DEFINABILITY IN THE LOCAL STRUCTURE OF THE ENUMERATION DEGREES

MARIYA SOSKOVA AND HRISTO GANCHEV

ABSTRACT. We prove that the class of total enumeration degrees and the class of low enumeration degrees are first order definable in the local structure of the enumeration degrees.

1. Introduction

The main focus in degree theory, established as one of the core areas in Computability theory, is to understand a mathematical structure, which arises as a formal way of classifying the computational strength of an object. The most studied examples of such structures are that of the Turing degrees, $\mathcal{D}_T$, based on the notion of Turing reducibility, as well as its local substructures, of the Turing degrees reducible to the first jump of the least degree, $\mathcal{D}_T(\leq \mathbf{0}_T{}')$, and of the computably enumerable degrees, $\mathcal{R}$. In investigating such a mathematical structure among the main questions that we ask is: how complex is this structure. The complexity of a structure can be inspected from many different aspects: how rich is it algebraically; how complicated is its theory; what sets are definable in it; does it have nontrivial automorphisms. The question about definability, in particular, is interrelated to all of the other questions, and can be seen as a key to understanding the natural concepts that are approximated by the corresponding mathematical formalism. Research of the Turing degrees has been successful in providing a variety of results on definability. For the global theory of the Turing degrees, among the most notable results is the definability of the jump operator by Slaman and Shore [16]. The method used in the proof of this result, as well as many other definability results in $\mathcal{D}_T$, leads Slaman and Woodin to conjecture that every definable set in second order arithmetic is definable in $\mathcal{D}_T$. This is a consequence of their Biinterpretability conjecture, which is shown to be equivalent to the rigidity of $\mathcal{D}_T$ [17]. In the local theory Nies, Shore and Slaman [12] show a weakening of the biinterpretability conjecture for the computably enumerable degrees and the Δ^0_2 Turing, and obtain from it the first order definition of the jump classes H_n and L_{n+1} in $\mathcal{D}_T(\leq \mathbf{0}_T{}')$ and in $\mathcal{R}$, for every natural number n. One class of degrees which has managed to elude every attempt at definability in the local structures is that of the low_1 degrees, L_1, the degrees whose jump is the least possible degree, $\mathbf{0}_T{}'$.

1991 *Mathematics Subject Classification.* 03D30.

Key words and phrases. Enumeration reducibility, first order definability, total enumeration degrees, low enumeration degrees.

This research was supported by an BNSF Grant No. D002-258/18.12.08 and by a Marie Curie European Reintegration Grant No. 239193 within the 7th European Community Framework Programme.

Another approach for understanding a structure, often used in mathematics, is to place this structure in a richer context, a context which would reveal new hidden relationships. The most promising candidate for such a larger context is the structure of the enumeration degrees, introduced by Friedberg and Rogers [4]. This structure is induced by a weaker form of relative computability: a set A is enumeration reducible to a set B if every enumeration of the set B can be effectively transformed into an enumeration of the set A. The induced structure of the enumeration degrees, $\mathcal{D}_e$, is an upper-semilattice with jump operation and least element. The Turing degrees can be embedded in the enumeration degrees via the standard embedding ι which maps the Turing degree of a set A to the enumeration degree of $A \oplus \overline{A}$. This embedding preserves the order, the least upper bound and the jump operation. The range of ι is therefore a substructure of $\mathcal{D}_e$, which is isomorphic to $\mathcal{D}_T$. This structure will be denoted by $\mathcal{TOT}$ and its elements will be called the total enumeration degrees. An important question, which immediately arises in this context, first set by Rogers [13], is whether $\mathcal{TOT}$ is first order definable in $\mathcal{D}_e$. Rozinas [14] proves that every enumeration degree is the greatest lower bound of two total enumeration degrees, thus the total enumeration degrees are an automorphism base for $\mathcal{D}_e$. This gives further motivation for studying the issue of the definability of $\mathcal{TOT}$ in $\mathcal{D}_e$, as it would provide a strong link between the automorphism problem for the structures of the Turing degrees and the enumeration degrees. If $\mathcal{TOT}$ is definable in $\mathcal{D}_e$ then a nontrivial automorphism of $\mathcal{D}_e$ would yield a nontrivial automorphism of $\mathcal{D}_T$.

Definability in the enumeration degrees has had its successes as well. Kalimullin [10] has shown that the enumeration jump is definable in $\mathcal{D}_e$. McEvoy [11] has shown that the range of the jump operator in the enumeration degrees coincides with the class of total degrees above $\mathbf{0}_e{}'$, thus an immediate corollary from these two results is that the class $\mathcal{TOT} \cap \{\mathbf{a} \mid \mathbf{a} \geq \mathbf{0}_e{}'\}$ is first order definable in $\mathcal{D}_e$. The method used in Kalimullin's proof is significantly less complex than that used to prove the definability of the jump operator in the Turing degrees. The definition of the enumeration jump is closer to the much sought natural definition, see Shore [15], and is based on the first order definability of the notion of a $\mathcal{K}$-pair.

Definition 1. *Let A and B be sets of natural numbers. We say that $\{A, B\}$ is a* $\mathcal{K}$-pair *if there is a c.e. set W, such that*

$$A \times B \subseteq W \ \&\ \overline{A} \times \overline{B} \subseteq \overline{W}.$$

The jump operation gives rise to a local structure in the enumeration degrees, $\mathcal{G}_e$, consisting of all enumeration degrees that are reducible to the first jump, $\mathbf{0}_e{}'$, of the least degree, $\mathbf{0}_e$. Cooper [2] has shown, that the elements of $\mathcal{G}_e$ are exactly the degrees which contain Σ^0_2 sets, thus $\mathcal{G}_e$ is often referred to as the structure of the Σ^0_2 enumeration degrees. As ι preserves the jump operation, it follows that $\mathcal{TOT} \cap \mathcal{G}_e$ is a structure, which is isomorphic to $\mathcal{D}_T(\leq \mathbf{0}_T{}')$. In [5] we have shown that $\mathcal{K}$-pairs are first order definable in $\mathcal{G}_e$, providing the first step in the investigation of the definability theme for the local structure of the enumeration degrees. The local definition of $\mathcal{K}$-pairs unlocked numerous further results in the study of $\mathcal{G}_e$. For example in [5] we show that the classes of the upwards properly Σ^0_2 enumeration degrees and the downwards properly Σ^0_2 enumeration degrees are first order definable in $\mathcal{G}_e$ and in [7] we show that the first order theory of true arithmetic can be interpreted in $\mathcal{G}_e$, using coding methods based on $\mathcal{K}$-pairs.

