Moschovakis Extension of Multi-Represented Spaces

Dimiter Skordev

Sofia University
Faculty of Mathematics and Informatics
Sofia, Bulgaria

CCA 2018, Kochel am See
August 8, 2018

The motivation

The present paper is related to certain works of Klaus Weihrauch, and the motivation is similar. Here is a quotation from one of them.

From the paper: Klaus Weihrauch. *The computable multi-functions on multi-represented sets are closed under programming.* J. of Univ. Computer Sci., **14**, 801–844, 2008.

The main motivation for this article came from the wish to make practical work in computable analysis easier. The results should simplify (or justify the current practice of) many proofs of computability in advanced applications. Although it does not intend to design a programming language this article can be considered as a step towards a higher level programming language for computable analysis.

A source for some notions that will be used

Some notions will be used from the following paper:

Yiannis N. Moschovakis. *Abstract first order computability. I.*
Trans. Amer. Math. Soc., **138**, 427–464, 1969.

The notion of Moschovakis extension

A quotation from Moschovakis' paper

Let B be an arbitrary set, let 0 be some object not in B, let

$$B^0 = B \cup \{0\}$$

We define the set B^* by the inductive clauses

$$\textit{if } x \in B^0, \ \textit{then } x \in B^*,$$

$$\textit{if } x, y \in B^*, \ \textit{then } (x, y) \in B^*.$$

Here (x, y) is the ordered pair of x and y and we assume that we have chosen a particular set-theoretic operation to represent the ordered pair so that no object in B^0 is an ordered pair. Thus if $z \in B^*$, then either $z \in B^0$ or $z = (x, y)$ with uniquely determined $x, y \in B^*$.

The above-mentioned object 0 will be denoted by o in this talk, thus $B^0 = B \cup \{o\}$ will hold. Let $B^+ = B^* \setminus B^0$. We will consider functions L^+ and R^+ from B^+ to B^* defined as follows: $L^+(z) = x$ and $R^+(z) = y$ if $z = (x, y)$ with $x, y \in B^*$. The identity mapping of any set Z will be denoted by I_Z.

Absolute prime computability

Certain relative computability notions for functions in B^* are introduced and studied in Moschovakis' paper. The considered functions are, in general, multi-valued. In the case of single-valued functions, one of these notions, namely absolute prime computability, seems to be able to cover any reasonable kind of deterministic computability by means of programs using some given functions.

The definition given by Moschovakis uses recursively an index construction described by means of a number of clauses. Functions of arbitrary number of variables are considered in this definition. On the other hand, it would be sufficient to confine ourselves to unary functions, because tuples of elements of B^* are representable by elements of B^*. For this case a simpler definition can be given in the spirit of functional programming.

Remark 1. Predicates on B^* can be represented by functions in B^* with values o and (o, o). For the representation of the natural numbers, *the Moschovakis injective mapping of* $\mathbb{N}$ *into* B^* can be used, i.e. the mapping ν defined inductively by setting

$$\nu(0) = o,\ \nu(k+1) = (\nu(k), o).$$

Basic functions for the simpler definition

Let $\mathcal{F}$ be the set of all partial multi-valued functions in B^* (they will be regarded as partial mappings of B^* into the set of the nonempty subsets of B^*, with the natural identification of the single-valued functions in B^* with such mappings).

We will use as basic functions the total single-valued functions $L, R \in \mathcal{F}$ defined as follows:

$$L(z) = L^+(z) \text{ and } R(z) = R^+(z) \text{ if } z \in B^+;$$
$$L(z) = R(z) = \begin{cases} o & \text{if } z = o, \\ (o, o) & \text{if } z \in B. \end{cases}$$

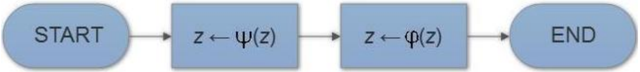
Moschovakis denotes these functions by π and δ, respectively, and adds the explanation below to their definition.

The difference in defining the components of z for the cases $z = 0$ and $z \in B$ is for technical reasons; one of its consequences is that the singleton $\{0\}$ will be a "primitive computable" subset of B^* without requiring a special axiom.

Basic operations for the simpler definition

Three binary operations in $\mathcal{F}$ will be the basic ones in the simpler definition: the usual relational composition specified below, as well as juxtaposition and a certain kind of while-iteration (they will be specified on the next slides).

