SOME PROPERTIES OF AN ALGEBRA OF ALL SETS OF NATURALS E-REDUCIBLE TO A FIXED SET *

Angel V. Ditchev

Faculty of Mathematics and Computer Science, Sofia University
5 James Bourchier Blvd., 1164 Sofia, Bulgaria
ditchev@fmi.uni-sofia.bg

Abstract

In this paper for any fixed set of natural numbers A the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is considered, where $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, $W_0, W_1, \ldots$ are the sequence of all c.e. sets considered as e-operators and Non is the predicate to be nonempty set. It is shown for any set of natural numbers A the algebra $\mathfrak{N}^A$ has the least enumeration, admits equivalent representation with 3 operators and it is finitely generated.

Mathematics Subject Classification: 03C50, 03D28, 03D30, 03D45, 03D60, 03D75

1 Introduction

In attempts to classify the family of all sets of naturals according to effective computability different kinds of reducibilities were introduced. Post first introduced so called "strong" reducibilities (m- ,tt- ,...) in [1] and a little later in [2] Turing reducibility. Every reducibility defines a pre-order. Thus in a natural way m-degrees, T-degrees, etc. were introduced. Enumeration

*Research partially supported by the Bulgarian science fund, Contract No. D002-258/18.12.2008 and Sofia University Science Fund, Contract No 176/2012

reducibility was introduced in 1959 by Friedberg and Rogers [3]. In [4] embedding of the semi-lattice of Turing degrees (T-degrees) into the semi-lattice of enumeration degrees (e-degrees) was found. It showed both semi-lattices were closely related and any result or question about one of them triggered a question of validity for the other. In 1966 Sacks [5] and in 1967 Rogers [6] stated the basic question about T-degrees if there were any nontrivial automorphisms in the upper semi-lattice of T-degrees. In case that there aren't nontrivial automorphisms in some upper semi-lattice we say it is rigid. The same question was stated about e-degrees, m-degrees, etc. This question is important because it is connected with definability in these semi-lattices. For m-degrees it was shown by Shore there exist $2^{2^{\aleph_0}}$ automorphisms.

In 1977 Jockusch and Solovay [7] and in 1979 Richter [8] and Epstein [9] proved that every automorphism is the identity on the cone above $0^{(3)}$ for Turing degrees. In 1986 Slaman and Woodin [10] approved the above result showing that every automorphism is the identity on the cone above $0''$. Using the connections between both T- and e-jumps in [11] Soskov and Ganchev obtained that every automorphism is the identity on the cone above $0^{(4)}$ for e-degrees.

Since the uppersemilattice of all e-degrees (e-degrees $\leq \mathbf{a}$) is defined by $\leq_e A$ in this paper for any fixed set of natural numbers A the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is considered. Here $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, $W_0, W_1, \ldots$ are the standard sequence of all c.e. sets considered as e-operators and Non is the predicate to be nonempty set. We would like to mention that the empty set plays special role and we differ from other c.e. sets. We modify insignificant the relation $\leq_e$ and show that the algebra $\mathfrak{N}^A$ has the least enumeration, admits equivalent representation with 3 operators ant it is finitely generated. We use so called unary partial structures without equality [12, 13].

In Section 2 we give all necessary definitions, notions and propositions concerning normal and least enumerations of unary partial structures. Here we modify a little the definitions of e-reducibility and e-operators, concerning the empty set. In Section 3 we prove the main results: The algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ admits the least enumeration. Then we prove that this algebra is recursively equivalent to an algebra with only 3 operators, and that the last algebra is finitely generated. At the end we see that among all algebras with different enumeration of all e-operators the standard one has the least enumeration.

2 Preliminaries

In this paper we use ω to denote the set of all natural numbers, $Dom(f)$, $Ran(f)$ and G_f to denote respectively the domain, the range and the graph of the function f, $\langle f\rangle$ or $\langle G_f\rangle$ to denote the set $\{\langle x_1,\ldots,x_n,y\rangle|(x_1,\ldots,x_n,y)\in G_f\}$, where $\langle .,\ldots,.\rangle$ is some fixed coding function for all finite sequences of natural numbers. We shall use $f(x)\downarrow$ to denote that $x\in Dom(f)$, as well; also we say that $f(x)$ *conditionally equal to* $g(x)$, or that the conditional equality $f(x)\cong g(x)$ is true iff

$$(f(x)\downarrow \& g(x)\downarrow \& f(x)=g(x))\vee(\neg(f(x)\downarrow)\&\neg(g(x)\downarrow)).$$

W_0, W_1,... denote the standard enumeration of all computably enumerable (c.e.) sets; E_v denote an effective coding of the family of all finite subsets of ω.

If W is c.e. set we denote $W_{[n]}=\{x|\langle n,x\rangle\in W\}$ and if A is arbitrary subset of ω, then by $W(A)$ we denote the set

$$W(A)=\{x|\exists v(\langle x,v\rangle\in W\&E_v\neq\emptyset\&E_v\subseteq A)\}.$$

Notice here is a little difference with the accepted sense of the term e-operator. It concerns $\emptyset$.

