$\mathcal{M}^2$-Computable Real Numbers

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Abstract

The paper concerns subrecursive computability of real numbers. Certain significant real numbers are shown to be $\mathcal{M}^2$-computable, and the set of the $\mathcal{M}^2$-computable real numbers is shown to be closed under the elementary functions of calculus.

Keywords: computable real number, computable real-valued function, subrecursive, $\mathcal{M}^2$, Δ_0 definable, elementary functions of calculus

1 Introduction

1.1 The subrecursive class $\mathcal{M}^2$

The class in question was implicitly introduced by Grzegorczyk in [4] when he formulated the problem whether it is possible to define the class $\mathcal{E}^2$ by using bounded least number operation instead of bounded primitive recursion.

Definition 1 *The class $\mathcal{M}^2$ is the smallest class of total functions in $\mathbb{N}$ which contains the projection functions, the constant 0 (as a function with no arguments), the successor function, the function $\lambda xy.x \dot{-} y$, the multiplication function, and is closed under substitution and bounded least number operation.*[1]

[1] There are different ways to define $(\mu t \le y)[f(\overline{x}, t) = 0]$ in the case when there is no $t \le y$ with $f(\overline{x}, t) = 0$, namely by using 0, y or $y + 1$ as the corresponding value. It does not matter which of them is accepted, and the function $\lambda xy.x \dot{-} y$ may be replaced with $\lambda xy.|x - y|$ in the above definition.

All functions from $\mathcal{M}^2$ are lower elementary in Skolem's sense, but it is an open problem whether the converse is true (it would be true iff $\mathcal{M}^2$ was closed under bounded summation).

The class $\mathcal{M}^2$ consists exactly of the total functions in $\mathbb{N}$ which are polynomially bounded and have Δ_0 definable graphs. Hence a relation in $\mathbb{N}$ is Δ_0 definable iff its characteristic function belongs to $\mathcal{M}^2$. This interconnection between $\mathcal{M}^2$ and Δ_0 definability is especially useful in combination with the result below due to Paris, Wilkie, Woods, Berarducci and D'Aquino (cf. [5, 1]).

Theorem 1 *If $N \in \mathbb{N}$, and the graph of a function $f : \mathbb{N}^{N+1} \to \mathbb{N}$ is Δ_0 definable, then so are the graphs of the functions $g(\overline{x}, y) = \sum_{k \leq \log_2(y+1)} f(\overline{x}, k)$ and $h(\overline{x}, y) = \prod_{k \leq y} f(\overline{x}, k)$.*

Corollary 1 *If $f : \mathbb{N}^{N+1} \to \mathbb{N}$ is in $\mathcal{M}^2$, and g, h are as above, then $g \in \mathcal{M}^2$ and $\lambda\overline{x}yz. \min(h(\overline{x}, y), z) \in \mathcal{M}^2$.*

Definition 2 *We will call* a convenient class *any class of total functions in $\mathbb{N}$ which contains $\mathcal{M}^2$ and is closed under substitution.*

Clearly $\mathcal{M}^2$ is the least convenient class. All good classes in the sense of [10] are convenient, and all convenient classes are acceptable in the sense of [6].[2]

1.2 $\mathcal{F}$-computability of real numbers

We will denote by $\mathbb{T}_k$ the set of all k-argument total functions from $\mathbb{N}$ to $\mathbb{N}$.

Definition 3 *Let $f, g, h \in \mathbb{T}_1$. We define a function $\langle f, g, h\rangle : \mathbb{N} \to \mathbb{Q}$ by setting*

$$\langle f, g, h\rangle(n) = \frac{f(n) - g(n)}{h(n) + 1}\,.$$

The triple (f, g, h) is called to name *a real number ξ if $|\langle f, g, h\rangle(n) - \xi| < \frac{1}{n+1}$ for all $n \in \mathbb{N}$.*

The mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_1$ will be called *k-ary operators*.