For any φ, ψ in $\mathcal{F}$, their *composition* $\varphi\psi$ describes the action on the variable z of the following program:

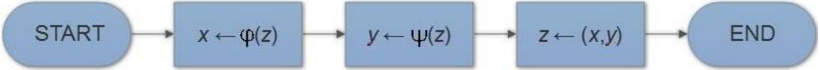


(all successful instances of termination are considered). Thus:

- $\mathrm{dom}(\varphi\psi) = \{z \in \mathrm{dom}(\psi) \mid \psi(z) \cap \mathrm{dom}(\varphi) \neq \varnothing\}$,
- $\varphi\psi(z) = \bigcup\{\varphi(z') \mid z' \in \psi(z) \cap \mathrm{dom}(\varphi)\}$ for all z in $\mathrm{dom}(\varphi\psi)$.

Juxtaposition in $\mathcal{F}$

For any φ, ψ in $\mathcal{F}$, their *juxtaposition* is the function which describes the action on the variable z of the following program:

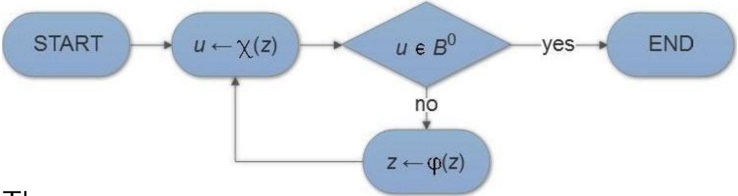


Thus:

- $\mathrm{dom}(\theta) = \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$,
- $\theta(z) = \varphi(z) \times \psi(z)$ for all z in $\mathrm{dom}(\theta)$.

Iteration in $\mathcal{F}$

This is the operation $\varphi, \chi \mapsto \iota$, where ι (*the iteration of* φ *controlled by* χ) describes the action on the variable z of the next program:



Thus:

- $z \in \mathrm{dom}(\iota)$ iff some finite sequence $z_0, z_1, \ldots, z_n$ of elements of $\mathrm{dom}(\chi)$ with $z_0 = z$ and $\chi(z_n) \cap B^0 \neq \varnothing$ exists such that $\chi(z_i) \cap B^+ \neq \varnothing$, $z_i \in \mathrm{dom}(\varphi)$ and $z_{i+1} \in \varphi(z_i)$ for all $i < n$,
- for any z in $\mathrm{dom}(\iota)$, $\iota(z)$ is the set of the last terms of the sequences $z_0, z_1, \ldots, z_n$ with the above properties.

Example 1

Let ι be the iteration of L controlled by L. Then $L^3\iota(z) = o$ for all z in B^*, the iteration of φ controlled by $L^3\iota$ is I_{B^*} for any φ in $\mathcal{F}$, and the iteration of I_{B^*} controlled by I_{B^*} is I_{B^0}.

The simpler definition

Definition 1

Let $\varphi_1, \ldots, \varphi_m$ be functions from $\mathcal{F}$. A function θ from $\mathcal{F}$ will be said to be *absolutely prime computable* in $\varphi_1, \ldots, \varphi_m$ if θ can be obtained from $\varphi_1, \ldots, \varphi_m, L, R$ by finitely many applications of the three above-mentioned operations.

Example 2

The constant function which maps all elements of B^* to o is absolutely prime computable in any $\varphi_1, \ldots, \varphi_m$ from B^*, and so are the mappings I_{B^*} and of I_{B^0}.

A proof of the equivalence of the above definition to the corresponding Moschovakis' one can be found, for instance, in the following book of mine:

Computability in Combinatory Spaces: An Algebraic Generalization of Abstract First Order Computability. Kluwer Academic Publishers, 1992.

Reducibility of iteration to a particular instance of it

Let $\varphi, \chi \in \mathcal{F}$, and let ι be the iteration of φ controlled by χ. If $\bar{\chi}$ is the juxtaposition of χ and I_{B^*} then

$$\iota = R\tilde{\iota}\bar{\chi},$$

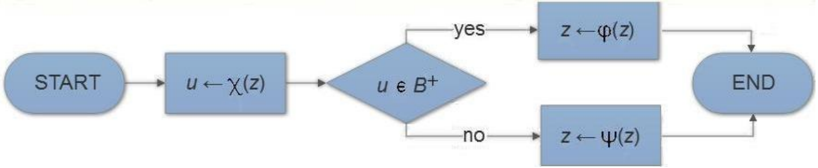
where $\tilde{\iota}$ is the iteration of $\bar{\chi}\varphi R$ controlled by L.