It is said A is e-reducible to B ($A\leq_e B$) iff there exists a c.e. set W such that $A=W(B)$; A is e-equivalent to B ($A\equiv_e B$) iff $A\leq_e B\&B\leq_e A$; $\mathbf{d}_e(A)=\{B|A\equiv_e B\}$. Thus we shall consider $\mathbf{0}_e$ to be the family of all nonempty c.e. sets and $\mathbf{\text{-}1}_e=\{\emptyset\}$.

For arbitrary sets A and B of naturals by $A\oplus B$ we denote the set

$\{2x|x\in A\}\cup\{2x+1|x\in B\}$; If A_0, A_1,... is a sequence of sets of naturals, by $\oplus_{i\in\omega}A_i$ we denote the set $\{\langle i,x\rangle|x\in A_i\}$.

We shall recall some definitions from [14, 15].

Let $\mathfrak{A}=\langle B;\theta_1,\ldots,\theta_n;R_1,\ldots,R_k\rangle$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1,\ldots,\theta_n$ are partial unary functions in B and $R_1,\ldots,R_k$ are unary partial predicates on B. We allow any of the sequences $\theta_1,\ldots,\theta_n$ and $R_1,\ldots,R_k$ to be infinite, as well. We call such structures unary. We identify the partial predicates with partial mapping taking values in $\{0,1\}$, writing 0 for true and 1 for false.

Let $\mathfrak{B}=\langle\omega;\varphi_1,\ldots,\varphi_n;\sigma_1,\ldots,\sigma_k\rangle$ be a partial structure over the set ω. By $\langle\mathfrak{B}\rangle$ we denote the set $\langle\varphi_1\rangle\oplus\cdots\oplus\langle\varphi_n\rangle\oplus\langle\sigma_1\rangle\oplus\cdots\oplus\langle\sigma_k\rangle$ (In case the set of functions or predicates are infinite we use the correspondent infinite version of $\bigoplus$).

Definition 1 *An enumeration of a structure* $\mathfrak{A}$ *is any ordered pair* $\langle \alpha, \mathfrak{B} \rangle$ *where* $\mathfrak{B} = \langle \omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k \rangle$ *is a partial unary structure on* ω *and* α *is a partial surjective mapping of* ω *onto* B *such that the following conditions hold:*

(i) $Dom(\alpha) \leq_e \langle \mathfrak{B} \rangle$*;*

(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ *for every* $x \in \omega$*,* $1 \leq i \leq n$*;*

(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ *for every* $x \in \omega$*,* $1 \leq j \leq k$*.*

An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *total* iff $Dom(\alpha) = \omega$.

Let $A \subseteq B$. The set A is called admissible in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff there exists a set W of naturals such that $W \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$, $x \in W \iff \alpha(x) \in A$.

A partial multiple-valued (p.m.v) function θ is called admissible in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a set $W \subseteq \omega^2$ such that $W \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$ and $t \in B$, the following equivalence is true:

$t \in \theta(\alpha(x)) \iff \exists y((x, y) \in W \& \alpha(y) = t)$.

In other notation the previous definition can be reformulated as follows.

A p.m.v function θ is called admissible in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a p.m.v function φ in ω such that $\langle G_\varphi \rangle \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$, $\alpha(\varphi(x)) = \theta(\alpha(x))$.

A set A or p.m.v function θ is called $\forall$-admissible in $\mathfrak{A}$ iff it is admissible in every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of the structure $\mathfrak{A}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of $\mathfrak{A}$ iff for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n unary functional symbols $\mathbf{f_1}, \dots, \mathbf{f_n}$, k unary predicate symbols $\mathbf{T_1}, \dots, \mathbf{T_k}$.

Let us fix some fixed denumerable set $X_1, X_2, \dots$ of variables. We shall use capital letters X, Y, Z and the same letters by indexes to denote variables.

The definition of a term in the language $\mathcal{L}$ is the usual: Every variable is a term; If τ is a term then $\mathbf{f_i}(\tau)$ is a term.

If τ is a term in the language $\mathcal{L}$, then we write $\tau(Y_1, \dots, Y_k)$ to denote that all variables which occur in the term τ are among $Y_1, \dots, Y_k$.

Termal predicate in the language $\mathcal{L}$ is defined by the following inductive clauses:

If $\mathbf{T} \in \{\mathbf{T_0}, \dots, \mathbf{T_k}\}$ and τ is a term, then $\mathbf{T}(\tau)$ and $\neg\mathbf{T}(\tau)$ are termal predicates. If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\mathfrak{B}$ be a structure, $a_1, \ldots, a_k$ are elements of B and $\tau(Y_1, \ldots, Y_k)$ is a term. By $\tau_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_k/a_k)$ we denote the value of the term τ in $\mathfrak{A}$ over the elements $a_1, \ldots, a_k$, if it exists.

Let $\Pi(Y_1, \ldots, Y_m)$ be a termal predicate whose variables are among $Y_1, \ldots, Y_m$ and $a_1, \ldots, a_m$ be elements of B. The value $\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m)$ of Π over $a_1, \ldots, a_r$ in $\mathfrak{A}$ is defined as follows:

If $\Pi = \mathbf{T}_j(\tau)$, $0 \leq j \leq k$, then

$\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong R_j(\tau_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m))$.

