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ON THE DURATION DOMAINS FOR THE INTERVAL TEMPORAL LOGIC

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The duration domains for the Interval Temporal Logic are characterized as the positive cones of the right-ordered groups.

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In 1985 B. Moszkowski [7] introduced a logical system called Interval Temporal Logic. Its semantics, proposed by B. Dutertre [2] in 1995, uses a kind of structures called *duration domains*. The same kind of structures have been used later also by D. Guelev for the semantics of other logical systems (cf., for example, [4, 5]). The structures in question can be defined as triples $(D, +, 0)$, where D is a set, $+$ is a binary operation in D, 0 is an element of D and the following five axioms are identically satisfied in D:

(D1) $(x+y)+z = x+(y+z)$,
(D2) $x+0 = 0+x = x$,
(D3) $x+z = y+z \Rightarrow x = y, \quad z+x = z+y \Rightarrow x = y$,
(D4) $x+y = 0 \Rightarrow x = y = 0$,
(D5) $\exists z(x+z = y \lor y+z = x), \quad \exists z(z+x = y \lor z+y = x)$.

The aim of the present paper is to characterize the duration domains as the positive cones of the right-ordered groups. This will be done by proving theorems 1 and 2 below.

A *right-ordered group* (cf. [1, 6]) is a structure $(G, +, 0, -, \geq)$, where $(G, +, 0, -)$ is a group (not necessarily abelian), $+$, 0, $-$ being respectively the binary group operation, the neutral element of the group and the unary operation of constructing the inverse element, and $\geq$ is a linear ordering in G such that for all x, y, z in G the following implication holds:

$$x \geq y \Rightarrow x + z \geq y + z$$

(as in [6], we assume the orderings reflexive, although the orderings in [1] are assumed to be irreflexive). *The positive cone* of such a structure is the set of all elements x of G that satisfy the condition $x \geq 0$. If P is the positive cone of a right-ordered group $(G, +, 0, -, \geq)$, then the following three conditions are satisfied for all x and y in G:

(P1) $x \in P \wedge -x \in P \Rightarrow x = 0$,
(P2) $x \in P \wedge y \in P \Rightarrow x + y \in P$,
(P3) $x \in P \vee -x \in P$.

Conversely, whenever $(G, +, 0, -)$ is a group and P is a subset of G with the properties (P1)–(P3), then a binary relation $\geq$ in G exists such that $(G, +, 0, -, \geq)$ is a right-ordered group with positive cone P.

Theorem 1. *Let $(G, +, 0, -, \geq)$ be a right-ordered group and P be its positive cone. Let $+_P$ be the restriction of the operation $+$ to P^2. Then $(P, +_P, 0)$ is a duration domain.*

Proof. The element 0 of G belongs to P by (P3), hence, taking into account also (P2), we may consider the structure $(P, +_P, 0)$. This structure obviously satisfies the axioms (D1)–(D3), and (D4) follows immediately from the property (P1). To verify (D5), suppose x and y are some elements of P. If we set $u = (-x) + y$, then the equalities $x + u = y$ and $y + (-u) = x$ hold, and, since some of the elements u and $-u$ belongs to P by (P3), this establishes the first statement of (D5). The second one can be established in a similar way. □

Remark. Under the assumptions of the above theorem, if the considered group is not abelian, then the operation $+_P$ is not commutative.[1] In fact, let x and y be elements of G such that $x + y \neq y + x$. By (P3) some of the elements x and $-x$ belongs to P and also some of the elements y and $-y$ belongs to P. Therefore it is sufficient to establish the inequalities

$$x + (-y) \neq (-y) + x, \quad (-x) + y \neq y + (-x), \quad (-x) + (-y) \neq (-y) + (-x).$$

To prove the first one, we suppose the equality $x + (-y) = (-y) + x$ and get $y + (x + (-y)) + y = y + ((-y) + x) + y$, i.e. $y + x = x + y$. In a similar way we

[1] Since there are non-abelian right-ordered groups (examples of such groups can be found, for instance, in [1] and [3, ch. 2]), this implies the existence of a duration domain with non-commutative addition operation.

show the impossibility of the equality $(-x)+y=y+(-x)$. Finally, if we suppose that $(-x)+(-y)=(-y)+(-x)$, then we get $-((-x)+(-y))=-((-y)+(-x))$, and this leads again to the contradictory equality $y+x=x+y$.

