \mathbf{b}\}$. Then $co(\mathcal{A}_{\mathbf{a}}) = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{b} \leq_e \mathbf{a}\}$.

Definition 2.5. *The co-spectrum of the structure* $\mathfrak{A}$ *is called the set* $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.

If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$. If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

We will give a normal form of the elements of the co-spectrum $CS(\mathfrak{A})$. A set A of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $A \leq_e f^{-1}(\mathfrak{A})$. Clearly $\mathbf{a} \in CS(\mathfrak{A})$ iff $\mathbf{a} = d_e(A)$ for some admissible set A.

Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part.* For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in \Psi_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Given an enumeration f of $\mathfrak{A}$, $e, x \in \mathbb{N}$, set

$$f \models F_e(x) \iff x \in \Psi_e(f^{-1}(\mathfrak{A})).$$

An enumeration f is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

A set A of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist finite part δ and natural number e s.t.

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Theorem 2.1. *Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *A is admissible.*
2. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$.*
3. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e B'$.*
4. *A is forcing definable.*

Example. (Richter 1981, [5]) Let $\mathfrak{A} = (\mathbb{N}; <)$ be a linear ordering. Then $DS(\mathfrak{A})$ contains a minimal pair of degrees and hence $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. Clearly $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. Therefore if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

Definition 2.6. *Let $n \geq 0$. The n-th jump spectrum of a structure $\mathfrak{A}$ is defined by $DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}$. Set $CS_n(\mathfrak{A}) = co(DS_n(\mathfrak{A}))$.*

Example. (Knight 1986, [2]) Consider again a linear ordering $\mathfrak{A}$. Then $CS_1(\mathfrak{A})$ consists of all Σ^0_2 sets. The co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.

Example. (Slaman 1998,Whener 1998) There exists an $\mathfrak{A}$ s.t.

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

Example. (based on Coles, Dawney, Slaman - 1998) Let G be a torsion free Abelian group of rank 1, i.e. G is a subgroup of Q. Let $a \neq 0 \in G$ and let p be a prime number.

Let $h_p(a)$ be the greatest k s.t. $(\exists x \in G)(p^k.x = a)$. Let

$$\chi(a) = (h_{p_0}(a), h_{p_1}(a), \dots) \text{ and}$$
$$S_a = \{\langle i, j\rangle : j \leq \text{the } i\text{-th member of } \chi(a)\}.$$

For $a, b \neq 0 \in G$, $S_a \equiv_e S_b$.

Set $\mathbf{s}_G = d_e(S_a)$. Then $DS(G) = \{\mathbf{b} : \mathbf{b} \text{ is total and } \mathbf{s}_G \leq_e \mathbf{b}\}$.

- The co-degree of G is $\mathbf{s}_G$.
- G has a degree iff $\mathbf{s}_G$ is total
- If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some G.

We can represent every coubntable non-principle countable ideal as co-spectra.

Example. Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

Definition 2.7. *Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is* upwards closed *if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.*

Let $\mathcal{A}$ be an upwards closed set of degrees. Note that if $\mathcal{B} \subseteq \mathcal{A}$, then $co(\mathcal{A}) \subseteq co(\mathcal{B})$. Selman [6] proved that if $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{a} \text{ is total}\}$ then $co(\mathcal{A}) = co(\mathcal{A}_t)$.

Propostion 2.1 (Selman's Theorem for Degree Spectra)**.** *Let $\mathbf{b}$ be an arbitrary enumeration degree and $n > 0$. Set $\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{b} \leq_e \mathbf{a}^{(n)}\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n})$.*

If $1 \leq n$ and $\mathbf{c} \in DS_n(\mathfrak{A})$ then $CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\})$.

Example. *(Upwards closed set for which the Theorem is not true)* Let $B \not\leq_e A$ and $A \not\leq_e B'$. Denote by $\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}$.

Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \&\ \mathbf{a}' = d_e(B)'\}$.

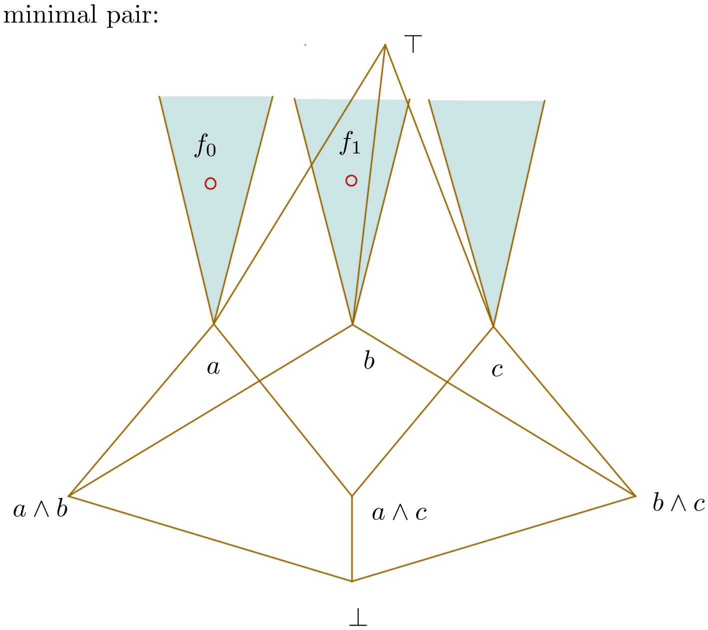
- $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.
- $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