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong \begin{cases} 1, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 0, \\ 0, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong \begin{cases} \Pi^2_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m), & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 0, \\ 1, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Formulae of the kind $\exists Y_1' \ldots \exists Y_l'(\Pi)$, where Π is a termal predicate are called *conditions*. Every variable which occurs in Π and is different from $Y_1', \ldots, Y_l'$ is called free in the condition $\exists Y_1' \ldots \exists Y_l'(\Pi)$.

Let $\exists Y_1' \ldots \exists Y_l'(\Pi)$ be a condition, all free variables in C are among $Y_1, \ldots, Y_m$, and $a_1, \ldots, a_m$ be elements of B. The value $C_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m)$ is defined by the equivalence:

$C_{\mathfrak{A}}(Y_1/a_1, \ldots, Y_m/a_m) \cong 0 \iff$

$\exists t_1 \ldots \exists t_l(\Pi_{\mathfrak{A}}(Y_1'/t_1, \ldots, Y_l'/t_l, Y_1/a_1, \ldots, Y_m/a_m) \cong 0)$.

We assume there is fixed some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$. We shall use superscripts to denote the corespondent codes.

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* (or just *definable*) in the structure $\mathfrak{A}$ iff there exists a recursive function γ of $r+1$ variables such that for all $n, x_1, \ldots, x_r$, $C^{\gamma(n,x_1,\ldots,x_r)}$ is a conditions with free variables among $Z_1, \ldots, Z_l, Y_1, \ldots, Y_m$ and for some fixed elements $t_1, \ldots, t_l$ of B the following equivalence is true:

$$(x_1, \ldots, x_r, a_1, \ldots, a_m) \in A \iff$$

$$\exists n \in \omega(C^{\gamma(n,x_1,\ldots,x_r)}_{\mathfrak{A}}(Z_1/t_1, \ldots, Z_l/t_l, Y_1/a_1, \ldots, Y_m/a_m) \cong 0).$$

If Π is a termal predicate and τ is a term, then $\exists Y_1' \dots \exists Y_l' (\Pi \supset \tau)$ is called a conditional expression.

Let $Q = \exists Y_1' \dots \exists Y_l' (\Pi \supset \tau)$ be a conditional expression with free variables among $X_1, \dots, X_a$, and $s_1, \dots, s_a \in B$. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \dots, X_a/s_a)$ of Q is the subset

$$\{\tau_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) | \Pi_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) \cong 0\}$$

of B.

Let θ be p.m.v. function in B. Then the function θ is called definable in $\mathfrak{A}$ iff for some c.e. set $\{Q^v\}_{v \in V}$ of conditional expressions with free variables among $X, Z_1, \dots, Z_r$ and for some fixed elements $t_1, \dots, t_r$ of B the following equivalence is true:

$$t \in \theta(s) \iff \exists v (v \in V \& t \in Q^v_{\mathfrak{A}}(Z_1/t_1, \dots, Z_r/t_r, X/s)).$$

In [14] Soskov has proved the following Theorem.

Theorem 1 *(Soskov [14]) Let θ be a unary p.m.v. function in B. Then θ is $\forall$-admissible in $\mathfrak{A}$ iff θ is definable in $\mathfrak{A}$.*

Define $f_i(p) = \langle i-1, p \rangle$, $i = 1, \dots, n$ and
$N_0 = \omega \setminus (Ran(f_1) \cup \dots \cup Ran(f_n))$. It is obvious that N_0 is an infinite recursive set and let $\{\mathbf{p_0}, \mathbf{p_1}, \dots\} = N_0$, where $\mathbf{p_i} < \mathbf{p_j}$ if $i < j$. In case the sequence f_i is infinite ($i \in \omega$) we can ensure N_0 to be infinite taking for example $f_i(p) = \langle i-1, p, 0 \rangle$.

We shall recall the definition and some properties of normal enumerations [14] only in case of total one. For every surjective mapping α^0 of N_0 onto B (called basis) we define a mapping α of ω onto B by the following inductive clauses:

(i) If $p \in N_0$, then $\alpha(p) = \alpha^0(p)$;

(ii) If $p = f_i(q)$, $\alpha(q) = a$ and $\theta_i(a) = b$, then $\alpha(p) = b$.

Let $\sigma_1, \dots, \sigma_k$ be the partial predicates, defined by the equalities
$\sigma_j(x) \cong R_j(\alpha(x))$, $j = 1, \dots, k$.

$\mathfrak{B}$ denote the partial structure $\langle \omega; f_1, \dots, f_n; \sigma_1, \dots, \sigma_k \rangle$.

It is well known that α is well defined and that the basis α^0 completely determines the normal enumeration $\langle \alpha, \mathfrak{B} \rangle$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal enumeration. We shall recall some obvious propositions for normal enumerations. The proofs are the same as in [14].

Proposition 1 *For every* $1 \leq i \leq n$ *and* $y \in \omega$, $\alpha(f_i(y)) = \theta_i(\alpha(y))$.

Corollary 1 *Let* $\tau(Y)$ *be a term, and* $y \in \omega$. *Then*

$$\alpha(\tau_{\mathfrak{B}}(Y/y)) = \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Proposition 2 *There exists an effective way for every* x *of* ω *to find* $y \in N_0$ *and a term* $\tau(Y)$ *such that* $x = \tau_{\mathfrak{B}}(Y/y)$.