Lemma 1 *Let the ternary operator K be defined by*

$$K(f, g, h)(n) = \left\lfloor (n + 1)\frac{f(2n + 1) \dot{-} g(2n + 1)}{h(2n + 1) + 1} + \frac{1}{2} \right\rfloor.$$

Then, whenever $f, g, h \in \mathbb{T}_1$ and $n \in \mathbb{N}$, some of the numbers $K(f, g, h)(n)$ and $K(g, f, h)(n)$ is 0, and if the triple $(f, g, h) \in \mathbb{T}_1^3$ names a real number ξ, then the triple $(K(f, g, h), K(g, f, h), \mathrm{id}_{\mathbb{N}})$ also names ξ.

[2] A *good class* is any class of total functions in $\mathbb{N}$ which contains the projection functions, the constant 0, the successor function, the function $\lambda xy.x \dot{-} y$, and is closed under substitution and bounded summation. An *acceptable class* is any class of total functions in $\mathbb{N}$ which contains the projection functions, the constant 0, the successor function, the function $\lambda xy.|x - y|$, the multiplication function, and is closed under substitution (the constant 0 is missing in the definition from [6], since only functions with arguments are considered there).

Proof. We note that, for all $x, y, z, r, s, t \in \mathbb{N}$, we have $z(x \dot{-} y) = zx \dot{-} zy$,

$$\left\lfloor \frac{r \dot{-} s}{t+1} + \frac{1}{2} \right\rfloor = 0 \text{ or } \left\lfloor \frac{s \dot{-} r}{t+1} + \frac{1}{2} \right\rfloor = 0, \quad \left| \left(\left\lfloor \frac{r \dot{-} s}{t+1} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{s \dot{-} r}{t+1} + \frac{1}{2} \right\rfloor \right) - \frac{r-s}{t+1} \right| \le \frac{1}{2}.$$

Let $f, g, h \in \mathbb{T}_1$. For any $n \in \mathbb{N}$, the above observation yields $K(f,g,h)(n) = 0$ or $K(g,f,h)(n) = 0$, $|(K(f,g,h)(n) - K(g,f,h)(n)) - (n+1)\langle f,g,h\rangle(2n+1)| \le \frac{1}{2}$. If (f,g,h) names a real number ξ then

$$|\langle K(f,g,h), K(g,f,h), \mathrm{id}_{\mathbb{N}}\rangle(n) - \xi| \le$$
$$|\langle K(f,g,h), K(g,f,h), \mathrm{id}_{\mathbb{N}}\rangle(n) - \langle f,g,h\rangle(2n+1)| + |\langle f,g,h\rangle(2n+1) - \xi|$$
$$< \frac{1}{2(n+1)} + \frac{1}{2n+2} = \frac{1}{n+1}. \quad \square$$

Definition 4 *Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. We will call an* $\mathcal{F}$-triple *any triple of elements of $\mathbb{T}_1 \cap \mathcal{F}$. The functions of the form $\langle f,g,h\rangle$, where (f,g,h) is an $\mathcal{F}$-triple, will be called* $\mathcal{F}$-expressible. *A real number ξ will be called* $\mathcal{F}$-computable *if there exists an $\mathcal{F}$-triple naming ξ. The set of the $\mathcal{F}$-computable real numbers will be denoted by $\mathbb{R}_{\mathcal{F}}$.*[3]

Remark 1 *If $\mathcal{F}$ is the class of all recursive functions then $\mathbb{R}_{\mathcal{F}}$ is the set of all computable real numbers.*

The next theorem is proved in [6].

Theorem 2 *Let $\mathcal{F}$ be an acceptable class of total functions in $\mathbb{N}$. Then $\mathbb{R}_{\mathcal{F}}$ is a field, and if $\mathcal{F}$ is closed under the bounded least number operation, then $\mathbb{R}_{\mathcal{F}}$ is a real closed field.*