Theorem 2. *Any duration domain can be obtained in the way from* Theorem 1 *at a convenient choice of some right-ordered group* $(G,+,0,-,\geq)$.

Proof. Let $(D,+,0)$ be a duration domain. To each element s of $D\setminus\{0\}$ we make to correspond an object $\overline{s}$ not belonging to D in such a way that $\overline{s}\neq\overline{t}$ whenever s and t are distinct elements of $D\setminus\{0\}$. Then we set

$$G=D\cup\{\overline{s}\mid s\in D\setminus\{0\}\},$$

and we define the inverse element of any element of G by setting $-0=0$ and

$$-s=\overline{s},\ -\overline{s}=s$$

for any s in $D\setminus\{0\}$. We extend the binary operation $+$ from D to G by stipulating the equalities

$$(z+s)+\overline{s}=z,\ \ x+\overline{t+x}=\overline{t},\ \ \overline{s}+(s+z)=z,\ \ \overline{y+t}+y=\overline{t},\ \ \overline{s}+\overline{t}=\overline{t+s}$$

for all x,y,z in D and all s,t in $D\setminus\{0\}$.[2] It follows immediately that

$$0+\overline{t}=\overline{t}+0=\overline{t},\ \ s+\overline{s}=\overline{s}+s=0$$

for all s,t in $D\setminus\{0\}$, hence

$$0+u=u+0=u,\ \ u+(-u)=(-u)+u=0$$

for all u in G. If we denote the set D by P, then the properties (P1)–(P3) will be obviously present. Therefore the proof will be completed if we show that the operation $+$ in G is associative. This reduces to showing that for all p,q,r in D the following seven implications hold:

(A1) $r\neq 0\ \Rightarrow\ (p+q)+\overline{r}=p+(q+\overline{r})$,
(A2) $q\neq 0\ \Rightarrow\ (p+\overline{q})+r=p+(\overline{q}+r)$,
(A3) $q\neq 0\wedge r\neq 0\ \Rightarrow\ (p+\overline{q})+\overline{r}=p+(\overline{q}+\overline{r})$,
(A4) $p\neq 0\ \Rightarrow\ (\overline{p}+q)+r=\overline{p}+(q+r)$,
(A5) $p\neq 0\wedge r\neq 0\ \Rightarrow\ (\overline{p}+q)+\overline{r}=\overline{p}+(q+\overline{r})$,
(A6) $p\neq 0\wedge q\neq 0\ \Rightarrow\ (\overline{p}+\overline{q})+r=\overline{p}+(\overline{q}+r)$,
(A7) $p\neq 0\wedge q\neq 0\wedge r\neq 0\ \Rightarrow\ (\overline{p}+\overline{q})+\overline{r}=\overline{p}+(\overline{q}+\overline{r})$.

Thus the remaining part of the proof decomposes into the verifications of (A1)–(A7), where p, q, r are arbitrary elements of D.

[2] To show that the above definition is a legitimate one, we use all axioms (D1)–(D5); in particular, the axiom (D4) is used for showing that it is not possible to have simultaneously two equalities $z+s=x,\ s=t+x$ or two equalities $s=y+t,\ s+z=y$, where $x,y,z\in D$, $s,t\in D\setminus\{0\}$, and the axiom (D5) is used for showing that the extension is defined for any pair of elements of G.

