On computability in topological spaces

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Overview

1. Computability and approximation systems in topological spaces
   - The computability notion
   - Approximation systems and their relation to computability
2. Computability and approximation systems in metric spaces
   - Computability and topologic approximation systems
   - Metric approximation systems

Computable functions in topological spaces

Let:

- $\mathbf{X}$ be a topological space with carrier X,
  $\mathcal{U} = \{U_i\}_{i \in I}$ with $I \subseteq \mathbb{N}$ be an indexed base of $\mathbf{X}$;
- $\mathbf{Y}$ be a topological space with carrier Y,
  $\mathcal{V} = \{V_j\}_{j \in J}$ with $J \subseteq \mathbb{N}$ be an indexed base of $\mathbf{Y}$;
- $f : E \to Y$, where $E \subseteq X$.

For any $x \in X$ and any $y \in Y$, we set

$$[x]_{\mathcal{U}} = \{i \in I \,|\, x \in U_i\}, \quad [y]_{\mathcal{V}} = \{j \in J \,|\, y \in V_j\}.$$

The function f will be called $(\mathcal{U}, \mathcal{V})$-*computable* if a recursive operator Γ exists such that, for any $x \in E$, Γ transforms all enumerations of $[x]_{\mathcal{U}}$ into enumerations of $[f(x)]_{\mathcal{V}}$.

Characterization by means of enumeration operators

Proposition (folklore?)

The function f is $(\mathcal{U},\mathcal{V})$-computable iff there exists an enumeration operator F such that $[f(x)]_{\mathcal{V}} = F([x]_{\mathcal{U}})$ for any $x \in E$.

Let $D_0, D_1, D_2, \ldots$ be an effective enumeration of the family of all finite subsets of $\mathbb{N}$.

A reformulation of the above proposition

The function f is $(\mathcal{U},\mathcal{V})$-computable iff there exists a r.e. subset W of $\mathbb{N}^2$ such that, for all $x \in E$ and all $j \in J$,

$$f(x) \in V_j \Leftrightarrow \exists u \left((u,j) \in W \,\&\, D_u \subseteq I \,\&\, x \in \bigcap_{i \in D_u} U_i\right).$$

One may regard W as a formal system with rules of the form

$$\frac{x \in U_{i_1}, \ldots, x \in U_{i_k}}{f(x) \in V_j},$$

where $i_1, \ldots, i_k \in I$, $j \in J$, and x ranges over E. What about using one-premise rules only?

Approximation systems

Definition

A $(\mathcal{U},\mathcal{V})$-*approximation system* for the function f is a subset R of the Cartesian product $I \times J$ such that, for all $x \in E$ and all $j \in J$,

$$f(x) \in V_j \Leftrightarrow \exists i\,\big((i,j) \in R \;\&\; x \in U_i\big).$$

If R is a r.e. set then it could be regarded as a formal system of the kind from the previous slide, but with one-premise rules only.

Remark. A subset R of $I \times J$ is an approximation system for f iff $f^{-1}(V_j) = \bigcup\{\, U_i \cap E \mid (i,j) \in R \,\}$ for all $j \in J$. If we set $\mathcal{U}^{-1} = \{(x,i) | i \in I \,\&\, x \in U_i\}$, $\mathcal{V}^{-1} = \{(y,j) | j \in J \,\&\, y \in V_j\}$ then R is an approximation system for f iff the following diagram in the category of relations is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \mathcal{U}^{-1}\restriction E}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{V}^{-1}} \\ I & \xrightarrow[\;R\;]{} & J \end{array}$$

Approximation systems and continuity

Proposition

A $(\mathcal{U}, \mathcal{V})$-approximation system for f exists iff f is continuous.

Proof. ($\Rightarrow$) If R is a $(\mathcal{U}, \mathcal{V})$-approximation system for f then f is continuous by the equality

$$f^{-1}(V_j) = \bigcup \{\, U_i \cap E \mid (i,j) \in R \,\}$$

from the remark on the previous slide. ($\Leftarrow$) If f is continuous then

$$\big\{ (i,j) \in I \times J \;\big|\; U_i \cap E \subseteq f^{-1}(V_j) \big\}$$

is a $(\mathcal{U}, \mathcal{V})$-approximation system for f. □

$(\mathcal{U},\mathcal{V})$-computability of the functions having r.e. $(\mathcal{U},\mathcal{V})$-approximation systems

Theorem

If there exists a r.e. $(\mathcal{U},\mathcal{V})$-approximation system for f then f is $(\mathcal{U},\mathcal{V})$-computable.