Corollary 2 *$\mathbb{R}_{\mathcal{M}^2}$ is a real closed field.*

It seems that many significant concrete real numbers are $\mathcal{M}^2$-computable. We showed that the numbers e and π, as well as Liouville's transcendental number $L = 1/10^{1!} + 1/10^{2!} + 1/10^{3!} + \cdots$ and several other mathematical constants belong to the set $\mathbb{R}_{\mathcal{M}^2}$ and that $\mathbb{R}_{\mathcal{M}^2}$ is closed under the elementary functions of calculus (we actually prove such a closedness of $\mathbb{R}_{\mathcal{F}}$ for any convenient class $\mathcal{F}$, thus strengthening a result of this kind from [10]). Unfortunately we do not know whether the Euler-Mascheroni constant is $\mathcal{M}^2$-computable (we know it is lower elementary computable).[4]

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[3] In the papers [6, 8]) the definition of $\mathcal{F}$-computability of a real number ξ uses a slightly weaker requirement, namely the existence of one-argument functions f, g, h from $\mathcal{F}$ such that (in the present notation) $|\langle f,g,h\rangle(n) - \xi| \le \frac{1}{n+1}$ for all $n \in \mathbb{N}$. This requirement is equivalent to the one imposed here (which is with $<$ instead of $\le$), whenever the class $\mathcal{F}$ contains the successor function and is closed under substitution.

[4] The study of $\mathcal{M}^2$-computability of real numbers began in June 2008 when the second author proved the $\mathcal{M}^2$-computability of the numbers e and L. He did this by improving an approach used in [7] for showing the $\mathcal{E}^2$-computability of the numbers in question, namely the construction of appropriate $\mathcal{E}^2$-expressible functions (regarded as sequences there) whose values are partial sums of expansions of these numbers (an $\mathcal{M}^2$-computability proof of this kind is given here in the next subsection). However, an accommodation of reasonings from [8] was needed for other constants, and such an accommodation was done in the Master Thesis [3] (defended in March 2009). The proof that $\mathbb{R}_{\mathcal{M}^2}$ is closed under the elementary functions of calculus was completed in August 2009 and refined several weeks later.

1.3 Proving $\mathcal{M}^2$-computability by using $\mathcal{M}^2$-expressible functions whose values are partial sums

Theorem 3 *Let $\xi = \sum_{k=0}^{\infty} \frac{1}{f(k)}$, where $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, $f(k)$ is a proper divisor of $f(k+1)$ for any $k \in \mathbb{N}$, and the graph of f is Δ_0 definable. Then $\xi \in \mathbb{R}_{\mathcal{M}^2}$.*

Proof. Let $\sigma_m = \sum_{k=0}^{m} \frac{1}{f(k)}$ for any $m \in \mathbb{N}$. Then $|\sigma_m - \xi| \le 2/f(m+1)$ for all $m \in \mathbb{N}$. Let $m_n = \min\{m \,|\, f(m+1) > 2n+2\}$ for any $n \in \mathbb{N}$. Then $|\sigma_{m_n} - \xi| < (n+1)^{-1}$ for all $n \in \mathbb{N}$. We will show that σ_{m_n} is an $\mathcal{M}^2$-expressible function of n. This will be done by using the equality $\sigma_{m_n} = f(m_n)\sigma_{m_n}/f(m_n)$ and proving that the functions $\lambda n.f(m_n)\sigma_{m_n}$ and $\lambda n.f(m_n)$ belong to $\mathcal{M}^2$. The second of them belongs to $\mathcal{M}^2$, since the equality $l = f(m_n)$ is equivalent to $(\exists k \le l)\big(l = f(k) \& (k = 0 \vee l \le 2n+2) \& (\forall j \le 2n+2)(j \ne f(k+1))\big)$, and this condition implies $l \le 2n + f(0) + 1$. To prove that $\lambda n.f(m_n)\sigma_{m_n} \in \mathcal{M}^2$, we note that $m_n \le \log_2(2n + f(0) + 1)$ and hence

$$f(m_n)\sigma_{m_n} = \sum_{k \le \log_2(2n+f(0)+1)} \big\lfloor f(m_n) \big/ \min(f(k), f(m_n)+1) \big\rfloor . \quad \square$$