Verification of (A1). Let $r \neq 0$. By axiom (D5), there is some element z of D such that $q = z + r$ or $r = z + q$. We choose such a z and we could assume that $z \neq 0$ in the second case, since if $z = 0$, then the second case is covered by the first one. If $q = z + r$, then

$$p + (q + \overline{r}) = p + ((z + r) + \overline{r}) = p + z, \quad (p + q) + \overline{r} = ((p + z) + r) + \overline{r} = p + z.$$

Consider now the case when $r = z + q$ and $z \neq 0$. Then

$$p + (q + \overline{r}) = p + (q + \overline{z + q}) = p + \overline{z}$$

and it is natural to apply the axiom (D5) again for choosing an element z' of G such that either $p = z' + z$ or $z = z' + p$, z' being distinct from 0 in the second case. If $p = z' + z$, then

$$(p + q) + \overline{r} = (z' + z + q) + \overline{r} = (z' + r) + \overline{r} = z', \quad p + (q + \overline{r}) = (z' + z) + \overline{z} = z'.$$

Otherwise, i.e. when $z = z' + p$ and $z' \neq 0$, we have

$$(p + q) + \overline{r} = (p + q) + \overline{z' + (p + q)} = \overline{z'}, \quad p + (q + \overline{r}) = p + \overline{z' + p} = \overline{z'}.$$

Verification of (A2). Let $q \neq 0$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. Choosing such a z, we shall have

$$(p + \overline{q}) + r = ((z + q) + \overline{q}) + r = z + r$$

in the first case and

$$(p + \overline{q}) + r = (p + \overline{z + p}) + r = \overline{z} + r$$

in the second one. By the same axiom, there is an element z' of D such that either $r = q + z'$ or $q = r + z'$, z' being distinct from 0 in the second case. Choosing such a z', we shall have

$$p + (\overline{q} + r) = p + (\overline{q} + (q + z')) = p + z'$$

in the first case and

$$p + (\overline{q} + r) = p + (\overline{r + z'} + r) = p + \overline{z'}$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $p = z + q$, $r = q + z'$. Then

$$(p + \overline{q}) + r = z + q + z', \quad p + (\overline{q} + r) = z + q + z'.$$

Combination 1.2: $p = z + q$, $q = r + z'$, $z' \neq 0$. Then

$$p + (\overline{q} + r) = ((z + r) + z') + \overline{z'} = z + r = (p + \overline{q}) + r.$$

Combination 2.1: $q = z + p$, $z \neq 0$, $r = q + z'$. Then

$$(p + \overline{q}) + r = \overline{z} + (z + (p + z')) = p + z' = p + (\overline{q} + r).$$

Combination 2.2: $q = z + p,\ z \neq 0,\ q = r + z',\ z' \neq 0$. Then $z + p = r + z'$. By axiom (D5), there is an element z'' of D such that either $r = z + z''$ or $z = r + z''$, z'' being distinct from 0 in the second case. In the first case we get

$$p = z'' + z',\ \ (p + \overline{q}) + r = \overline{z} + (z + z'') = z'',\ \ p + (\overline{q} + r) = (z'' + z') + \overline{z'} = z''.$$

In the second one we have

$$z'' + p = z',\ \ (p + \overline{q}) + r = \overline{r + z''} + r = \overline{z''},\ \ p + (\overline{q} + r) = p + \overline{z'' + p} = \overline{z''}.$$

Verification of (A3). Let $q \neq 0$, $r \neq 0$. Then $p + (\overline{q} + \overline{r}) = p + \overline{r + q}$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. In the first case we get

$$(p + \overline{q}) + \overline{r} = ((z + q) + \overline{q}) + \overline{r} = z + \overline{r},\ \ p + (\overline{q} + \overline{r}) = z + (q + \overline{r + q}) = z + \overline{r}.$$

In the second one we have

$$(p + \overline{q}) + \overline{r} = (p + \overline{z + p}) + \overline{r} = \overline{z} + \overline{r} = \overline{r + z},\ \ p + (\overline{q} + \overline{r}) = p + \overline{(r + z) + p} = \overline{r + z}.$$

Verification of (A4). Similar to the verification of (A1).