Hence iteration can be replaced in Definition 1 with iteration controlled by L if one adds I_{B^*} to the list

$$\varphi_1, \ldots, \varphi_m, L, R.$$

On branching in $\mathcal{F}$

There are some natural operations of *branching* in $\mathcal{F}$, for instance the ternary operation $\chi, \varphi, \psi \mapsto (\chi \supset \varphi, \psi)$, where $(\chi \supset \varphi, \psi)$ is the action on the variable z of the following program:



Using a technique by Böhm and Jacopini, we may express this operation through the chosen basic functions and operations, thus showing that $(\chi \supset \varphi, \psi)$ is uniformly absolutely prime computable in the functions χ, φ, ψ. Indeed, if the functions $\bar{\chi}, \bar{\varphi}, \bar{\psi}$ from $\mathcal{F}$ are defined by setting

$$\bar{\chi}(z) = \chi(z) \times \{((o,o),z)\},$$

$$\bar{\varphi}(z) = \{o\} \times \left(\{o\} \times \varphi R^2(z)\right), \ \bar{\psi}(z) = \{o\} \times \psi R(z)$$

then $(\chi \supset \varphi, \psi) = R\tilde{\psi}R\tilde{\varphi}\bar{\chi}$, where $\tilde{\varphi}$ is the iteration of $\bar{\varphi}$ controlled by L, and $\tilde{\psi}$ is the iteration of $\bar{\psi}$ controlled by L.

Iteration in $\mathcal{F}$ as least fixed point

Definition 2

We define the partial ordering $\geq$ in $\mathcal{F}$ by adopting that $\tau_1 \geq \tau_2$ iff $\mathrm{dom}(\tau_1) \supseteq \mathrm{dom}(\tau_2)$ and $\tau_1(z) \supseteq \tau_2(z)$ for all $z \in \mathrm{dom}(\tau_2)$.

The following holds:

For any φ and χ in $\mathcal{F}$, if ι is the iteration of φ controlled by χ then $\iota = (\chi \supset \iota\varphi, I_{B^*})$. Moreover, ι is the least τ in $\mathcal{F}$ satisfying the inequality $\tau \geq (\chi \supset \tau\varphi, I_{B^*})$.

Absolute prime computability and recursion

First Recursion Theorem for absolute prime computability

Suppose $\varphi_1, \ldots, \varphi_m \in \mathcal{F}$, and Γ is a mapping of $\mathcal{F}$ into $\mathcal{F}$ such that $\Gamma(\tau)$ is absolutely prime computable in $\varphi_1, \ldots, \varphi_m, \tau$ uniformly with respect to τ when τ ranges over $\mathcal{F}$. Then the inequality $\tau \geq \Gamma(\tau)$ has a least solution, this solution is absolutely prime computable in $\varphi_1, \ldots, \varphi_m$, and it is a fixed point of Γ (the least one).

The case when $\varphi_1, \ldots, \varphi_m$ are single-valued and total is settled in Moschovakis' paper. The validity of the theorem in the general case follows, for instance, from results established in the book mentioned above.

The notion of multi-representation

There are several versions of the notion. In all of them, the multi-representations of a set Z are multi-valued functions with range Z. The difference between the versions concerns the admissible domains of these functions The most often used definition for multi-representation assumes a finite alphabet Σ is given which contains at least the symbols 0 and 1. This definition has two variants:

1. Any set of infinite sequences of symbols from Σ is admitted.
2. In addition, any set of finite sequences of such symbols is also admitted.

From the point of view of the corresponding computability notions, these two variants are equipollent, and the concrete choice of the alphabet Σ is inessential. They are equipollent in this respect also to another version – it does not use Σ, and the subsets of $\mathbb{N}^{\mathbb{N}}$ are the admissible domains of the multi-representations of Z (of course, $\mathbb{N}^{\mathbb{N}}$ is the set of all infinite sequences of natural numbers).