If $\langle \alpha, \mathfrak{B} \rangle$ is a normal enumeration, then by R_α we denote the set $\cup_{j=1}^k \{\langle j, x, z \rangle | \sigma_j(x) = z\}$. In the general case we have to add some additional members, but in our case the functions f_i are totally defined and we don't need it. It is clear that for every $W \subseteq \omega$, $W \leq_e R_\alpha$ iff $W \leq_e \langle \mathfrak{B} \rangle$.

Proposition 3 *There exists an effective way for every natural* u *to find elements* $y_1, \ldots, y_m \in N_0$ *and a termal predicate* $\Pi(Y_1, \ldots, Y_m)$ *such that for every normal enumeration* $\langle \alpha, \mathfrak{B} \rangle$,

$$u \in R_\alpha \iff \Pi_{\mathfrak{A}}(Y_1/\alpha(y_1), \ldots, Y_m/\alpha(y_m)) \cong 0.$$

Proposition 4 *There exists an effective way for every code* v *of a finite set* E_v *to find elements* $y_1^v, \ldots, y_{m_v}^v \in N_0$ *and a termal predicate* $\Pi^v(Y_1, \ldots, Y_{m_v})$ *such that for every normal enumeration* $\langle \alpha, \mathfrak{B} \rangle$,

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^v(Y_1/\alpha(y_1^v), \ldots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

To be precise we need to mention that in the previous proposition, instead of using $\Pi^{\gamma(v)}$ for some recursive function γ for the sake of simplicity we used just Π^v. We shall use it from now on, as well.

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k \rangle$ be an unary partial structure.

Type of the sequence $b_1, \ldots, b_m$ of elements of B we call the set $\{v | \Pi_{\mathfrak{A}}^v(X_1/b_1, \ldots, X_m/b_m) \cong 0 \ \& \Pi^v$ is a termal predicate with variables among $X_1, \ldots, X_m\}$. The type of the sequence $b_1, \ldots, b_m$ we shall denote by $[b_1, \ldots, b_m]_{\mathfrak{A}}$. Call type of the element a of B the type of the sequence a.

Call a condition simple if it does not contain free variables and it is in the form $\exists X_1 \Pi$, where Π is a termal predicate. Let $V_0^{\mathfrak{A}} = \{v | C_{\mathfrak{A}}^v \cong 0 \ \& \ C^v$ be a simple condition $\}$.

Definition 2 *Let* $\mathcal{A}$ *be a family of subsets of* ω. *It is said that a set* $U \subseteq \omega^2$ *is universal for the family* $\mathcal{A}$ *iff the following conditions hold:*

a) For every fixed $e \in \omega$, $\{x_1 | (e, x_1) \in U\} \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, *then there exists* e *such that* $A = \{x_1 | (e, x_1) \in U\}$.

Theorem 2 *Let $\mathfrak{A}$ be an unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$ is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

3 The main result

We shall consider the *standard structure* $\mathfrak{N} = \langle \mathcal{P}(\omega); W_0, W_1, \dots; Non \rangle$, where $\mathcal{P}(\omega)$ is the family of all subsets of ω, $W_0, W_1, \dots$ are the sequence of all c.e. sets considered as functions (e-operators) and Non is the family of all nonempty sets of naturals. To be more precise Non is a partial unary predicate defined as follows: $Non(A) = 0$, if $A \neq \emptyset$ and $Non(\emptyset) \uparrow$.

First we shall consider the structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \dots; Non \rangle$, where $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$ which we call standard as well. Let us mention the functions $W_0, W_1, \dots$ are totally defined as e-operators and we do not use the equality among predicates. Let in addition $\mathbf{W}$ be the family of all c.e. sets considered as e-operators.

Let $\mathcal{L}^*$ be the first order language $\langle \mathbf{f_0}, \mathbf{f_1}, \dots; \mathbf{T} \rangle$, containing a countable set of unary functional symbols $\mathbf{f_0}, \mathbf{f_1}, \dots$ and a unary predicate symbol $\mathbf{T}$. We call $\overline{\mathfrak{A}}$ a *generalized structure* if $\overline{\mathfrak{A}} = \langle B; \mathbf{\Theta}; R \rangle$ where B is a denumerable set, $\mathbf{\Theta}$ – denumerable set of unary functions on B and R is an unary predicate on B. When we consider structures with finite functions and finite predicates, the considerations don't depend on enumerations of the functions and predicates. In case when we consider denumerable set of functions the things are different.

Enumeration of the family $\mathbf{\Theta}$ of functions is any sequence $\theta_0, \theta_1, \dots$ such that $\mathbf{\Theta} = \{\boldsymbol{\theta_0}, \boldsymbol{\theta_1}, \dots\}$. We don't require all members of the sequence $\theta_0, \theta_1, \dots$ to be different.

Let us fix some enumeration $\theta_0^0, \theta_1^0, \dots$ of the family $\mathbf{\Theta}$ and consider the structure $\mathfrak{A}_0 = \langle B; \theta_0^0, \theta_1^0, \dots; R \rangle$.