Proof. If we set $W = \{(u,j)\in\mathbb{N}^2 \,|\, \exists i((i,j)\in R \,\&\, D_u=\{i\})\}$ then $\exists u\,((u,j)\in W \,\&\, D_u\subseteq I \,\&\, x\in\bigcap_{i\in D_u} U_i) \Leftrightarrow \exists i\,((i,j)\in R \,\&\, x\in U_i)$. □

Remark. The converse theorem is not generally true. To construct a counter-example to it, we partition $\mathbb{N}$ into disjoint two-elements sets $P_0, P_1, P_2, \ldots$ such that $\max P_x - \min P_x = x+1$ for any $x\in\mathbb{N}$ and no recursive choice function exists for this partition. Let $X = Y = I = J = \mathbb{N}$. For any $x\in\mathbb{N}$, let $U_i=\{x\}$ for $i\in P_x$, $V_x=\{x\}$. Then the function $\mathrm{id}_{\mathbb{N}}$ is $(\mathcal{U},\mathcal{V})$-computable (via $W = \{(u,j)\in\mathbb{N}^2 \,|\, \exists i(D_u=\{i,i+j+1\})\}$), but no r.e. $(\mathcal{U},\mathcal{V})$-approximation system exists for it.

Indexed bases with computable meet operation

Definition

The indexed base $\mathcal{U} = \{U_i\}_{i \in I}$ will be said to be *with computable meet operation* if a r.e. subset H of I^3 exists such that $U_{i_1} \cap U_{i_2} = \bigcup \{ U_i \mid (i_1, i_2, i) \in H \}$ for all $i_1, i_2 \in I$.

The computability of the meet operation is essentially the main requirement in the definition of computable topological space in joint papers of Weihrauch and Grubba from 2007 and 2009.

Lemma

If $\mathcal{U}$ is with computable meet operation then a r.e. subset H^* of $\mathbb{N}^2$ exists such that $D_u \cap \{i\} \subseteq I$, whenever $(u, i) \in H^*$, and $\bigcap_{i \in D_U} U_i = \bigcup \{ U_i \mid (u, i) \in H^* \}$ for all $u \in \mathbb{N}$ with $D_u \subseteq I$.

A partial conversion of the previous theorem

Theorem (improves a result of Korovina and Kudinov)

Let the indexed base $\mathcal{U}$ be with computable meet operation, and J be a r.e. set. If the function f is $(\mathcal{U},\mathcal{V})$-computable then there exists a r.e. $(\mathcal{U},\mathcal{V})$-approximation system for f.

Proof. Let H^* be a r.e. subset of $\mathbb{N}^2$ with the properties from the Lemma, i.e. $D_u \cap \{i\} \subseteq I$, whenever $(u,i) \in H^*$, and

$$\bigcap_{i\in D_U} U_i = \bigcup \{ U_i \mid (u,i) \in H^* \}$$

for all $u \in \mathbb{N}$ with $D_u \subseteq I$. Let W be a r.e. subset of $\mathbb{N}^2$ such that

$$f(x) \in V_j \Leftrightarrow \exists u \left((u,j) \in W \,\&\, D_u \subseteq I \,\&\, x \in \bigcap_{i\in D_u} U_i \right)$$

for all $x \in \mathbb{N}$ and all $j \in J$. Then, for all $x \in \mathbb{N}$ and all $j \in J$,

$$f(x) \in V_j \Leftrightarrow \exists i \big(\exists u((u,j) \in W \,\&\, (u,i) \in H^*) \,\&\, x \in U_i\big),$$

hence $\{(i,j) \mid j \in J \,\&\, \exists u((u,j) \in W \,\&\, (u,i) \in H^*)\}$ is a r.e. approximation system for f. □

Topological and metrical computability in metric spaces

Suppose now the topologies of **X** and **Y** are generated by a metric d in X and a metric e in Y, respectively. Let $\alpha : K \to X$, $\beta : L \to Y$, where $K \subseteq \mathbb{N}$, $L \subseteq \mathbb{N}$, be such that $\mathrm{rng}(\alpha)$ is dense in (X, d), and $\mathrm{rng}(\beta)$ is dense in (Y, e). Corresponding indexed bases $\mathcal{U}$ of **X** and $\mathcal{V}$ of **Y** are defined as follows: we set

$$I = \{\langle k, m\rangle | k \in K, m \in \mathbb{N}\},\ J = \{\langle l, n\rangle | l \in l, n \in \mathbb{N}\},$$

where $(s, t) \mapsto \langle s, t\rangle$ is a computable bijection from $\mathbb{N}^2$ to $\mathbb{N}$,
$U_{\langle k,m\rangle} = \{x \in X | d(x, \alpha(k)) < 2^{-m}\}$, $V_{\langle l,n\rangle} = \{y \in Y | d(y, \beta(l)) < 2^{-n}\}$.
Besides the topological $(\mathcal{U}, \mathcal{V})$-computability, a metric one called (α, β)-computability can be considered.