In this article we give two more examples of classes of degrees with natural first order definitions in $\mathcal{G}_e$. The first one gives a positive answer to the local version of Rogers' question.

Theorem 1. *The set of total Σ^0_2 enumeration degrees is first order definable in $\mathcal{G}_e$.*

In view of the above discussed results by Kalimullin [10] a corollary of Theorem 1 is that the class of total degrees which are comparable with $\mathbf{0}_e{}'$ is first order definable in the global structure $\mathcal{D}_e$.

The second example supplies further evidence that studying the structure of the Turing degrees within the larger context of the enumeration degrees can provide us with more insight. Combined with Theorem 1 it gives the first instance of a local first order definition of an isomorphic copy of the low Turing degrees.

Theorem 2. *The set of low enumeration degrees is first order definable in $\mathcal{G}_e$.*

2. Preliminaries

We refer to Cooper [3] for a survey of basic results on the structure of the enumeration degrees and to Sorbi [18] for a survey of basic results on the local structure $\mathcal{G}_e$. We outline here basic definitions and properties of the enumeration degrees used in this article.

Definition 2. *A set A is* enumeration reducible *($\leq_e$) to a set B if there is a c.e. set Φ such that:*

$$A = \Phi(B) = \{n \mid \exists u(\langle n, u\rangle \in \Phi \ \&\ D_u \subseteq B)\},$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an enumeration operator*.*

A set A is *enumeration equivalent* ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree $d_e(A)$ of A.

The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq\rangle$ is the class of all enumeration degrees with relation $\leq$ defined by $d_e(A) \leq d_e(B)$ if and only if $A \leq_e B$. It has a least element $\mathbf{0}_e = d_e(\emptyset)$, the set of all c.e. sets.

We can define a least upper bound operation, by setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$. Here $A \oplus B = \{2a \mid a \in A\} \cup \{2b+1 | b \in B\}$.

The enumeration jump of a set A, denoted by $J_e(A)$ is defined by Cooper [2] as $L_A \oplus \overline{L_A}$, where $L_A = \{\ n|\ n \in \Phi_n(A)\}$, where $\{\Phi_n\}_{n<\omega}$ is an effective listing of all enumeration operators. The enumeration jump operator is defined by $d_e(A)' = d_e(J_e(A))$.

Definition 3. *A set A is called* total *if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is called* total *if it contains a total set. The collection of all total degrees is denoted by $\mathcal{TOT}$.*

As noted above, the structure $\mathcal{TOT}$ is an isomorphic copy of the Turing degrees. The map ι, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A})$$

is an embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$, which preserves the order, the least upper bound and the jump operation.

The local structure of the enumeration degrees, denoted by $\mathcal{G}_e$, is the substructure with domain, consisting of all enumeration degrees, which are reducible to $\mathbf{0}_e{}'$.

As noted above, the elements of $\mathcal{G}_e$ are the enumeration degrees which contain Σ^0_2 sets, or equivalently, which consist entirely of Σ^0_2 sets.

Definition 4. *A set A will be called low if $J_e(A) \equiv_e J_e(\emptyset)$. An enumeration degree $\mathbf{a} \in \mathcal{G}_e$ is low, if $\mathbf{a}' = \mathbf{0}_e{}'$.*

3. $\mathcal{K}$-pairs in $\mathcal{G}_e$

We start this section with two examples of pairs of sets of natural numbers, which form a $\mathcal{K}$-pair. Recall that by Definition 1 for sets of natural numbers A and B, $\{A, B\}$ is a $\mathcal{K}$-pair if there is a c.e. set W, such that:

$$A \times B \subseteq W \;\&\; \overline{A} \times \overline{B} \subseteq \overline{W}.$$

As a first example of a $\mathcal{K}$-pair, consider a c.e. set U and an arbitrary nonempty set of natural numbers A. Then U and A form a $\mathcal{K}$-pair via the c.e. set $U \times \mathbb{N}$. $\mathcal{K}$-pairs of this sort we will consider trivial and we will not be interested in them. A $\mathcal{K}$-pair $\{A, B\}$ is nontrivial if A and B are not c.e.

A more interesting example is given by Jockusch's semi-recursive sets, [9].

Definition 5 (Jockusch)**.** *We say that a set of natural numbers, A, is semi-recursive if there is a computable function $s_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that for any $x, y \in \mathbb{N}$, $s_A(x, y) \in \{x, y\}$ and whenever $\{x, y\} \cap A \neq \emptyset$, $s_A(x, y) \in A$. The function s_A is called a selector function for A.*

It is easy to see that if A is semi-recursive, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair. Indeed let s_A be the selector function for A and let

$$\overline{s_A}(n, m) = \begin{cases} n \text{ , if } s_A(n, m) = m \\ m \text{ , if } s_A(n, m) = n. \end{cases}$$

Now consider the c.e. set $W = \{(s_A(n, m), \overline{s_A}(n, m)) \mid n, m \in \mathbb{N}\}$ and notice that $A \times \overline{A} \subseteq W$ and $\overline{A} \times \overline{\overline{A}} = \overline{A} \times A \subseteq \overline{W}$. The following theorem by Jockusch [9] yields the existence of nontrivial $\mathcal{K}$-pairs.

Theorem 3 (Jockusch)**.** *Every nonzero Turing degree contains a semi-recursive set A, such that both A and $\overline{A}$ are not c.e.*

Kalimullin [10] has shown that the property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in the global structure, $\mathcal{D}_e$.

Theorem 4 (Kalimullin)**.** *A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if and only if*

$$\forall \mathbf{x} \in \mathcal{D}_e[\, \mathbf{x} = (\mathbf{x} \vee \mathbf{d}_e(A)) \wedge (\mathbf{x} \vee \mathbf{d}_e(B)) \,].$$

Thus we can lift the notion of a $\mathcal{K}$-pair to the enumeration degrees. A pair of enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ shall be called a $\mathcal{K}$-pair if every member of $\mathbf{a}$ forms a $\mathcal{K}$-pair with every member of $\mathbf{b}$.

The universal quantifier in the definition above makes it nontrivial to show that the class of $\mathcal{K}$-pairs is first order definable in the local structure $\mathcal{G}_e$. In [5] we show that this is nevertheless true.