Corollary 3 *The numbers e and L are $\mathcal{M}^2$-computable.*

Proof. We take $f(k) = k!$ for e and $f(k) = 10^{(k+1)!}$ for L. $\square$

2 Stronger tools for proving $\mathcal{M}^2$-computability of real numbers

2.1 $\mathcal{F}$-computability of real-valued functions with natural arguments

Definition 5 *Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$, and let $N \in \mathbb{N}$. A partial function θ from $\mathbb{N}^N$ to $\mathbb{R}$ is called* $\mathcal{F}$-computable *if there exist $N+1$-argument functions $f, g, h \in \mathcal{F}$ such that the triple $(\lambda n.f(\overline{x}, n), \lambda n.g(\overline{x}, n), \lambda n.h(\overline{x}, n))$ names the number $\theta(\overline{x})$ for any $\overline{x} \in \mathrm{dom}(\theta)$.*

Example 1 *A real number α is $\mathcal{F}$-computable iff the argumentless function with value α is $\mathcal{F}$-computable.*

Example 2 *Let $\theta_1 : \mathbb{N}^2 \times (\mathbb{N} \setminus \{0\}) \to \mathbb{Q}$ and $\theta_2 : \mathbb{N} \times (\mathbb{N} \setminus \{0\}) \to \mathbb{R}$ be defined by*

$$\theta_1(x_1, x_2, x_3) = \frac{x_1 - x_2}{x_3}, \quad \theta_2(x_1, x_2) = \sqrt{x_1/x_2}\,.$$

Then θ_1 and θ_2 are $\mathcal{M}^2$-computable.

Proof. Take $f(x_1, x_2, x_3, n) = x_1$, $g(x_1, x_2, x_3, n) = x_2$, $h(x_1, x_2, x_3, n) = x_3 \dot{-} 1$ for θ_1, and $f(x_1, x_2, n) = \min\{y \,|\, y = (n+1)x_1 \vee (y+1)^2 x_2 > (n+1)^2 x_1\}$, $g(x_1, x_2, n) = 0$, $h(x_1, x_2, n) = n$ for θ_2. $\square$

Example 3 *Let $f_0, g_0 : \mathbb{N}^N \to \mathbb{N}$ belong to $\mathcal{M}^2$, h_0 be a partial function from $\mathbb{N}^N$ to $\mathbb{N}$ whose graph is Δ_0 definable, and θ_1 be as in Example 2. Then the partial function $\lambda\overline{x}.\theta_1(f_0(\overline{x}), g_0(\overline{x}), h_0(\overline{x}))$ is $\mathcal{M}^2$-computable.*

Proof. We may take $f(\overline{x}, n) = f_0(\overline{x})$, $g(\overline{x}, n) = g_0(\overline{x})$, and
$h(\overline{x}, n) = \min\{y \,|\, y = (n+1)\,|f_0(\overline{x}) - g_0(\overline{x})| \dot{-} 1 \ \vee \ y+1 = h_0(\overline{x})\}$. $\square$

Obviously, any restriction of an $\mathcal{F}$-computable function of the considered type will be again $\mathcal{F}$-computable. The next two statements are also obvious.

Lemma 2 *Let $\mathcal{F}$ be an acceptable class, and θ be an $\mathcal{F}$-computable real-valued partial function with natural arguments. Then all substitutions of functions from $\mathcal{F}$ into θ produce again $\mathcal{F}$-computable functions. In particular,* $\mathrm{range}(\theta) \subseteq \mathbb{R}_{\mathcal{F}}$.

Lemma 3 *If $\mathcal{F}$ is a convenient class then definition by cases using $\mathcal{F}$-computable real-valued partial functions with natural arguments and controlled by predicates of the class $\mathcal{F}$ produces again a function of this kind.*

In next three lemmas, N can be any natural number (including 0).