We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of the generalized structure $\overline{\mathfrak{A}}$ iff for every enumeration $\theta_0, \theta_1, \dots$ of $\mathbf{\Theta}$ and every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A} = \langle B; \theta_0, \theta_1, \dots; R \rangle$ the inequality $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$ holds.

Let us consider the structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \dots; Non \rangle$ for the language $\mathcal{L}^*$ and define the m.v.f. $\Phi^A : \mathcal{P}(\omega)^A \setminus \{\emptyset\} \to \mathcal{P}(\omega)^A \setminus \{\emptyset\}$ as follows: $\Phi^A(B) = \{C | C \leq_e B \& C \neq \emptyset\}$ for nonempty B.

Proposition 5 *The m.v.f. Φ^A is definable in the structure $\mathfrak{N}^A$.*

Proof: Let Q^n be the conditional expression $\mathbf{T}(X)\&\mathbf{T}(\mathbf{f_n}(X)) \supset \mathbf{f_n}(X)$ and notice that the sequence $\{Q^n\}_{n\in\omega}$ is c.e. and

$C \in Q^n_{\mathfrak{N}^A}(X/B) \iff Non(B)\&Non(W_n(B))\&C = W_n(B)$.

Then $C \in \Phi^A(B) \iff C \leq_e B\&C \neq \emptyset\&B \neq \emptyset \iff$
$\exists n(W_n(B) = C\&C \neq \emptyset\&B \neq \emptyset) \iff \exists n(C \in Q^n_{\mathfrak{N}^A}(X/B))$. $\dashv$

Let $L_A = \{\langle n, x\rangle | x \in W_n(A)\}$. The following lemma is well-known and is a simple use of the S^m_n-theorem.

Lemma 1 *There exists a recursive function δ of two variables such that for all naturals m, n and set C of naturals the following equality is true $W_m(W_n(C)) = W_{\delta(m,n)}(C)$.*

Let us fix a function δ from the lemma above and define the pair $\langle\alpha_0, \mathfrak{B}_0\rangle$ as follows:

$\alpha_0(n) = W_n(A)$, $\mathfrak{B}_0 = \langle\omega; \varphi^0_0, \varphi^0_1, \ldots; \sigma^0\rangle$, where $\varphi^0_i(x) = \delta(i, x)$, $i, x \in \omega$, $\sigma^0(x) \cong 0 \iff W_x(A) \neq \emptyset$ and $\sigma^0(x)\uparrow$ if $W_x(A) = \emptyset$.

Lemma 2 *The pair $\langle\alpha_0, \mathfrak{B}_0\rangle$ is an enumeration of the structure $\mathfrak{N}^A$.*

Proof: $W_i(\alpha_0(x)) = W_i(W_x(A)) = W_{\delta(i,x)}(A) = \alpha_0(\delta(i, x)) = \alpha_0(\varphi^0_i(x))$.
$Non(\alpha_0(x)) \cong 0 \iff W_x(A) \neq \emptyset \iff \sigma^0(x) \cong 0$. $\dashv$

Let $W_A = \{n|\exists x(\langle n, x\rangle \in L_A)\} = \{n|W_n(A) \neq \emptyset\} = \{n|\sigma^0(n) \cong 0\}$.

Proposition 6 *$W_A \equiv_e A$.*

Proof: Let n_0 be a fixed element of ω and define the set B by the following equivalence: $\langle\langle n, x\rangle, m\rangle \in B \iff \langle n, x\rangle \in L_A\&m = n_0$.

Obviously, $B \leq_e L_A \equiv_e A$. Therefore, using S^m_n-theorem we obtain $\langle\langle n, x\rangle, m\rangle \in B$

$\iff \exists v(\langle\langle\langle n, x\rangle, m\rangle, v\rangle \in W_a\&\emptyset \neq E_v \subseteq A)$(for some fixed natural a)
$\iff \exists v(\langle\langle m, v\rangle, \langle n, x\rangle\rangle \in W_b\&\emptyset \neq E_v \subseteq A)$(for some fixed natural b)
$\iff \exists v(\langle m, v\rangle \in W_{\gamma(\langle n,x\rangle)}\&\emptyset \neq E_v \subseteq A)$
(for some fixed recursive function γ) $\iff m \in W_{\gamma(\langle n,x\rangle)}(A)$.

We shall show that $L_A \leq_m W_A$ by recursive function γ.

Let us assume $\langle n, x\rangle \in L_A$. Then $\langle\langle n, x\rangle, n_0\rangle \in B$, thus $n_0 \in W_{\gamma(\langle n,x\rangle)}(A)$, i.e. $W_{\gamma(\langle n,x\rangle)}(A) \neq \emptyset$, hence $\gamma(\langle n, x\rangle) \in W_A$.

Let us suppose $\gamma(\langle n,x\rangle) \in W_A$. Then $\exists m(m \in W_{\gamma(\langle n,x\rangle)}(A))$, thus $n_0 \in W_{\gamma(\langle n,x\rangle)}(A)$. Therefore $\langle\langle n,x\rangle, n_0\rangle \in B$ and $\langle n,x\rangle \in L_A$.

We proved the equivalence $\langle n,x\rangle \in L_A \iff \gamma(\langle n,x\rangle) \in W_A$, i.e. $L_A \leq_m W_A$. Therefore, $L_A \leq_e W_A$.