Theorem 5. *There is a first order formula $\mathcal{L}K$, such that for any pair of Σ^0_2 sets A and B, $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{L}K(\mathbf{d}_e(A), \mathbf{d}_e(B))$.*

4. Definability via $\mathcal{K}$-pairs

The example of a nontrivial $\mathcal{K}$-pair given in the previous section, by a semi-recursive set and its complement, hints towards a strong connection between total enumeration degrees and $\mathcal{K}$-pairs. Let us investigate this connection further. To do this we will need the following three properties of $\mathcal{K}$-pairs:

Theorem 6 (Kalimullin[10]). *The the following assertions hold for Σ^0_2 sets A and B:*

1. *If A and B form a nontrivial $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$;*
2. *The enumeration degrees $d_e(A)$ and $d_e(B)$ are incomparable and quasiminimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.*
3. *The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.*

In view of the properties listed above, let us consider again the special case of a $\mathcal{K}$-pair given by a semi-recursive non c.e. set and its non-c.e. complement, say $\{A, \overline{A}\}$. This $\mathcal{K}$-pair can be considered as a maximal $\mathcal{K}$-pair, in the sense that for every $\mathcal{K}$-pair $\{C, D\}$, if $A \leq_e C$ and $\overline{A} \leq_e D$ then $A \equiv_e C$ and $\overline{A} \equiv_e D$. Indeed suppose there were a $\mathcal{K}$-pair $\{C, D\}$, such that, say, $A \leq_e C$ but $\overline{A} <_e D$. By the third property, as $A \leq_e C$ and $\{C, D\}$ is a $\mathcal{K}$-pair, A would also form a $\mathcal{K}$-pair with D. By the first property $D \leq_e \overline{A}$, contradicting the strong inequality $\overline{A} <_e D$. Let us generalize the notion of a maximal $\mathcal{K}$-pair.

Definition 6. *We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.*

Using the second property in Theorem 6, we can restate Jockusch's Theorem 3 as follows:

Corollary 1. *Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.*

To prove Theorem 1 we only need to show that, the opposite is true as well: the join of every maximal $\mathcal{K}$-pair of Σ^0_2 sets is enumeration equivalent to a total set. We prove something a little bit stronger. We prove that every nontrivial $\mathcal{K}$-pair $\{A, B\}$ can be extended to a maximal $\mathcal{K}$-pair of the form $\{C, \overline{C}\}$.

Theorem 7. *For every nontrivial Σ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.*

The proof of Theorem 7 is presented in the next section. To complete the proof of Theorem 1 consider the formula:

$$\begin{aligned}\mathcal{TOT}(\mathbf{x}) \iff \mathbf{x} = \mathbf{0}_e \ \vee\ \exists \mathbf{a} \exists \mathbf{b} [\mathcal{LK}(\mathbf{a}, \mathbf{b})\ \&\ \mathbf{x} = \mathbf{a} \vee \mathbf{b}\ \&\\ \forall \mathbf{c} \forall \mathbf{d} (\mathcal{LK}(\mathbf{c}, \mathbf{d})\ \&\ \mathbf{c} \geq \mathbf{a}\ \&\ \mathbf{d} \geq \mathbf{b} \rightarrow \mathbf{c} = \mathbf{a}\ \&\ \mathbf{d} = \mathbf{b})]\end{aligned}$$

A Σ^0_2 enumeration degree $\mathbf{x}$ is total if and only if $\mathcal{G}_e \models \mathcal{TOT}(\mathbf{x})$.

Now we turn to the local definability of the low enumeration degrees. For this we need two more ingredients. The first one is proved by Giorgi, Sorbi and Yang [8] and involves the notion of a downwards properly Σ^0_2 enumeration degree.

Definition 7. *A Σ^0_2 set A is called downwards properly Σ^0_2 if for every non c.e. set B, such that $B \leq_e A$, B is not Δ^0_2. A degree $\mathbf{a}$ is downwards properly Σ^0_2 if it contains a downwards properly Σ^0_2 set.*

In [5] we show that the class of downwards properly Σ^0_2 degrees is first order definable in $\mathcal{G}_e$. Let $\mathcal{DP}\Sigma^0_2$ denote the first order formula, such that $\mathbf{x}$ is downwards properly Σ^0_2 if and only if $\mathcal{G}_e \models \mathcal{DP}\Sigma^0_2(\mathbf{x})$.

Theorem 8 (Giorgi, Sorbi, Yang). *Every non-low total Σ^0_2 enumeration degree bounds a downwards properly Σ^0_2 enumeration degree.*

This result combined with the first order definability of the class of the total enumeration degrees and of the class of the downwards properly Σ^0_2 enumeration degrees already gives the first order definition of the low *total* enumeration degrees. To complete the prove we use a special case of a much more general jump inversion theorem, proved by Soskov [19]:

Theorem 9 (Soskov). *For every enumeration degree $\mathbf{x}$ there is a total enumeration degree $\mathbf{y}$, such that $\mathbf{x} < \mathbf{y}$ and $\mathbf{x}' = \mathbf{y}'$.*

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it. The low Σ^0_2 enumeration degrees are defined by the following formula:

$$\mathcal{LOW}(\mathbf{x}) \iff \exists \mathbf{a}[\, \mathbf{x} \leq \mathbf{a} \;\&\; \mathcal{TOT}(\mathbf{a}) \;\&\; \forall \mathbf{b} \leq \mathbf{a}(\neg \mathcal{DP}\Sigma^0_2(\mathbf{b}))].$$

5. Proof of Theorem 7

To prove Theorem 7 we will use the following dynamic characterization of $\mathcal{K}$-pairs, proved in [10].

Lemma 1 (Kalimullin). *A pair of non-c.e. Σ^0_2 sets forms a $\mathcal{K}$-pair if and only if there are Δ^0_2 approximations $\{A_i\}_{i<\omega}$ to A and $\{B_i\}_{i<\omega}$ to B, such that:*

$$\forall i(A_i \subseteq A \;\vee\; B_i \subseteq B).$$

Approximations to A and B with the property above will be called $\mathcal{K}$-approximations.

Fix a nontrivial Σ^0_2 $\mathcal{K}$-pair $\{A, B\}$ and let $\{A_i\}_{i<\omega}$ and $\{B_i\}_{i<\omega}$ be their respective Δ^0_2 $\mathcal{K}$-approximations. We shall build two Σ^0_2 sets C and D which shall satisfy the following requirements:

(R1) $A = \{x \mid \exists j[2\langle x, j\rangle \in C]\}$, $B = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D]\}$;
(R2) C and D are Δ^0_2;
(R3) $\overline{C} = D$;
(R4) $\{C, D\}$ is a $\mathcal{K}$-pair.