Verification of (A5). Let $p \neq 0$, $r \neq 0$. By Axiom (D5), there is an element z of D such that either $q = p + z$ or $p = q + z$, z being distinct from 0 in the second case. Choosing such a z, we shall have

$$(\overline{p} + q) + \overline{r} = (\overline{p} + (p + z)) + \overline{r} = z + \overline{r}$$

in the first case and

$$(\overline{p} + q) + \overline{r} = (\overline{q + z} + q) + \overline{r} = \overline{z} + \overline{r} = \overline{r + z}$$

in the second one. By the same axiom, there is an element z' of D such that either $q = z' + r$ or $r = z' + q$, z' being distinct from 0 in the second case. Choosing such a z', we shall have

$$\overline{p} + (q + \overline{r}) = \overline{p} + ((z' + r) + \overline{r}) = \overline{p} + z'$$

in the first case and

$$\overline{p} + (q + \overline{r}) = \overline{p} + (q + \overline{z' + q}) = \overline{p} + \overline{z'} = \overline{z' + p}$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $q = p + z,\ q = z' + r$. Then $p + z = z' + r$. By Axiom (D5), there is an element z'' of D such that either $z = z'' + r$ or $r = z'' + z$, z'' being distinct from 0 in the second case. In the first case we get

$$p + z'' = z',\ \ (\overline{p} + q) + \overline{r} = (z'' + r) + \overline{r} = z'',\ \ \overline{p} + (q + \overline{r}) = \overline{p} + (p + z'') = z''.$$

In the second one we have

$$p = z' + z'',\ \ (\overline{p} + q) + \overline{r} = z + \overline{z'' + z} = \overline{z''},\ \ \overline{p} + (q + \overline{r}) = \overline{z' + z''} + z' = \overline{z''}.$$

Combination 1.2: $q = p + z,\ r = z' + q,\ z' \neq 0$. Then

$$(\overline{p} + q) + \overline{r} = z + \overline{(z' + p) + z} = \overline{z' + p} = \overline{p} + (q + \overline{r}).$$

Combination 2.1: $p = q + z,\ z \neq 0,\ q = z' + r$. Then

$$\overline{p} + (q + \overline{r}) = \overline{z' + (r + z)} + z' = \overline{r + z} = (\overline{p} + q) + \overline{r}.$$

Combination 2.2: $p = q + z,\ z \neq 0,\ r = z' + q,\ z' \neq 0$. Then

$$(\overline{p} + q) + \overline{r} = \overline{z' + q + z},\ \ \overline{p} + (q + \overline{r}) = \overline{z' + q + z}.$$

Verification of (A6). Similar to the verification of (A3).

Verification of (A7). Let $p \neq 0,\ q \neq 0,\ r \neq 0$. Then

$$(\overline{p} + \overline{q}) + \overline{r} = \overline{q + p} + \overline{r} = \overline{r + q + p},\ \ \overline{p} + (\overline{q} + \overline{r}) = \overline{p} + \overline{r + q} = \overline{r + q + p}. \ \square$$

APPENDIX

The proof of Theorem 2 makes use of the existence of some set that has the same cardinality as $D \setminus \{0\}$ and does not meet D. The existence of such a set can be obtained as a particular case of the statement that for any sets A and B there is a set having the same cardinality as A and not meeting B. This statement follows immediately from certain facts of the cardinal arithmetic, but some of them in the final analysis are based on the Axiom of Choice. Here is a direct proof of the statement without using that axiom. Let

$$C = (A \times \mathcal{P}(B)) \cap B,$$

where $\mathcal{P}(B)$ is the set of the subsets of B. Let f be the projection mapping of C into $\mathcal{P}(B)$ defined by the equality

$$f(x, Y) = Y.$$

Since C is a subset of B, the range of f is a proper subset of $\mathcal{P}(B)$ (as the well-known diagonal argument shows, the set $\{z \in C \mid z \notin f(z)\}$ is an element of $\mathcal{P}(B)$ not belonging to the range of f). If Y_0 is an element of $\mathcal{P}(B) \setminus \mathrm{range}(f)$, then the set $A \times \{Y_0\}$ does not meet B, and clearly $A \times \{Y_0\}$ has the same cardinality as A.

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