Conversely, $n \in W_A \iff \exists x(\langle n,x\rangle \in L_A) \iff \exists x(x \in W_n(A))$
$\iff \exists x \exists v(\langle x,v\rangle \in W_n \& \emptyset \neq E_v \subseteq A)$
$\iff \exists v(\exists x(\langle n,v\rangle \in W_{\gamma_1(x)}) \& \emptyset \neq E_v \subseteq A)$
$\iff \exists v(\langle n,v\rangle \in W_a) \& \emptyset \neq E_v \subseteq A) \iff n \in W_a(A)$, for some fixed recursive function γ_1 and fixed natural a. Consequently, $W_A \leq_e A$. $\dashv$

Lemma 3 *There exists a recursive function γ_0 such that for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v the equality $\tau^v_{\mathfrak{N}^A}(X/A) = W_{\gamma_0(v)}(A)$ holds.*

Proof: Decode $\tau^v(X)$ as a sequence of $f_{i_1}, f_{i_2}, \ldots, f_{i_p}$ and variable X. Then consider the composition of the operators $W_{i_1}, W_{i_2}, \ldots, W_{i_p}$ over A and use recursive function δ. Thus there exists an effective way for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v to find a natural number n such that $\tau^v_{\mathfrak{N}^A}(X/A) = W_n(A)$. $\dashv$

Lemma 4 $[A]_{\mathfrak{N}^A} \equiv_m W_A$.

Proof: $[A]_{\mathfrak{N}^A} = \{v | \tau^v_{\mathfrak{N}^A}(X/A) \neq \emptyset\}$ and let γ_0 be the recursive function from the previous lemma. Then $v \in [A]_{\mathfrak{N}^A} \iff \tau^v_{\mathfrak{N}^A}(X/A) \neq \emptyset \iff W_{\gamma_0(v)}(A) \neq \emptyset \iff \sigma^0(\gamma_0(v)) \cong 0 \iff \gamma_0(v) \in W_A$. Thus, $[A]_{\mathfrak{N}^A} \leq_m W_A$.

Conversely, $n \in W_A \iff W_n(A) \neq \emptyset \iff$ the term $\mathbf{f}_n(X)$ with code $v(n)$ satisfies $(\mathbf{f}_n(X))^{v(n)}_{\mathfrak{N}^A}(X/A) \neq \emptyset$, i.e. $W_A \leq_m [A]_{\mathfrak{N}^A}$. $\dashv$

Theorem 3 *The enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$ is the least one for the structure $\mathfrak{N}^A$.*

Proof: According to Theorem 2, having in mind $W_A = V_0^{\mathfrak{N}^A}$, we need to show that all types of elements B such that B is a set of naturals and $B \leq_e A$ satisfies the condition $[B]_{\mathfrak{N}^A} \leq_e [A]_{\mathfrak{N}^A}$ and that there exists an universal set with e-degree $deg_e(A)$ for all types $[B]_{\mathfrak{N}^A}$.

Let $B \leq_e A$. Then there exists an e-operator W_n such that $W_n(A) = B$. Consequently, $v \in [B]_{\mathfrak{N}^A} \iff$ the code v_1 of the term $\mathbf{f_n}(\tau^v)$ belongs to $[A]_{\mathfrak{N}^A}$, thus $[B]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{N}^A}$. Further using the type $[A]_{\mathfrak{N}^A}$, we shall define the set U^A by the equivalence: $(n,v)\in U^A \iff \exists v_1(\tau^{v_1}=\mathbf{f_n}(\tau^v)\&v_1\in[A]_{\mathfrak{N}^A})$. Actually, we could define U^A by the equivalence: $(n,v)\in U^A \iff \langle n,v\rangle\in L_A$, as well. It is obvious U^A is universal for the family of all types of the structure $\mathfrak{N}^A$. $\dashv$

Let us consider the structure $\mathfrak{D}^A=\langle\mathcal{P}(\omega)^A;\Phi^A\rangle$. The following definition is natural but it is not used because we don't normally consider structures with p.m.v. functions.

Definition 3 *Enumeration of the structure $\mathfrak{D}^A$ we call $\langle\alpha,\mathfrak{B}\rangle$ where $\alpha:\omega\to\mathcal{P}(\omega)^A$, $\mathfrak{B}=\langle\omega;\varphi\rangle$ and φ is a partial m.v.f. in ω such that for all natural n the equality $\alpha(\varphi(n))=\Phi^A(\alpha(n))$ holds (We mean the equality between sets here).*

Proposition 7 *There exists an enumeration $\langle\alpha_0,\mathfrak{B}'\rangle$ of the structure $\mathfrak{D}^A$ such that $\langle\mathfrak{B}'\rangle\equiv_e A$.*

Proof: Let us recall $\alpha_0(n)=W_n(A)$ and define the partial m.v.f. φ^0 as follows: $m\in\varphi^0(n) \iff \exists k(\sigma^0(m)\cong 0\&\sigma^0(n)\cong 0\&\delta(k,n)=m)$. It is clear that $\langle G_\varphi\rangle\leq_e A$.