Definition

The function f will be called (α, β)-*computable* if a recursive operator Γ exists such that, whenever $x \in E$, $\kappa : \mathbb{N} \to K$ and $d(\alpha(\kappa(m)), x) < 2^{-m}$ for all $m \in \mathbb{N}$, then $\Gamma(\kappa) : \mathbb{N} \to L$ and $e(\beta(\Gamma(\kappa)(n)), f(x)) < 2^{-n}$ for all $n \in \mathbb{N}$.

The interrelation between $(\mathcal{U}, \mathcal{V})$- and (α, β)-computability

In general, none of the two computabilities in question implies the other one. However, it is known that they are equivalent in the case when (X, d, α) and (Y, e, β) are semi-computable metric spaces, i.e. the sets

$$\left\{ (k_1, k_2, p, q) \in K^2 \times \left(\mathbb{N}^+\right)^2 \,\middle|\, d(\alpha(k_1), \alpha(k_2)) < \frac{p}{q} \right\},$$

$$\left\{ (l_1, l_2, p, q) \in L^2 \times \left(\mathbb{N}^+\right)^2 \,\middle|\, e(\beta(l_1), \beta(l_2)) < \frac{p}{q} \right\}$$

are recursively enumerable (cf. K. Weihrauch, Log. Meth. Comput. Sci. **9(3:5)** (2013)). In particular, the two computabilities are equivalent in the case when $X = Y = \mathbb{R}$, both d and e are the usual metric in $\mathbb{R}$, α and β are computable enumerations of $\mathbb{Q}$.

Topological approximation systems in the metric case

In the considered metric case, a subset R of $I \times J$ is a $(\mathcal{U}, \mathcal{V})$-approximation system for f iff

$$e(\beta(l), f(x)) < 2^{-n} \Leftrightarrow \exists (k,m) \in K \times \mathbb{N}((\langle k,m\rangle, \langle l,n\rangle) \in R \;\&\; d(\alpha(k),x) < 2^{-m}). \quad (1)$$

for all $x \in E$, all $l \in L$ and all $n \in \mathbb{N}$. If $R \subseteq I \times J$, and we set $S = \{(k,m,l,n) \in \mathbb{N}^4 | (\langle k,m\rangle, \langle l,n\rangle) \in R\}$, then $S \subseteq K \times \mathbb{N} \times L \times \mathbb{N}$ and the condition (1) is equivalent to the following one:

$$e(\beta(l), f(x)) < 2^{-n} \Leftrightarrow \exists k,m((k,m,l,n) \in S \;\&\; d(\alpha(k),x) < 2^{-m}). \quad (2)$$

Definition

A *topological* (α, β)*-approximation system* for f is any subset S of $K \times \mathbb{N} \times L \times \mathbb{N}$ such that (2) holds for all $x \in E$, all $l \in L$ and all $n \in \mathbb{N}$.

Clearly the existence of a $(\mathcal{U}, \mathcal{V})$-approximation system for f is equivalent to the existence of an (α, β)-approximation system for f (the same for r.e. approximation systems).

(α, β)-computability and topological approximation systems

Theorem

Let (X, d, α) and (Y, e, β) be semi-computable metric spaces. Then the function f is metrically (α, β)-computable iff there exists a r.e. topological (α, β)-approximation system for f.

Proof. Consider the following conditions: (a) f is (α, β)-computable, (b) f is $(\mathcal{U}, \mathcal{V})$-computable, (c) there exists a r.e. $(\mathcal{U}, \mathcal{V})$-approximation system for f, (d) there exists a r.e. topological (α, β)-approximation system for f. As seen from the previous slides, (a) and (c) are equivalent to (b) and (d), respectively. In addition, (b) is equivalent to (c), since the semi-computability of (X, d, α) implies that $\mathcal{U}$ is with computable meet operation (cf. K. Weihrauch, Log. Meth. Comput. Sci. **9(3:5)** (2013)), and the semi-computability of (Y, e, β) implies that L is a r.e. set, hence J is also a r.e. set. $\square$

A simple characterization of the computable functions in $\mathbb{R}$

Suppose now that $X = Y = \mathbb{R}$, both d and e are the usual metric in $\mathbb{R}$, $K = L = \mathbb{N}$, α and β are computable enumerations of $\mathbb{Q}$. Any subset T of $\mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\times\mathbb{N}$ such that, for all $x\in E$, $b\in\mathbb{Q}$, $n\in\mathbb{N}$,

$$|b - f(x)| < 2^{-n} \Leftrightarrow \exists a\, \exists m\big((a, m, b, n) \in T \,\&\, |a - x| < 2^{-m}\big)$$

will be called a *topological approximation net* for f.