In the slides that follow, except for one of them, it does not matter which of the variants is accepted.

TTE computability w.r.t. multi-representations

We will consider TTE computability via single-valued realizations. For completeness sake, we give below some needed definitions.

Definition 3

Let $\alpha_1, \ldots, \alpha_n, \beta$ be multi-representations of the sets $X_1, \ldots, X_n, Y$, respectively. A partial multi-valued function φ from $X_1 \times \cdots \times X_n$ to Y will be said to be $(\alpha_1, \ldots, \alpha_n, \beta)$- *computable* if a computable partial mapping F of $\mathrm{dom}(\alpha_1) \times \cdots \times \mathrm{dom}(\alpha_n)$ into $\mathrm{dom}(\beta)$ exists such that $(p_1, \ldots, p_n) \in \mathrm{dom}(F)$ and $\beta(F(p_1, \ldots, p_n)) \cap \varphi(x_1, \ldots, x_n) \neq \varnothing$ whenever $(p_1, \ldots, p_n) \in \mathrm{dom}(\alpha_1) \times \cdots \times \mathrm{dom}(\alpha_n)$ and $(x_1, \ldots, x_n) \in (\alpha_1(p_1) \times \cdots \times \alpha_n(p_n)) \cap \mathrm{dom}(\varphi)$.

Definition 4

Let α be a multi-representation of a set X. An element of X is said to be α-*computable* if it belongs to $\alpha(p)$ for some computable element p of $\mathrm{dom}(\alpha)$. A mapping φ of X into $\mathbb{N}$ is said to be α-*computable* if a computable mapping F of $\mathrm{dom}(\alpha)$ into $\mathbb{N}$ exists such that $F(p) = \varphi(x)$ whenever $p \in \mathrm{dom}(\alpha)$ and $x \in \alpha(p)$.

Acceptable multi-representations of B^*

Definition 5

A multi-representation δ of the set B^* will be called *acceptable* if it satisfies the following conditions:

1. The element o is δ-computable, and so is the mapping of B^* into $\mathbb{N}$ which maps o, the elements of B and the ones of B^+ into 0, 1 and 2, respectively.
2. The functions L^+ and R^+ are (δ, δ)-computable.
3. The ordered pair operation in B^* is (δ, δ, δ)-computable.

Remark 2. One gets a definition equivalent to the above one by replacing the first two conditions with the following ones:

1. The element o is δ-computable, and so is the mapping of B^* into $\mathbb{N}$ which maps the elements of B^0 and the ones of B^+ into 0 and 1, respectively.
2. The functions L and R are (δ, δ)-computable.

Acceptable multi-representations of B^* which are in accordance with a given multi-representation of B

Definition 6

Let $Z \subseteq Z'$ and γ be a multi-representation of Z. A multi-representation δ of Z' will be said to be *in accordance with* γ if the mapping I_Z is both (γ, δ)- and (δ, γ)-computable (if $Z = Z'$ then γ and δ are said to be *equivalent* in such a case).

Remark 3. Let $Z \subseteq Z'$. If a multi-representation δ of Z' is in accordance with a multi-representation γ of Z, and $\alpha_1, \ldots, \alpha_n, \beta \in \{\gamma, \delta\}$, then the $(\alpha_1, \ldots, \alpha_n, \beta)$-computability of an n-ary partial multi-valued function in Z is equivalent to its $(\gamma, \ldots, \gamma, \gamma)$-computability. Obviously any multi-representation of Z' is in accordance with some multi-representation of Z, and any two such multi-representations of Z are equivalent.

Theorem 1

For any multi-representation γ of B, an acceptable multi-representation of B^* exists which is in accordance with γ, and any two such multi-representations of B^* are equivalent.

Proof of the first part of the statement of Theorem 1

In the proof below, the multi-representations of a set are assumed to be the surjective partial multi-valued mappings from $\mathbb{N}^{\mathbb{N}}$ to it.

Proof.

Let γ be a multi-representation of B. We choose a computable injective mapping C of $\mathbb{N}^2$ into $\mathbb{N}$ such that $2C(i,j) \geq \max(i,j)$ for all i,j in $\mathbb{N}$, and define the partial mapping δ from $\mathbb{N}^{\mathbb{N}}$ into the set of the nonempty subsets of B^* inductively as follows:

- $\delta(\lambda k.2p(k)+2) = \gamma(p)$ for any p in $\mathrm{dom}(\gamma)$;
- $\delta(\lambda k.0) = \{o\}$;
- $\delta(\lambda k.2C(q(k), r(k)) + 1) = \delta(q) \times \delta(r)$ for all q, r in $\mathrm{dom}(\delta)$.