Lemma 4 *Let $\mathcal{F}$ be a convenient class, and θ be an $\mathcal{F}$-computable partial function from $\mathbb{N}^N$ to $\mathbb{R}$. Then there exist $N+1$-argument functions $f, g \in \mathcal{F}$ such that the triple $(\lambda n.f(\overline{x}, n), \lambda n.g(\overline{x}, n), \mathrm{id}_{\mathbb{N}})$ names $\theta(\overline{x})$ for any $\overline{x} \in \mathrm{dom}(\theta)$.*

Proof. By Lemma 1 and Proposition 1, making use of the fact that $\mathcal{F}$ is closed under the operator K from Lemma 1. $\square$

Lemma 5 *Let a partial function θ from $\mathbb{N}^{N+1}$ to $\mathbb{R}$ be $\mathcal{M}^2$-computable. Then so is the partial function defined by $\theta^{\Sigma}(\overline{x}, y) = \sum_{k \le \log_2(y+1)} \theta(\overline{x}, k)$.*

Proof. By Lemma 4, $N+2$-argument functions $f, g, h \in \mathcal{M}^2$ exist such that $(\lambda n.f(\overline{x}, k, n), \lambda n.g(\overline{x}, k, n), \mathrm{id}_{\mathbb{N}})$ names $\theta(\overline{x}, k)$ for any $(\overline{x}, k) \in \mathrm{dom}(\theta)$. Then, for any $(\overline{x}, y) \in \mathrm{dom}(\theta^{\Sigma})$, the triple of the functions

$$\lambda n. \sum_{k \le \log_2(y+1)} f(\overline{x}, k, h^{\Sigma}(\overline{x}, y, n)), \quad \lambda n. \sum_{k \le \log_2(y+1)} g(\overline{x}, k, h^{\Sigma}(\overline{x}, y, n)),$$
$$\lambda n.h^{\Sigma}(\overline{x}, y, n)),$$

where $h^{\Sigma}(\overline{x}, y, n) = (n+1)(\lfloor \log_2(y+1) \rfloor + 1) - 1$, names the number $\theta^{\Sigma}(\overline{x}, y)$. $\square$

Lemma 6 *Let θ be an $\mathcal{M}^2$-computable partial function from $\mathbb{N}^{N+1}$ to $\mathbb{R}$, and σ be a partial function from $\mathbb{N}^N$ to $\mathbb{R}$ such that $\mathrm{dom}(\sigma) \times \mathbb{N} \subseteq \mathrm{dom}(\theta)$ and, for any $\overline{x} \in \mathrm{dom}(\sigma)$, the equality $\sigma(\overline{x}) = \sum_{k=0}^{\infty} \theta(\overline{x}, k)$ holds. Let there exist an $N+1$-argument function $p \in \mathcal{M}^2$ such that $\left|\sum_{k > log_2(p(\overline{x}, n)+1)} \theta(\overline{x}, k)\right| \le \frac{1}{n+1}$ for any $\overline{x} \in \mathrm{dom}(\sigma)$ and any $n \in \mathbb{N}$. Then σ is also $\mathcal{M}^2$-computable.*

Proof. Let θ^Σ be as in Lemma 5, and $f^\Sigma, g^\Sigma, h^\Sigma$ be $N+2$-argument functions from $\mathcal{M}^2$ such that $(\lambda n.f^\Sigma(\overline{x},y,n), \lambda n.g^\Sigma(\overline{x},y,n), \lambda n.h^\Sigma(\overline{x},y,n))$ names $\theta^\Sigma(\overline{x},y)$ for any $(\overline{x},y) \in \mathrm{dom}(\theta^\Sigma)$. Then the triple of the functions

$$\lambda n.f^\Sigma(\overline{x},p(\overline{x},2n+1),2n+1), \quad \lambda n.g^\Sigma(\overline{x},p(\overline{x},2n+1),2n+1),$$
$$\lambda n.h^\Sigma(\overline{x},p(\overline{x},2n+1),2n+1)$$

names the number $\sigma(\overline{x})$ for any $\overline{x} \in \mathrm{dom}(\sigma)$. □