Then

$C\in\alpha_0(\varphi^0(n)) \iff \exists m(m\in\varphi(n)\&\alpha_0(m)=C) \iff$
$\exists m(\exists k(\sigma^0(m)\cong 0\&\sigma^0(n)\cong 0\&\delta(k,n)=m)\&W_m(A)=C) \iff$
$\exists m\exists k(W_m(A)=W_k(W_n(A))\&C=W_m(A)\neq\emptyset\&W_n(A)\neq\emptyset) \iff$
$\exists m(W_m(A)\leq_e W_n(A)\&C=W_m(A)\neq\emptyset\&W_n(A)\neq\emptyset) \iff$
$\exists m(C=W_m(A)\in\Phi^A(W_n(A)) \iff C\in\Phi^A(\alpha_0(n))$. Therefore $\langle\alpha_0,\mathfrak{B}'\rangle$ is an enumeration of $\mathfrak{D}^A$.

Further, let us fix some a such that $\alpha_0(a)=A$. Then $W_n(A)=W_n(W_a(A))=W_{\delta(n,a)}(A)$ and hence $W_A=\{n|W_n(A)\neq\emptyset\}\equiv_e\{\delta(n,a)|W_{\delta(n,a)}(A)\neq\emptyset\}=\{\delta(n,a)|\sigma^0(\delta(n,a))\cong 0\}\equiv_e\{\delta(n,a)|\delta(n,a)\in\varphi^0(a)\}\leq_e\langle G_\varphi\rangle\equiv_e\langle\mathfrak{B}'\rangle$ $\dashv$

Lemma 5 *There exist c.e. sets $V^{[n]},V',V^{[S]}$ such that the effective sequence of compositions $\{V^{[0]}(V^{[S]})^nV'\}_{n\in\omega}$ is recursively isomorphic to the sequence $\{W_n\}_{n\in\omega}$.*

Proof: Let us notice first that $V^{[0]}(V^{[S]})^nV'$ means the following:

$V^{[0]}(V^{[S]})^0V'=V^{[0]}V'$; $V^{[0]}(V^{[S]})^{n+1}V'=((V^{[0]}(V^{[S]})^n)V^{[S]})V'$.

Let us denote $V^{[n]}=\{\langle x,v\rangle|x\in\omega\&E_v=\{\langle n,x\rangle\}\}$ and
$V^{[S]}=\{\langle\langle n,x\rangle,v\rangle|n,x\in\omega\&E_v=\{\langle n+1,x\rangle\}\}$. Further, let

$V = \{\langle n, x\rangle | x \in W_n\}$ and $V' = \{\langle\langle k, x\rangle, v\rangle | \langle k, \langle x, v\rangle\rangle \in V\}$.
Then
$x \in V^{[n]}V'(X) \iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \subseteq V'(X))$
$\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \& \langle n, x\rangle \in V'(X))$
$\iff \exists v(\langle\langle n, x\rangle, v\rangle \in V' \& \emptyset \neq E_v \subseteq X)$
$\iff \exists v(\langle x, v\rangle \in V_{[n]} \& \emptyset \neq E_v \subseteq X) \iff x \in V_{[n]}(X)$.
$x \in V^{[n]}V^{[S]}(X) \iff$
$\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \subseteq V^{[S]}(X))$
$\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \& \langle n, x\rangle \in V^{[S]}(X))$
$\iff \exists v(\langle\langle n, x\rangle, v\rangle \in V^{[S]} \& E_v = \{\langle n+1, x\rangle\} \subseteq X)$
$\iff \exists v(\langle x, v\rangle \in V^{[n+1]} \& E_v = \{\langle n+1, x\rangle\} \subseteq X) \iff x \in V^{[n+1]}(X)$.
We shall prove by induction the equivalence

$$x \in V^{[0]}(V^{[S]})^n V'(X) \iff x \in V_{[n]}(X). \qquad (*)$$

Indeed, $x \in V^{[0]}(V^{[S]})^0 V'(X) \iff x \in V^{[0]}V'(X) \iff x \in V_{[0]}(X)$.
Let us assume the equivalence $(*)$ is true. Then
$x \in V^{[0]}(V^{[S]})^{n+1} V'(X) \iff x \in V^{[n+1]}V'(X) \iff x \in V_{[n+1]}(X)$. $\dashv$

The next two corollaries are obvious.

Corollary 2 *The structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is equivalent to the structure $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$, where $V^{[0]}, V^{[S]}, V'$ is the c.e. sets from the previous Lemma.*

Corollary 3 *For any set A of naturals the set $\mathcal{P}(\omega)^A$ is finitely generated in the structure $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$ from the single element A.*

Proposition 8 *For any enumeration $\{V_0, V_1, \ldots\}$ of the family $\boldsymbol{W}$ the structure $\mathfrak{M}^A = \langle \mathcal{P}(\omega)^A; V_0, V_1, \ldots; Non\rangle$ admits a least enumeration $\langle \alpha, \mathfrak{B}\rangle$ such that $A \leq_e \langle \mathfrak{B}\rangle$.*

Proof: Let $\alpha^0 : N_0 \to \mathcal{P}(\omega)^A$ be defined as follows: $\alpha^0(\mathbf{p_n}) = V_n(A)$. Take α^0 as a basis of a normal enumeration $\langle \alpha, \mathfrak{B}\rangle$, where $\mathfrak{B} = \langle \omega; \varphi_0, \varphi_1, \ldots; \sigma\rangle$ and $\varphi_i(x)$ is a computable function of both variables i, x. According to Proposition 2 there exists an effective way for any x to find $y = \mathbf{p_n} \in N_0$ and a term τ such that $x = \tau_{\mathfrak{B}}(Y/y)$; thus $\alpha(x) = \tau_{\mathfrak{A}}(Y/\alpha(y)) = \tau_{\mathfrak{A}}(Y/\alpha^0(\mathbf{p_n})) = \tau_{\mathfrak{A}}(Y/V_n(A)) = \tau'_{\mathfrak{A}}(Y/A)$, where $\tau' = \tau(\mathbf{f}_n(Y))$.