To ensure these conditions, we shall construct respective Σ^0_2 approximations $\{C_i\}_{i<\omega}$ and $\{D_i\}_{i<\omega}$, which will have the following properties:

(P1) $A_i = \{x \mid \exists j[2\langle x, j\rangle \in C_i]$ and $B_i = \{x \mid \exists j[2\langle x, j\rangle + 1 \in D_i]$. Assuming that the approximations are Δ^0_2 this will ensure that $A \supseteq \{x \mid \exists j[2\langle x, j\rangle \in C]\}$ and $B \supseteq \{x \mid \exists j[2\langle x, j\rangle + 1 \in D]\}$.

(P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$. This property will ensure that the constructed approximations are Δ^0_2, i.e. (R2) holds. Indeed, if we assume that for some $d \notin D$ the set $I(d) = \{i \mid d \in D_i\}$ is infinite, then $\{A_i\}_{i \in I(d)}$ is a c.e. approximation to A contradicting that A is not c.e. Furthermore together with (P1) it ensures the inclusions $A \subseteq \{x \mid \exists j[2\langle x, j\rangle \in C]\}$ and $B \subseteq \{x \mid \exists j[2\langle x, j\rangle + 1 \in D]\}$. The argument is similar: there are infinitely many stages i, such that $B_i \not\subseteq B$, otherwise B_i

would turn out c.e. So for every $x \in A$ we can find a stage i such that $x \in A_i$ and $B_i \nsubseteq B$. By (P1) there is a number j, such that $2\langle x, j\rangle \in C_i$. By (P2) $C_i \subseteq C$. The second inclusion is proved in a similar way.

(P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets. This will ensure (R3);

(P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$. This will ensure (R4).

Note that the property (P2) is a consequence of properties (P1) and (P4), so let us consider in more detail what the property (P4) is expressing. Suppose that $x \notin C$, but for some i, $x \in C_i$. Then i is a bad stage for C, i.e. $C_i \nsubseteq C$, and we must ensure that all the elements in D_i are ultimately enumerated in D. Thus from this point on the element x is connected to all elements in D_i, in the sense that we should not enumerate x in D_k at a further stage $k > i$ unless we also ensure that $D_i \subseteq D_k$. This suggests the following relations for every stage j:

$$r_j(x, y) \iff \exists i \leq j[x \in C_i \ \&\ y \in D_i].$$

The main property (MP) of the construction is as follows: for every stage j and every x and y, if $r_j(x, y)$, then

$$x \in D_j \Longrightarrow y \in D_j \quad \text{and} \quad y \in C_j \Longrightarrow x \in C_j.$$

Note that (MP) automatically ensures that the two constructed approximations have the $\mathcal{K}$-pair property. The construction must therefore ensure that (P1), (P3) and (MP) are true. The other properties are implied by these.

5.1. **Construction.** We introduce the following piece of notation: with $\mathbf{c}_i^a$ we shall denote the natural number $2\langle a, i\rangle$, and by $\mathbf{d}_i^b$ the natural number $2\langle b, i\rangle+1$. If $a \in A_i$ we shall say that $\mathbf{c}_i^a$ is a follower of a, and similarly if $b \in B_i$ we shall say that $\mathbf{d}_i^b$ is a follower of b. Note that by the properties of the construction we will have that $a \in A$ if and only if at least one of its followers is in C and $b \in B$ if and only if at least one of its followers is in D. During the construction each follower will have one of the following two states: *free* or *not free*. Intuitively a follower is free if it is not currently enumerated in either of the sets C or D. By $Free$ we denote the set of all followers that are currently free.

The construction will be carried out in stages. Every stage consists of two parts - *Extracting* and *Adding*. We shall describe the construction formally and supply a brief description of the intuition for every action. The main intuition of the construction is that followers $\mathbf{c}_j^a$ *want* to end up in the set D and followers $\mathbf{d}_j^b$ *want* to end up in the set C. A follower $\mathbf{c}_j^a$ remains in the set C ($\mathbf{d}_j^b$ remains in the set D) only if it is forced to do so by other followers to which it is connected.

Start of construction.

We set $C_0 = \{\mathbf{c}_a^0 \mid a \in A_0\}$ and $D_0 = \{\mathbf{d}_b^0 \mid b \in B_0\}$. At stage $i > 0$ we construct C_i and D_i by modifying C_{i-1} and D_{i-1} respectively as follows:

Initially we set $C_i = C_{i-1}$ and $D_i = D_{i-1}$.

Part 1: Extracting

(E1) For all $\mathbf{c}_j^a \in C_i$ such that $a \notin A_i$ we extract $\mathbf{c}_j^a$ from C_i.
For all $\mathbf{d}_j^b \in D_i$ such that $b \notin B_i$ we extract $\mathbf{d}_j^a$ from D_i.

Intuition: This action ensures that $\{a \mid \exists j[\mathbf{c}_j^a \in C_i]\} \subseteq A_i$ and that $\{b \mid \exists j[\mathbf{d}_j^b \in D_i]\} \subseteq B_i$.

(E2) For all $\mathbf{d}_j^b \in C_i$ such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \not\subseteq C_i$ we extract $\mathbf{d}_j^b$ from C_i.
For all $\mathbf{c}_j^a \in D_i$ such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \not\subseteq D_i$ we extract $\mathbf{c}_j^a$ from D_i.

Intuition: The follower $\mathbf{d}_j^b$ is only allowed to remain in C_i if all of the elements to which it has been connected at a previous stage, i.e. $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\}$ are still in C_i. These elements we can consider as *requested* in C_i by $\mathbf{d}_j^b$. If one of these requests cannot be fulfilled (due to the properties of A_i for example and rule (E1)), $\mathbf{d}_j^b$ must also be extracted from C_i. Similar reasoning is applied to followers $\mathbf{c}^a$ and their membership to D_i. These actions ensure that the main property of the construction is true.

(E3) For all $\mathbf{c}_j^a \in C_i$ such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \cap C_i = \emptyset$ we extract $\mathbf{c}_j^a$ from C_i.
For all $\mathbf{d}_j^b \in D_i$ such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \cap D_i = \emptyset$ we extract $\mathbf{d}_j^b$ from D_i.