Theorem

The function f is (α,β)-computable iff there exists a r.e. topological approximation net for f.

Proof. The mappings $S \mapsto \{(\alpha(k), m, \beta(l), n) | (k, m, l, n) \in S\}$ and $T \mapsto \{(k, l, m, n) | (\alpha(k), m, \beta(l), n) \in T\}$ transform, respectively, r.e. topological (α,β)-approximation systems for f into r.e. topological approximation nets for f and vice versa. □

Some examples of r.e. topological approximation nets (I)

Example 1. Let $E = \mathbb{R}$, $f(x) = x^2$ for all $x \in E$. Let

$$T = \left\{(a, m, b, n) \in \mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\times\mathbb{N} \;\middle|\; 2^{-m}(2|a|+2^{-m}) + \left|b - a^2\right| \leq 2^{-n}\right\}.$$

Then T is a r.e. topological approximation net for f.

Example 2. Let $E = \mathbb{R} \setminus \{0\}$, $f(x) = \dfrac{1}{x}$ for all $x \in E$. Let T be the set of all $(a, m, b, n) \in \mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\times\mathbb{N}$ satisfying the inequalities 0

$$|a| > 2^{-m}, \quad \frac{2^{-m}}{|a|(|a| - 2^{-m})} + \left|b - \frac{1}{a}\right| \leq 2^{-n}.$$

Then T is a r.e. topological approximation net for f.

Some examples of r.e. topological approximation nets (II)

Example 3. Let $E = \mathbb{R}$, $f(x) = \cos x$ for all $x \in E$. For any $k \in \mathbb{N}$, let $\sigma_k : \mathbb{Q} \to \mathbb{Q}$ be defined by

$$\sigma_k(a) = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots + (-1)^{k-1}\frac{a^{2k-2}}{(2k-2)!} + (-1)^k \frac{a^{2k}}{2(2k)!},$$

and T_k be the set of all $(a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N}$ such that

$$a^2 \leq (2k+1)(2k+2), \quad 2^{-m} + \frac{a^{2k}}{2(2k)!} + |b - \sigma_k(a)| \leq 2^{-n}.$$

Then $\bigcup_{k=0}^{\infty} T_k$ is a r.e. topological approximation net for f.

Example 4. Let again $E = \mathbb{R}$, and let f be the van der Waerden's non-differentiable function defined by $f(x) = \sum_{k=0}^{\infty} 4^{-k} u_0(4^k x)$, where $u_0(t)$ is the distance from t to $\mathbb{Z}$. Then

$$\left\{ (a, m, b, n) \in \mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\times\mathbb{N} \;\middle|\; \frac{m+1}{2^m} + \frac{1}{6 \cdot 4^m} + \left| b - \sum_{k=0}^{m} \frac{u_0(4^k a)}{4^k} \right| \leq 2^{-n} \right\}$$

is a r.e. topological approximation net for f.

Definition of the notion of metric approximation system

In the situation from the previous subsection, another kind of (α, β)-approximation system will be considered besides the topological ones.

Definition

A *metric* (α, β)*-approximation system for the function* f is any subset S of $K \times \mathbb{N} \times L \times \mathbb{N}$ such that, for any $x \in E$,

$$\forall (k, m, l, n) \in S\, (d(\alpha(k), x) < 2^{-m} \Rightarrow e(\beta(l), f(x)) < 2^{-n}),$$
$$\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall k \in K\, \big(d(\alpha(k), x) < 2^{-m} \Rightarrow \exists l\big((k, m, l, n) \in S\big)\big).$$

Both the topological and the metrical (α, β)-approximation systems for S can be regarded as systems of rules of the form

$$\frac{d(\alpha(k), x) < 2^{-m}}{e(\beta(l), f(x)) < 2^{-n}},$$

where $k \in K$, $l \in L$, $m, n \in \mathbb{N}$ and x ranges over E, but the ways they are connected with the function f are somewhat different.