Theorem 2.2 (The minimal pair theorem)**.** *Let $\mathbf{c} \in DS_2(\mathfrak{A})$. There exist $\mathbf{f}$,$\mathbf{g} \in DS(\mathfrak{A})$ s.t. $\mathbf{f}$,$\mathbf{g}$ are total, $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ and $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.*

Notice that for every enumeration degree $\mathbf{a}$ there exists a structure $\mathfrak{A}_{\mathbf{a}}$ s. t. $DS(\mathfrak{A}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{a} <_e \mathbf{x}\}$. As a corollary we receive

Corollary 2.1. *(Rozinas) For every $\mathbf{b} \in \mathcal{D}_e$ there exist total $\mathbf{f}, \mathbf{g}$ below $\mathbf{b}''$ which are a minimal pair over $\mathbf{b}$.*

The next example shows that not every upwards closed set of enumeration degrees has a minimal pair:



Definition 2.8. *Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:*

- $\mathbf{q} \notin co(\mathcal{A})$.
- *If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.*
- *If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.*

If $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.

Theorem 2.3. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary 2.2. *(Slaman and Sorbi) Let I be a countable ideal of enumeration degrees. There exist an enumeration degree $\mathbf{q}$ s.t.*

(1) *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
(2) *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if $(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a})$.

Theorem 2.4. *Let $\mathcal{A}$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.*

As a corollary we have that a total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.

If we consider the set of two incomparable degrees and the cones of all total degrees over them then this set is an example of a upwards closed set which is not a degree spectrum, since it has a countable base but it has no degree.

3. Jump spectra

Definition 3.1. *The n-th jump spectrum of a structure $\mathfrak{A}$ is the set*

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$ then $\mathbf{a}$ is called n-th jump degree of $\mathfrak{A}$. For every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subseteq DS(\mathfrak{A})$. It is not known if for every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subset DS(\mathfrak{A})$. Probably the answer is "no".

We will show that every jump spectrum is spectrum of a total structure. Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_n)$. Consider a new element $\bar{0} \notin \mathbb{N}$. Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle .,. \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and N^* be the least set containing $\mathbb{N}_0$ and closed under $\langle .,. \rangle$.

Definition 3.2. Moschovakis' extension [4] *of $\mathfrak{A}$ is the structure*

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \dots, R_n, \mathbb{N}_0, G_{\langle .,. \rangle}).$$

It is easy to see that $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$. And $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$.

Theorem 3.1. *The structure $\mathfrak{A}'$ is total and $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.*

We will present an analogue of the Jump Inversion Theorem for degree spectra. Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem 3.2. *There exists a structure* $\mathfrak{C}$ *s.t.* $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ *and* $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.

Corollary 3.1. [8] *Let* $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. *Then there exists a structure* $\mathfrak{C}$ *s.t.* $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ *and* $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.

Corollary 3.2. *Suppose that* $DS(\mathfrak{B})$ *consists of total degrees greater than or equal to* $\mathbf{0}'$. *Then there exists a total structure* $\mathfrak{C}'$ *such that* $DS(\mathfrak{B}) = DS(\mathfrak{C}')$.

Theorem 3.3. *Let* $n \geq 1$. *Suppose that* $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. *There exists a structure* $\mathfrak{C}$ *s.t.* $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.

Corollary 3.3. *Suppose that* $DS(\mathfrak{B})$ *consists of total degrees greater than or equal to* $\mathbf{0}^{(n)}$. *Then there exists a total structure* $\mathfrak{C}$ *s.t.* $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.

Example. Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

(1) $DS(\mathfrak{B})$ has not least element.
(2) $\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.
(3) All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.

Consider a set B satisfying: B is quasi-minimal above $\mathbf{0}^{(n)}$ and $B' \equiv_e \mathbf{0}^{(n+1)}$.

Let G be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a} \text{ and } \mathbf{a} \text{ is total}\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Example. Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} | \mathbf{0}^{(n)} <_e \mathbf{a}\}$.

It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.

This is done by Whener's construction using a special family of sets: There exists a family $\mathcal{F}$ of sets of natural number s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a recursive in X set U satisfying the equivalence:

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a,x) \in U\}).$$

But there is no c.e. in $\mathbf{0}^{(n)}$ such U.

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Faculty of Mathematics and Computer Science, Sofia University, 5 James Bourchier Blvd, 1164 Sofia, Bulgaria

E-mail address: `asoskova@fmi.uni-sofia.bg, soskov@fmi.uni-sofia.bg`