Intuition: A follower $\mathbf{c}_j^a$ was forced into C_{i-1} because of a request by some $\mathbf{d}_k^b$, to which it is connected. However at this stage the follower that made this request is not any longer in C_i, (it was extracted under (E2) as one of its other requests was not fulfilled). In other words $\mathbf{c}_j^a$ is not requested any longer in C_i, so it is free to leave and attempt entering D_i.

All extracted elements become free.

Part 2 (Adding)

(A1) For all free $\mathbf{d}_j^b$ such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq C_i \cup Free$ and $\{a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq A_i$ we enumerate $\mathbf{d}_j^b$ and $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\}$ in C_i.
All enumerated elements become *not free*.
For all free $\mathbf{c}_j^a$ such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i \cup Free$ and $\{b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq B_i$ we enumerate $\mathbf{c}_j^a$ and $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ in D_i.
All enumerated elements become *not free*.

Intuition: This is the action that allows followers $\mathbf{d}_j^b$ to enter C_i and respectively $\mathbf{c}_j^a$ to enter D_i. This can be done only if all of their requests can be fulfilled at the same time. These requests must also not injure the actions of rule (E1).

(A2) For all $a \in A_i$ we enumerate $\mathbf{c}_i^a$ in C_i.
For all $b \in B_i$ we enumerate $\mathbf{d}_i^b$ in D_i.
All enumerated elements become *not free*.

Intuition: This action ensures that $\{a \mid \exists j[\mathbf{c}_j^a \in C_i]\} \supseteq A_i$ and that $\{b \mid \exists j[\mathbf{d}_j^b \in D_i]\} \supseteq B_i$ and together with (E1), property (P1).

(A3) For all $a, j \leq i$ such that $a \notin A_j$ we enumerate $\mathbf{c}_j^a \in D_i$.
For all $b, j \leq i$ such that $b \notin B_j$ we enumerate $\mathbf{d}_j^b \in C_i$.

Intuition: This action handles elements that are not followers. As our aim is to construct D as $\overline{C}$, these elements also need to be enumerated in one of the two constructed sets. Note that even elements are enumerated in D_i and odd elements are enumerated in C_i. At the following stage an even number $\mathbf{c}_j^a$, which was enumerated in D_i under this action, cannot be extracted under rules (E1) and (E2). Furthermore as $\mathbf{c}_j^a$ has never been enumerated into an approximating set to C, the set $\{\mathbf{d}_k^a \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ is empty, so it cannot be extracted under rule (E2). Thus this element remains in D_k at all further stages $k > i$. Similar reasoning is applied to odd numbers, enumerated in D_i under this action.

End of construction.

5.2. **Verification of the construction.** We prove that the described construction produces sets C and D, which have the properties listed as (P1)-(P4) and (MP). We start with the easiest property: (P1).

Proposition 1. *For every i, $A_i = \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i = \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$.*

Proof. The claims of the proposition follow directly from rules (E1), (A1) and (A2). Indeed (A2) guarantees the inclusion $\subseteq$, as $A_i = \{a \mid \mathbf{c}_i^a \in C_i\}$ and $B_i = \{b \mid \mathbf{d}_i^b \in D_i\}$. On the other hand (E1) and (A1) enforce that $A_i \supseteq \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i \supseteq \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$.

□

The following proposition is a direct consequence of the construction. We state it nevertheless for completeness.

Proposition 2. *For all i, $C_i \cap D_i = \emptyset$.*

Next we turn to the main property of the construction (MP). One particular case of it will be used frequently in the rest of the proof and we will state and prove it here separately.

Proposition 3. *If $\mathbf{c}_j^a$ is a follower and $\mathbf{c}_j^a \in D_i$ then $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$;*
If $\mathbf{d}_j^b$ is a follower and $\mathbf{d}_j^b \in C_i$ then $\{\mathbf{c}_k^a \mid r_i(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq C_i$.

Proof. We prove the first statement. The second statement is proved similarly. Let $\mathbf{c}_j^a$ be a follower, (i.e. $a \in A_j$), such that $\mathbf{c}_j^a \in D_i$. If $\mathbf{c}_j^a$ is enumerated in D_i at stage i under rule (A1) then by construction the set $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ is also enumerated in D_i. As no more elements are extracted from D_i after the execution of step $A1$, it follows that $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$.

The other possibility is that $\mathbf{c}_j^a \in D_{i-1}$ and during stage i, $\mathbf{c}_j^a$ is not extracted from D_i. But then the prerequisites of rule (E2) are not valid for $\mathbf{c}_j^a$ at stage i and hence before starting the execution of (E3) it is true that $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$. During the execution of (E3) it is the case that for every $\mathbf{d}_k^b \in \{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$, $\mathbf{c}_j^a \in \{\mathbf{c}_l^\alpha \mid r_{i-1}(\mathbf{c}_l^\alpha, \mathbf{d}_k^b)\} \cap D_i$. By ($E3$) this intersection must be empty in order to extract $\mathbf{d}_k^b$ from D_i, so none of the elements in $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ are extracted from D_i during the execution of (E3). Thus finally $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$.

□

We are now ready to prove the main property (MP).

Lemma 2 (Main Lemma)**.** *Let x and y be natural numbers, such that $r_i(x, y)$, for some natural number i. Then the following two conditions are true.*

(C1) $x \in D_i \Longrightarrow y \in D_i$.
(C2) $y \in C_i \Longrightarrow x \in C_i$.

Proof. The claim of the lemma is trivial when either x or y are not followers, as every such element is only enumerated once under (A3) in its corresponding set and is never extracted. For followers x and y we shall consider three different cases.

Case 1. $x = \mathbf{c}^a_j$ and $y = \mathbf{d}^b_k$. This is a direct consequence of Proposition 3.

Case 2. $x = \mathbf{c}^a_j$ and $y = \mathbf{c}^\alpha_l$ (or $x = \mathbf{d}^b_k$ and $y = \mathbf{d}^\beta_l$). Let s be the least natural number for which $r_s(x, y)$. We shall prove simultaneously claims (C1), (C2) and that

$$\{\mathbf{d}^b_k \mid r_i(y, \mathbf{d}^b_k)\} \subseteq \{\mathbf{d}^b_k \mid r_i(x, \mathbf{d}^b_k)\} \tag{1}$$

by induction on $i \geq s$. For $i = s$ claims (C1) and (C2) are trivially true, as by the definition of the relation r_s and the choice of s we have $x \in C_s$ and $y \in D_s$. For claim (1) suppose that $\mathbf{d}^b_k$ is such that $r_s(y, \mathbf{d}^b_k)$. Since $y \in D_s$, Proposition 3 implies that $\mathbf{d}^b_k \in D_s$ and hence from $x \in C_s$ we obtain $r_s(x, \mathbf{d}^b_k)$.