Then δ is an acceptable multi-representation of B^* which is in accordance with γ. □

The proof of the second part is more complicated, therefore it will be omitted in this presentation.

Some results about preservation of TTE computability

In the sequel, an acceptable multi-representation δ of B^* is supposed to be given.

Theorem 2

Let $\varphi_1, \ldots, \varphi_m$ be (δ, δ)-computable single-valued functions from $\mathcal{F}$. Then all functions from $\mathcal{F}$ absolutely prime computable in $\varphi_1, \ldots, \varphi_m$ are also (δ, δ)-computable.

The proof is based on Definition 1. We show that:

1. The basic functions L and R are (δ, δ)-computable.
2. The considered three basic operations in $\mathcal{F}$ transform (δ, δ)-computable single-valued functions again in such ones.

Corollary 1

Under the assumption of Theorem 2, let Γ be a mapping of $\mathcal{F}$ into $\mathcal{F}$ such that $\Gamma(\tau)$ is absolutely prime computable in $\varphi_1, \ldots, \varphi_m, \tau$ uniformly with respect to τ when τ ranges over $\mathcal{F}$. Then the least fixed point of Γ w.r.t. the partial ordering $\geq$ is single-valued and (δ, δ)-computable.

The conclusion of Theorem 2 fails without the single-valuedness assumption

Definition 7

Let α be a multi-representation of a set Z. A mapping φ of $\mathbb{N}$ into Z is said to be *α-computable* if a computable mapping F of $\mathbb{N}$ into $\mathrm{dom}(\alpha)$ exists such that $\varphi(k) \in \alpha(F(k))$ for all k in $\mathbb{N}$.

Let ν be *the Moschovakis injective mapping of $\mathbb{N}$ into B^**.

Lemma

The mappings ν and ν^{-1} are δ-computable.

Example 3

Let f be a non-computable function from $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$, and τ be the function from $\mathrm{rng}(\nu)$ to $\mathrm{rng}(\nu)$ defined by setting $\tau(\nu(k)) = \nu(f(k))$. Let $\varphi = I_{B^+}$, $\mathrm{dom}(\psi) = \mathrm{rng}(\nu)$ and $\psi(z) = \{o, \tau(z)\}$ for all $z \in \mathrm{dom}(\psi)$. Then $\varphi\psi = \tau$, hence $\varphi\psi$ is not (δ, δ)-computable, although φ and ψ are (δ, δ)-computable.

Secure composition and secure iteration in $\mathcal{F}$

Definition 8

The *secure composition* is the operation $\varphi, \psi \mapsto \varphi \circ \psi$ (used, e.g., in Klaus Weihrauch's book *Computable Analysis*), where $\varphi \circ \psi$ is the restriction of $\varphi\psi$ to the set $\{z \in \mathrm{dom}(\psi) \mid \psi(z) \subseteq \mathrm{dom}(\varphi)\}$.

Definition 9

Let $\varphi, \chi \in \mathcal{F}$. The *secure iteration of* φ *controlled by* χ is the restriction of the one considered till now to the set of the elements z of B^* with the following properties:

- no infinite sequence $z_0, z_1, z_2, \ldots$ of elements of B^* exists with $z_0 = z$ such that
  $$z_i \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\chi) \ \& \ \chi(z_i) \cap B^+ \neq \varnothing \ \& \ z_{i+1} \in \varphi(z_i) \quad (1)$$
  for all i;
- $z_n \in \mathrm{dom}(\chi) \ \& \ (\chi(z_n) \cap B^+ \neq \varnothing \Rightarrow z_n \in \mathrm{dom}(\varphi))$ whenever $z_0, z_1, \ldots, z_n$ is a finite sequence of elements of B^* with $z_0 = z$ and (1) holding for all $i < n$.