The existence of a r.e. metric (α, β)-approximation system implies (α, β)-computability

Let S be a r.e. metric (α, β)-approximation system for the function f. A 5-argument primitive recursive function χ can be found such that

$$(k, m, l, n) \in S \Leftrightarrow \exists s \in \mathbb{N} \big(\chi(k, m, l, n, s) = 0 \big)$$

for all $k, m, l, n \in \mathbb{N}$. Let π_1, π_2, π_3 be unary primitive recursive functions such that

$$\{(\pi_1(t), \pi_2(t), \pi_3(t)) \mid t \in \mathbb{N}\} = \mathbb{N}^3,$$

and let the recursive operator Γ be defined as follows:

$$\Gamma(\kappa)(n) = \pi_2(\mu t[\,\chi(\kappa(\pi_1(t)), \pi_1(t), \pi_2(t), n, \pi_3(t)) = 0\,]).$$

It can be shown that, whenever $x \in E$, $\kappa : \mathbb{N} \to K$ and $d(\alpha(\kappa(m)), x) < 2^{-m}$ for all $m \in \mathbb{N}$, then $\Gamma(\kappa) : \mathbb{N} \to L$ and $e(\beta(\Gamma(\kappa)(n)), f(x)) < 2^{-n}$ for all $n \in \mathbb{N}$.

Some r.e. metric approximation systems fail to be topological ones

On two previous slides, several examples of r.e. topological approximation nets for real functions were given. The mapping $T \mapsto \{(k,l,m,n) | (\alpha(k), m, \beta(l), n) \in T\}$ transforms each of these approximation nets into a r.e. set which is both a topological (α, β)-approximation system and a metric (α, β)-approximation system for the corresponding function. However, any of the approximation nets in question has a subset whose image under the same mapping is a r.e. metric (α, β)-approximation system for the function in question without being a topological one. For instance, such is the situation with the subset

$$\{(a,m,b,n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N} \mid b = a^2 \ \& \ 2^{-m}(2|a| + 2^{-m}) \leq 2^{-n}\}$$

of the topological (α, β)-approximation system

$$\{(a,m,b,n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N} \mid 2^{-m}(2|a| + 2^{-m}) + |b - a^2| \leq 2^{-n}\}$$

from Example 1.

Some r.e. topological approximation systems fail to be metric ones

Let $X = Y = \mathbb{R}$, both d, e be the usual metric in $\mathbb{R}$, $K = L = \mathbb{N}$, α and β be computable enumerations of $\mathbb{Q}$. Let $E=\{0\}$, $f(0)=0$,

$$S = \left\{(k_0, 0, l, n) \,\middle|\, l \in \mathbb{N} \,\&\, n \in \mathbb{N} \,\&\, |\beta(l)| < 2^{-n}\right\},$$

where $k_0 \in \mathbb{N}$, $\alpha(k_0) = 0$. Then S is a r.e. topological (α, β)-approximation system for f. However, S is not a metric (α, β)-approximation system for f. Suppose it is. Then a natural number m must exist such that

$$\forall k \in \mathbb{N} \left(|\alpha(k)| < 2^{-m} \Rightarrow \exists l\big((k, m, l, n) \in S\big)\right),$$

hence k_0 must be the only $k \in \mathbb{N}$ with $|\alpha(k)| < 2^{-m}$, and this is a contradiction.

Transforming topological approximation systems into metric ones and vice versa

Theorem

Let S be a topological (α, β)-approximation system for the function f. Then the set of all $(k', m', l, n) \in \mathbb{N}^4$ satisfying the condition

$$k' \in K \,\&\, \exists k, m \left((k, m, l, n) \in S \,\&\, d(\alpha(k'), \alpha(k)) < 2^{-m} - 2^{-m'} \right)$$

is a metric (α, β)-approximation system for f.

Theorem

Let S be a metric (α, β)-approximation system for the function f. Then the set of all $(k, m, l', n') \in \mathbb{N}^4$ satisfying the condition

$$l' \in L \,\&\, \exists l, n \left((k, m, l, n) \in S \,\&\, e(\beta(l), \beta(l')) < 2^{-n'} - 2^{-n} \right)$$

is a topological (α, β)-approximation system for f.

Two corollaries

Corollary

If there exists a metric (α, β)-aproximation system for f then f is continuous. If there exists a r.e. metric (α, β)-aproximation system for f, and (Y, e, β) is a semi-computable metric space, then f is $(\mathcal{U}, \mathcal{V})$-computable.

Corollary

If (X, d, α) and (Y, e, β) are semi-computable metric spaces then the following conditions are equivalent:

- The function f is $\mathcal{U}, \mathcal{V}$-computable.
- The function f is (α, β)-computable.
- There exists a r.e. topological (α, β)-approximation system for f.
- There exists a r.e. metric (α, β)-approximation system for f.

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