Now let $i > s$. In order to prove (C1) suppose that $x \in D_i$. Then according to Proposition 3, $\{\mathbf{d}^b_k \mid r_i(x, \mathbf{d}^b_k)\} \subseteq D_i$. Now using the induction hypothesis for (1) and that $\{\mathbf{d}^b_k \mid r_{i-1}(x, \mathbf{d}^b_k)\} \subseteq \{\mathbf{d}^b_k \mid r_i(x, \mathbf{d}^b_k)\}$ we obtain $\{\mathbf{d}^b_k \mid r_{i-1}(y, \mathbf{d}^b_k)\} \subseteq D_i$. As by Proposition 2 we have that $D_i \cap C_i = \emptyset$, it follows that at stage i when we reach step (E3), $\{\mathbf{d}^b_k \mid r_i(y, \mathbf{d}^b_k) \cap C_i = \emptyset$, which implies that $y \notin C_i$. This means that if y is not already in D_i, it is free during the execution of (A1) and we would enumerate it in D_i.

In order to prove (C2) suppose that $y \in C_i$. Then there is a $\mathbf{d}^b_k \in D_i$ such that $r_{i-1}(y, \mathbf{d}^b_k)$, since otherwise y would have been extracted under (E3). From the induction hypothesis for (1) we obtain that $r_{i-1}(x, \mathbf{d}^b_k)$ and hence $x \in C_i$ in by Proposition 3.

Finally let us prove (1). We consider two cases. First suppose that $y \notin C_i$. Then $\{\mathbf{d}^b_k \mid r_i(y, \mathbf{d}^b_k)\} = \{\mathbf{d}^b_k \mid r_{i-1}(y, \mathbf{d}^b_k)\}$. On the other hand $\{\mathbf{d}^b_k \mid r_{i-1}(x, \mathbf{d}^b_k)\} \subseteq \{\mathbf{d}^b_k \mid r_i(x, \mathbf{d}^b_k)\}$ and now the claim follows from the induction hypothesis. Secondly let $y \in C_i$. Then

$$\{\mathbf{d}^b_k \mid r_i(y, \mathbf{d}^b_k)\} = \{\mathbf{d}^b_k \mid r_{i-1}(y, \mathbf{d}^b_k)\} \cup \{\mathbf{d}^b_k \mid \mathbf{d}^b_k \in D_i\}.$$

On the other hand by (C2) we have $x \in C_i$ and hence

$$\{\mathbf{d}^b_k \mid r_i(x, \mathbf{d}^b_k)\} = \{\mathbf{d}^b_k \mid r_{i-1}(x, \mathbf{d}^b_k)\} \cup \{\mathbf{d}^b_k \mid \mathbf{d}^b_k \in D_i\}$$

and again the claim follows from the induction hypothesis.

Case 3. $x = \mathbf{d}^b_k$ and $y = \mathbf{c}^a_j$. Let s be again the least stage for which $r_s(x, y)$. In particular $x \in C_s$ and $y \in D_s$. We shall prove simultaneously (C1), (C2) and for all $i \geq s$:

$$\{\mathbf{d}^\beta_l \mid r_i(y, \mathbf{d}^\beta_l)\} \subseteq \{\mathbf{d}^\beta_l \mid r_i(x, \mathbf{d}^\beta_l)\} \tag{2}$$

$$\{\mathbf{c}^\alpha_l \mid r_i(\mathbf{c}^\alpha_l, x)\} \subseteq \{\mathbf{c}^\alpha_l \mid r_i(\mathbf{c}^\alpha_l, y)\} \tag{3}$$

by induction on i. For $i = s$ claims (C1) and (C2) are trivial. In order to prove (2) suppose that $\mathbf{d}^\beta_l$ is such that $r_s(y, \mathbf{d}^\beta_l)$. Then according to Proposition 3, $\mathbf{d}^\beta_l \in D_s$ and hence $r_s(x, \mathbf{d}^\beta_l)$. The proof of (3) is analogous.

Now let $i > s$. In order to prove (C1) suppose that $x \in D_i$. Then according steps (E3) and (A1) of the construction there is a $\mathbf{c}^\alpha_l \in D_i$, such that $r_{i-1}(\mathbf{c}^\alpha_l, x)$. The induction hypothesis for (3) implies $r_{i-1}(\mathbf{c}^\alpha_l, y)$. Now from claim (C1) of *Case 2* and $\mathbf{c}^\alpha_l \in D_i$ we obtain $y \in D_i$. The proof of (C2) is analogous.

Now let us prove (2). Suppose that for some $\mathbf{d}^\beta_l$, $r_i(y, \mathbf{d}^\beta_l)$. We shall consider two cases. First suppose that $y \notin C_i$. Then it should be the case $r_{i-1}(y, \mathbf{d}^\beta_l)$ which together with the induction hypothesis implies $r_{i-1}(x, \mathbf{d}^\beta_l)$ and hence $r_i(x, \mathbf{d}^\beta_l)$. Now let $y \in C_i$. If $r_{i-1}(y, \mathbf{d}^\beta_l)$ we reason in the same way as above, so suppose that

$r_{i-1}(y, \mathbf{d}_l^\beta)$ is not true. Then it should be the case $\mathbf{d}_l^\beta \in D_i$. On the other hand (C2) implies $x \in C_i$ and hence $r_i(x, \mathbf{d}_l^\beta)$.

Claim (3) is proved analogously.

□

Next we show that property (P2) is true.

Proposition 4. *For every i the following holds:*

- $A_i \not\subseteq A \Longrightarrow D_i \subseteq D$;
- $B_i \not\subseteq B \Longrightarrow C_i \subseteq C$.

Furthermore

- $a \in A_i \setminus A \Longrightarrow \mathbf{c}_i^a \in D$;
- $b \in B_i \setminus B \Longrightarrow \mathbf{d}_i^b \in C$.