A modification of absolute prime computability

Definition 10

Let $\varphi_1, \ldots, \varphi_m$ be functions from $\mathcal{F}$. A function θ from $\mathcal{F}$ will be said to be *regularly computable* in $\varphi_1, \ldots, \varphi_m$ if θ can be obtained from $\varphi_1, \ldots, \varphi_m, L, R$ by finitely many applications of secure composition, juxtaposition and secure iteration.

Remark 4. On single-valued functions, the secure composition and the secure iteration coincide with the initially considered ones. Therefore regular computability in single-valued functions is equivalent to absolute prime computability in them. However, neither of these two computability properties implies the other one in the general case.

Remark 5. Clearly a function regularly computable in certain functions is always a restriction of some function absolutely prime computable in these functions. Moreover, each function regularly computable in certain functions is absolutely prime computable in some restrictions of these functions.

Reducibility of secure iteration to a particular instance of it

Let $\varphi, \chi \in \mathcal{F}$, and let ι be the secure iteration of φ controlled by χ. If $\bar{\chi}$ is the juxtaposition of χ and I_{B^*} then

$$\iota = R \circ \tilde{\iota} \circ \bar{\chi},$$

where $\tilde{\iota}$ is the secure iteration of $\bar{\chi} \circ \varphi \circ R$ controlled by L.

Hence secure iteration can be replaced in Definition 10 with secure iteration controlled by L if one adds I_{B^*} to the list

$$\varphi_1, \ldots, \varphi_m, L, R.$$

A counter-example to the implication from regular computability to absolute prime computability

Example 4

Functions φ and ψ from $\mathcal{F}$ will be constructed such that $\varphi \circ \psi$ is not absolutely prime computable in φ, ψ. Let φ be the same as in Example 3, i.e. $\varphi = I_{B^+}$. For the definition of ψ, we take a two-argument primitive recursive function f such that the set $\{k \in \mathbb{N} \,|\, \forall l \in \mathbb{N}(f(k,l) \neq 0)\}$ is not recursively enumerable. Let $\mathrm{dom}(\psi) = \mathrm{rng}(\nu)$, where ν is the Moschovakis injective mapping of $\mathbb{N}$ into B^*, and let

$$\psi(\nu(k)) = \{\nu(f(k,0)), \nu(f(k,1)), \nu(f(k,2)), \ldots\}$$

for all k in $\mathbb{N}$. Then $\{k \in \mathbb{N} \,|\, \nu(k) \in \mathrm{dom}(\varphi \circ \psi)\}$ is not recursively enumerable, and this can be used for showing that $\varphi \circ \psi$ is not absolutely prime computable in φ, ψ.

A counter-example to the converse implication will be given a little later.

Secure branching in $\mathcal{F}$

An appropriate *secure branching* operation in $\mathcal{F}$ can be defined as follows: it is the operation $\chi, \varphi, \psi \mapsto (\chi \sqsupset \varphi, \psi)$, where $(\chi \sqsupset \varphi, \psi)$ is the restriction of $(\chi \supset \varphi, \psi)$ to the set

$$\{z \in \mathrm{dom}(\chi) \mid (\chi(z) \cap B^+ \neq \varnothing \Rightarrow z \in \mathrm{dom}(\varphi)) \\ \& \ (\chi(z) \cap B^0 \neq \varnothing \Rightarrow z \in \mathrm{dom}(\psi))\}.$$

This operation turns out to be expressible through L, R, secure composition, juxtaposition and secure iteration, i.e. $(\chi \sqsupset \varphi, \psi)$ is uniformly regularly computable in the functions χ, φ, ψ (the expression for $(\chi \sqsupset \varphi, \psi)$ can be obtained from the Böhm-Jacopini style expression for $(\chi \supset \varphi, \psi)$ by replacing usual composition and iteration with secure ones).

Secure iteration as a least fixed point

Definition 11

We define the partial ordering $\trianglerighteq$ in $\mathcal{F}$ by adopting that $\tau_1 \trianglerighteq \tau_2$ iff $\mathrm{dom}(\tau_1) \supseteq \mathrm{dom}(\tau_2)$ and $\tau_1(z) \subseteq \tau_2(z)$ for all $z \in \mathrm{dom}(\tau_2)$.

The following holds:

For any φ and χ in $\mathcal{F}$, if ι is the secure iteration of φ controlled by χ then $\iota = (\chi \sqsupset \iota \circ \varphi, I_{B^*})$. Moreover, ι is the least τ in $\mathcal{F}$ satisfying the inequality $\tau \trianglerighteq (\chi \sqsupset \tau \circ \varphi, I_{B^*})$.