Let us denote $V_A = \{n | \sigma(n) \cong 0\}$. Then, using the term τ' obtained above, $x \in V_A \iff \sigma(x) \cong 0 \iff \alpha(x) \neq \emptyset \iff \tau'_{\mathfrak{A}}(Y/A) \neq \emptyset \iff v' \in$

$[A]_{\mathfrak{M}^A}$ for the code v' of the term τ'. Thus, having in mind we can find v' effectively from x, we have proved $V_A \leq_m [A]_{\mathfrak{M}^A}$.

Analogously, let $v' \in [A]_{\mathfrak{M}^A}$, $\tau^{v'} = \tau^{v'}(Y)$ and n be a fixed natural such that $\alpha^0(\mathbf{p}_n) = V_n(A) = A$, where $y = \mathbf{p}_n \in N_0$. Then $\tau^{v'}_{\mathfrak{A}}(Y/A) = \alpha(\tau^{v'}_{\mathfrak{B}}(Y/y) \neq \emptyset$ and let $x = \tau^{v'}_{\mathfrak{B}}(Y/y)$. Then $\sigma(x) \cong 0$ and $x \in V_A$. Therefore, $[A]_{\mathfrak{M}^A} \leq_m V_A$.

Hence, $[A]_{\mathfrak{M}^A} \equiv_m V_A$ and $\langle \mathfrak{B} \rangle \equiv_e [A]_{\mathfrak{M}^A} \equiv_e V_A$.

Corollary 4 $W_A \leq_e V_A$

Proof: Let $V_{i_0} = V^{[0]}$, $V_{i_1} = V^{[S]}$ and $V_{i_2} = V'$ and consider the sequence of terms $\tau^{v(n)}$, where $\tau^{v(n)} = \mathbf{f}_{i_0}((\mathbf{f}_{i_1})^n(\mathbf{f}_{i_2}(X)))$. Here, $(\mathbf{f}_{i_1})^n$ means n times the term $\mathbf{f}_{i_1}$. Then it is easy to check that $n \in [A]_{\mathfrak{N}^A} \iff v(n) \in [A]_{\mathfrak{M}^A}$. Thus we have proved $[A]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{M}^A}$, hence $W_A \leq_e V_A$. $\dashv$ $\dashv$

Corollary 5 *The enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is the least one for the generalized structure $\overline{\mathfrak{N}^A} = \langle \mathcal{P}(\omega)^A; \boldsymbol{W}; Non \rangle$.*

References

[1] E.L. Post, Recursively enumerable sets of positive integers and their decision problems, Bull. Am. Math. Soc. 50, 1944, 284–316

[2] E.L. Post, Degrees of recursive unsolvability, Bull. Amer. Math. Sos. 54, 1948, 641–642

[3] R.M. Friedberg, H. Rogers, Reducibility and completness for sets of integers, **Zeit. Math. Log. Grund. Math.** 5(1959) 117-125

[4] J. Myhill, Note on degrees of partial functions, Proc. Am. Math. Soc., 12, 1961, 519–521

[5] G.E. Sacks, Degrees of Unsolvabilities, Ann. of Math. Stud. 55, Princeton University Press, Princeton, NJ, 1966

[6] H. Rogers, Theory of Recursive Functions and Effective Computability, McGraw-Hill, New York, 1967

[7] C. Jockusch, R.M. Solovay, Fixed-point of jump-preserving automorphisms of degrees, Isr. J. Math. 26, 1977, 91–94

[8] L. Richter, On automorphisms of the degrees that preserve jumps, Isr. J. Math. 32, 1979, 27–31

[9] R.L. Epstein, Degrees of unsolvability: structure and theory, Springer Lect. Not. Math. 759, 1979

[10] T.A. Slaman, W.H. Woodin, Illinois J. Math., 30, 1986, 320–334

[11] I.N. Soskov, H. Ganchev, The jump operator on the ω-enumeration degres, to appear in Annals of Pure and Applied Logic

[12] A.V. Ditchev, Least enumerations of unary structures, Submitted for publication

[13] A.V. Ditchev, Least enumerations of partial structures, Accepted for publication in J. Logic and Computation (Special issue of Logic Colloquium 2009)

[14] I.N. Soskov, Definability via enumerations, J. Symb. Log., 54, 1989, 428–440.

[15] A.V. Ditchev, On Skordev's Conjecture (in Russian), Algebra & Logic 24, No 4 (1985), 379–391.

[16] S.B. Cooper, Local Degree Theory, Handbook of Computability Theory, Elsevier, Amsterdam, 1999, 123–153