Proof. Fix an i such that $A_i \not\subseteq A$ and let $a \in A_i \setminus A$. Consider the follower $\mathbf{c}_i^a$. According to (A2) $\mathbf{c}_i^a \in C_i$, so that for all $y \in D_i$, $r_i(\mathbf{c}_i^a, y)$ and hence $r_j(\mathbf{c}_i^a, y)$ for $j \geq i$. Let $s_1 > i$ be the least stage, such that for all $j \geq s_1$, $a \notin A_j$ (such stage exists since $\{A_i\}_{i<\omega}$ is a Δ_2^0 approximation). Then according to rule (E1) for each $j \geq s_1$, $\mathbf{c}_j^a \notin C_j$ and hence for $j \geq s_1$, $\{\mathbf{d}_k^b \mid r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)\} \cap C_j = \emptyset$. Thus for $j \geq s_1$, $\{\mathbf{d}_k^b \mid r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)\} \subseteq D_j \cup Free$. Now consider the set $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\}$. Note that as for all $j \geq s_1$, $\mathbf{c}_j^a \notin C_j$, it follows that this set is finite and does not change. We claim that

$$\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B. \tag{4}$$

Indeed, $r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)$ implies that for some l, $\mathbf{c}_i^a \in C_l$ and $\mathbf{d}_k^b \in D_l$, and in the particular $a \in A_l$ and $b \in B_l$. Thus $A_l \not\subseteq A$, so that by our choice of $\mathcal{K}$-approximations to A and B, it must be true that $B_l \subseteq B$ and hence $b \in B$.

Fix the least stage $s_2 \geq s_1$, such that for all $j \geq s_2$, $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B_j$ (such a stage exists in virtue of (4)). Then for $j \geq s_2$, $\{\mathbf{d}_k^b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq D_j \cup Free$ and $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B_j$, so that (A1) implies $\mathbf{c}_i^a \in D_j$. Thus $\mathbf{c}_i^a \in D$.

Finally since for all $y \in D_i$ and all $j \geq s_2$, $r_j(\mathbf{c}_i^a, y)$, Lemma 2 implies $D_i \subseteq D_j$ and hence $D_i \subseteq D$.

□

Corollary 2. *$\{C_i\}_{i<\omega}$ and $\{D_i\}_{i<\omega}$ are Δ_2^0 approximations to C and D respectively.*

Proof. Towards a contradiction assume that $\{D_i\}_{i<\omega}$ is not a Δ_2^0 approximation to D. Then there is an element $y \notin D$ such that the set $I(y) = \{i \mid y \in D_i\}$ is infinite. Every $i \in I(y)$ is a bad stage for D and hence according to Proposition 4 it is a good stage for A. On the other hand $I(y)$ is computable and hence

$$A = \{a \mid \exists i[i \in I(y) \,\&\, a \in A_i]\}.$$

This contradicts that A is not c.e.

□

Corollary 3. $A = \{a \mid \exists j[\mathbf{c}_j^a \in C]\}$ *and* $B = \{b \mid \exists j[\mathbf{d}_j^b \in D]\}$.

Proof. By Proposition 1 for every i, $A_i = \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i = \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$. Hence if $a \notin A$ there is a stage i_a such that $a \notin A_i$ for all $i > i_a$ and hence for all j and all $i > i_a$, $\mathbf{c}_j^a \notin C_i$. This yields $A \supseteq \{a \mid \exists j[\mathbf{c}_j^a \in C]\}$.

Now let $a \in A$. Let i_a be a stage such that $a \in A_i$ for all $i > i_a$. Let $j > i_a$ be a stage such that $B_j \not\subseteq B$. Such a stage exists, as if we assume otherwise, i.e. that for all $j > i_a$, $B_j \subseteq B$, it would follow that B is c.e. contrary to what is given. At stage j, as $a \in A_j$, $\mathbf{c}^a_j \in C_j$ by (A2). By Proposition 4, as $B_j \not\subseteq B$, $C_j \subseteq C$. So $\mathbf{c}^a_j \in C$ and $A \subseteq \{a \mid \exists j[\mathbf{c}^a_j \in C]\}$. That $B = \{b \mid \exists j[\mathbf{d}^b_j \in D]\}$ is proved similarly. □

To complete the verification of the construction in the last two propositions we prove that properties (P3) and (P4) are true.

Proposition 5. $D = \overline{C}$.

Proof. First we claim that $C \cap D = \emptyset$. Indeed, at each stage the rules of the construction guarantee the $C_i \cap D_i = \emptyset$. This together with the fact that $\{C_i\}_{i<\omega}$ and $\{D_i\}_{i<\omega}$ are Δ^0_2 approximations implies $C \cap D = \emptyset$.

Next we prove that $C \cup D = \mathbb{N}$. Fix a natural $x \in \mathbb{N}$. Suppose that x is not a follower. Without loss of generality we may assume that $x = \mathbf{c}^a_i$ for some natural numbers i and a. Then at stage $s = \max\{i, a\}$, x is enumerated in D_s under rule (A3). It is never extracted from D. Indeed it could be extracted at a stage j only under rule (E2), because this is the only rule which extracts an even number from D. However the set $\{\mathbf{d}^b_k \mid r_{j-1}(x, \mathbf{d}^b_k)\} = \emptyset$ so rule (E2) does not apply. Thus $x \in D$.

Now suppose that x is a follower. If $x = \mathbf{c}^a_i$ for some $a \notin A$, or $x = \mathbf{d}^b_i$ for some $b \notin B$ then according to Proposition 4, $x \in D$ or $x \in C$ respectively. So let $x = \mathbf{c}^a_i$ for some $a \in A$ and suppose that $x \notin C$. Then according to Proposition 4 for every j if $x \in C_j$, then $B_j \subseteq B$. Thus if $r_j(x, \mathbf{d}^b_k)$, then $b \in B$. Let s_1 be the least stage, such that for $j \geq s_1$, $x \notin C_j$. Then for $j \geq s_1$ we have $r_j(x, \mathbf{d}^b_k) \iff r_{s_1}(x, \mathbf{d}^b_k)$. Furthermore Proposition 3 implies that for $j \geq s_1$, $\{\mathbf{d}^b_k \mid r_j(x, \mathbf{d}^b_k)\} \cap C_j = \emptyset$ and hence $\{\mathbf{d}^b_k \mid r_j(x, \mathbf{d}^b_k)\} \subseteq D_j \cup Free$. Let $s_2 \geq s_1$ be the least stage such that for $j \geq s_2$, $\{b \mid \exists k[r_j(x, \mathbf{d}^b_k)]\} \subseteq B_j$. Then at stage s_2, x is enumerated in D_{s_2} under rule (A1) and is never extracted from D.