Regular computability and recursion

First Recursion Theorem for regular computability

Suppose $\varphi_1, \ldots, \varphi_m \in \mathcal{F}$, and Γ is a mapping of $\mathcal{F}$ into $\mathcal{F}$ such that $\Gamma(\tau)$ is regularly computable in $\varphi_1, \ldots, \varphi_m, \tau$ uniformly with respect to τ when τ ranges over $\mathcal{F}$. Then the inequality $\tau \trianglerighteq \Gamma(\tau)$ has a least solution, this solution is regularly computable in $\varphi_1, \ldots, \varphi_m$, and it is a fixed point of Γ (the least one).

The validity of the theorem follows from results established in my book mentioned earlier.

Regular computability preserves (δ, δ)-computability

Theorem 3

Let $\varphi_1, \ldots, \varphi_m$ be (δ, δ)-computable functions from $\mathcal{F}$. Then all functions from $\mathcal{F}$ regularly computable in $\varphi_1, \ldots, \varphi_m$ are also (δ, δ)-computable.

The proof is based on Definition 10. We show that secure composition, juxtaposition and secure iteration transform (δ, δ)-computable functions again in such ones.

Example 5

In Example 3, (δ, δ)-computable functions φ and ψ from $\mathcal{F}$ have been constructed such that $\varphi\psi$ is not (δ, δ)-computable. By Theorem 3, $\varphi\psi$ is not regularly computable in φ, ψ.

Corollary 2

Under the assumption of Theorem 3, let Γ be a mapping of $\mathcal{F}$ into $\mathcal{F}$ such that $\Gamma(\tau)$ is regularly computable in $\varphi_1, \ldots, \varphi_m, \tau$ uniformly with respect to τ when τ ranges over $\mathcal{F}$. Then the least fixed point of Γ w.r.t. the partial ordering $\trianglerighteq$ is (δ, δ)-computable.

Brattka style TTE computability

In the case of represented spaces, another TTE computability notion for multi-valued functions is defined in the paper

Vasko Brattka. *Computability over topological structures.* In: *Computability and Models*, eds. S. B. Cooper and S. S. Goncharov, Kluwer Academic Publishers, 2003, 93-136.

The definition below is an adaptation of Brattka's one for the case of multi-represented spaces.

Definition 12

Let α and β be multi-representations of the sets X and Y, respectively. A partial multi-valued function φ from X to Y will be said to be *Brattka style* (α, β)*-computable* if a computable partial mapping F of $\mathrm{dom}(\alpha) \times \mathbb{N}^{\mathbb{N}}$ into $\mathrm{dom}(\beta)$ exists such that the following holds whenever $p \in \mathrm{dom}(\alpha)$ and $x \in \alpha(p) \cap \mathrm{dom}(\varphi)$:

1. $(p, q) \in \mathrm{dom}(F)$ and $\beta(F(p, q)) \cap \varphi(x) \neq \varnothing$ for all $q \in \mathbb{N}^{\mathbb{N}}$;
2. $\varphi(x) \subseteq \bigcup_{q \in \mathbb{N}^{\mathbb{N}}} \beta(F(p, q))$.

Preservation of Brattka style (δ, δ)-computability under regular computability

Theorem 3′

Let $\varphi_1, \ldots, \varphi_m$ be Brattka style (δ, δ)-computable functions from $\mathcal{F}$. Then all functions from $\mathcal{F}$ regularly computable in $\varphi_1, \ldots, \varphi_m$ are also Brattka style (δ, δ)-computable.

The proof is based on Definition 10. We show that secure composition, juxtaposition and secure iteration transform Brattka style (δ, δ)-computable functions again in such ones.

Corollary 2′

Under the assumption of Theorem 3′, let Γ be a mapping of $\mathcal{F}$ into $\mathcal{F}$ such that $\Gamma(\tau)$ is regularly computable in $\varphi_1, \ldots, \varphi_m, \tau$ uniformly with respect to τ when τ ranges over $\mathcal{F}$. Then the least fixed point of Γ w.r.t. the partial ordering $\trianglerighteq$ is Brattka style (δ, δ)-computable.

Thank you for your attention!