Analogously we may prove that if $x = \mathbf{d}^b_k$ for some $b \in B$ and $x \notin D$ then $x \in C$. □

Proposition 6. *For every i, either $C_i \subseteq C$ or $D_i \subseteq D$*

Proof. Suppose that for some i, $C_i \not\subseteq C$ and let $x \in C_i \setminus C$. Fix a stage s such that for each $j \geq s$, $x \in D_j$ (such a stage exists since $D = \overline{C}$ and the approximation to D is Δ^0_2). Take an arbitrary $y \in D_i$. Then for each $j \geq i$, $r_j(x, y)$ and hence according to claim (C1) of Lemma 2 we obtain that for $j \geq s$, $y \in D_j$. Thus $y \in D$ and hence $D_i \subseteq D$. □

6. Final remark

The proof of Theorem 1 can be relativized above any total degree. Consider the following generalization of a $\mathcal{K}$-pair, originally known as a U-e-ideal and introduced by Kalimullin [10].

Definition 8. *Let A, B and U be sets of natural numbers. We say that $\{A, B\}$ is a $\mathcal{K}$-pair over U if there is a set $W \leq_e U$, such that*

$$A \times B \subseteq W \;\&\; \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Kalimullin shows that a pair of sets A and B for a $\mathcal{K}$-pair over U if and only if their degrees $\mathbf{a} = d_e(A)$, $\mathbf{b} = d_e(B)$ and $\mathbf{u} = d_e(U)$ satisfy the formula:

$$\forall \mathbf{x}[\mathbf{x} \vee \mathbf{u} = (\mathbf{x} \vee \mathbf{a} \vee \mathbf{u}) \wedge (\mathbf{x} \vee \mathbf{b} \vee \mathbf{u})].$$

In [6] we relativize the dynamic characterization of $\mathcal{K}$-pairs as follows:

Lemma 3. *Let G be a total set and let A and B be sets, such that $G <_e A \leq_e J_e(G)$ and $G <_e B \leq_e J_e(G)$. A and B form a $\mathcal{K}$-pair over G if and only if A and B have $\Delta^0_2(G)$ approximations $\{A_i\}_{i<\omega}$ and $\{B_i\}_{i<\omega}$ such that for every i either $A_i \subseteq A$ or $B_i \subseteq B$.*

This relativized dynamic characterization allows us to carry out the construction in Section 5 relative to any total set G, thus proving that every $\mathcal{K}$-pair over G can be lifted to a maximal $\mathcal{K}$-pair over G. Since every $\mathcal{K}$-pair over $\emptyset$ is also a $\mathcal{K}$-pair over G, Jouckusch's theorem is still applicable in relativized form - every total set T, such that $G <_e T \leq_e G'$ is the join of a maximal $\mathcal{K}$-pair over G. Thus we obtain the following corollary:

Corollary 4. *For every total degree $\mathbf{a}$ the class $\mathcal{TOT} \cap [\mathbf{a}, \mathbf{a}']$ is first order definable in $\mathcal{D}_e$ with parameter $\mathbf{a}$.*

References

[1] S. B. Cooper, *Minimal degrees and the jump operator*, J. Symbolic Logic **38** (1973), 249–271.
[2] ———, *Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense*, J. Symbolic Logic **49** (1984), 503–513.
[3] ———, *Enumeration reducibilty, nondeterministic computations and relative computability of partial functions*, Recursion theory week, Oberwolfach 1989, Lecture notes in mathematics (Heidelberg) (K. Ambos-Spies, G. Muler, and G. E. Sacks, eds.), vol. 1432, Springer-Verlag, 1990, pp. 57–110.
[4] R. M. Friedberg and Jr. H. Rogers, *Reducibility and completeness for sets of integers*, Z. Math. Logik Grundlag. Math. **5** (1959), 117–125.
[5] H. Ganchev and M. I. Soskova, *Cupping and definability in the local structure of the enumeration degrees*, to appear in J. Symb. Logic.
[6] ———, *The high/low hierarchy in the local structure of the ω-enumeration degrees*, accepted in Annals of Pure and Applied Logic.
[7] ———, *Interpreting true arithmetic in the local structure of the enumeration degrees*, submitted.
[8] M. Giorgi, A. Sorbi, and Y. Yang, *Properly Σ^0_2 enumeration degrees and the high/low hierracrhy*, J. Symbolic Logic **71** (2006), 1125–1144.
[9] C. G. Jockusch, *Semirecursive sets and positive reducibility*, Trans. Amer. Math. Soc. **131** (1968), 420–436.
[10] I. Sh. Kalimullin, *Definability of the jump operator in the enumeration degrees*, Journal of Mathematical Logic **3** (2003), 257–267.
[11] K. McEvoy, *Jumps of quasi-minimal enumeration degrees*, J. Symbolic Logic **50** (1985), 839–848.
[12] A. Nies, R. A. Shore, and T. A. Slaman, *Interpretability and definability in the recursively enumerable degrees*, Proc. London Math. Soc. **77** (1998), 241–249.
[13] H. Rogers Jr., *Theory of recursive functions and effective computability*, McGraw-Hill Book Company, New York, 1967.
[14] M. Rozinas, *The semi-lattice of e-degrees*, Recursive functions (Ivanovo), Ivano. Gos. Univ., 1978, Russian, pp. 71–84.
[15] R. A. Shore, *Degree structures: Local and global investigations*, Bull. Symb Logic **12** (2006), 369–389.
[16] R. A. Shore and T. A. Slaman, *Defining the Turing jump*, Math. Res. Lett. **6** (1999), 711–722.

[17] T. A. Slaman, *Global properties of the turing degrees and the turing jump*, Computational Prospects of Infinity. Part I: Tutorials (C. Chong, Qi Feng, T. A. Slaman, W. H. Woodin, and Y. Yang, eds.), World Scientific, 2005, pp. 83–101.

[18] A. Sorbi, *The enumeration degrees of the* Σ^0_2 *sets*, Complexity, Logic and Recursion Theory (New York) (A. Sorbi, ed.), Marcel Dekker, 1997, pp. 303–330.

[19] I. N. Soskov, *A jump inversion theorem for the enumeration jump*, Arch. Math. Logic **39** (2000), 417–437.

Faculty of Mathematics, Sofia University, 5 James Bourchier blvd., 1164 Sofia
E-mail address: `ganchev@fmi.uni-sofia.bg, msoskova@fmi.uni-sofia.bg`