Example 4 *The number π is $\mathcal{M}^2$-computable.*

Proof. Since $\pi = 4(\arctan\frac{1}{2} + \arctan\frac{1}{3})$, it is sufficient to show that $\arctan\frac{1}{m}$ is in $\mathbb{R}_{\mathcal{M}^2}$ for any natural number m, greater than 1. Let $m \in \mathbb{N}$ and $m > 1$. Then we can apply Lemma 6 to the expansion $\arctan\frac{1}{m} = \sum_{k=0}^{\infty}\theta(k)$, where

$$\theta(k) = \frac{(-1)^k}{(2k+1)m^{2k+1}}$$

(the requirements of the lemma are satisfied by [1], Example 3, the equality $(-1)^k = (k+1) \bmod 2 - k \bmod 2$ and the inequality $|\theta(k)| \leq 1/2^{2k+1}$). □

Example 5 *Let $\sigma : \mathbb{N} \smallsetminus \{0,1\} \to \mathbb{R}$ be defined by $\sigma(x) = -\ln\left(1-\frac{1}{x}\right)$. Then σ is $\mathcal{M}^2$-computable.*

Proof. For any $x \in \mathrm{dom}(\sigma)$, we have $\sigma(x) = \sum_{k=0}^{\infty}\theta(x,k)$ with

$$\theta(x,k) = \frac{1}{(k+1)x^{k+1}}\,. \quad \square$$

Note. By application of Lemma 6 with $N = 0$ to appropriate expansions, the $\mathcal{M}^2$-computability of the following constants is also shown in [3]: the Erdös-Borwein constant E, the logarithm of the golden mean φ, the paper folding constant σ. The expansions in question are

$$E = \sum_{k=1}^{\infty}\frac{1}{2^k-1}, \qquad 2(\ln\varphi)^2 = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2\binom{2k}{k}}, \qquad \sigma = \sum_{k=0}^{\infty}2^{-2^k}\left(1-2^{-2^{k+2}}\right)^{-1}.$$

2.2 Uniformly $\mathcal{F}$-computable real functions

We are going to define a notion of uniform $\mathcal{F}$-computability for real-valued functions of real arguments in such a way that:

- in the situation considered in [10], the uniform $\mathcal{F}$-computability of a function is equivalent to being uniformly in $\mathcal{F}$ (cf. Remark 4 for a proof of the easier direction of this equivalence), and the uniform $\mathcal{M}^2$-computability of partial functions from $\mathbb{N}^N$ to $\mathbb{R}$ is equivalent to their $\mathcal{M}^2$-computability;
- the elementary functions of calculus are uniformly $\mathcal{M}^2$-computable at least after restriction to compact subsets of their domains;

- all uniformly $\mathcal{M}^2$-computable functions are computable in the sense of [11], they preserve $\mathcal{F}$-computability of real numbers for any convenient class $\mathcal{F}$, and the uniform $\mathcal{M}^2$-computability is preserved at substitution and at restriction to arbitrary subsets of the domain of the function.

A class of operators needed for the definition will be introduced firstly.

Definition 6 *Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. For any $k \in \mathbb{N}$, some k-ary operators will be called* $\mathcal{F}$-substitutional, *the notion being introduced inductively as follows:*

1. *The k-ary operator F, defined by the equality $F(f_1,\ldots,f_k)(n) = n$, is $\mathcal{F}$-substitutional.*
2. *For any $i \in \{1,\ldots,k\}$, if F_0 is an $\mathcal{F}$-substitutional k-ary operator, then so is the operator F, defined by $F(f_1,\ldots,f_k)(n) = f_i(F_0(f_1,\ldots,f_k)(n))$.*
3. *For any natural number l, if $f : \mathbb{N}^l \to \mathbb{N}$ belongs to $\mathcal{F}$, and $F_1,\ldots,F_l$ are $\mathcal{F}$-substitutional k-ary operators, then so is the operator F, defined by $F(f_1,\ldots,f_k)(n) = f(F_1(f_1,\ldots,f_k)(n),\ldots,F_l(f_1,\ldots,f_k)(n))$.*

Intuitively, a k-ary operator F is $\mathcal{F}$-substitutional iff there is an expression for $F(f_1,\ldots,f_k)(n)$ build from the variable n by using function symbols $f_1,\ldots,f_k$ and function symbols for functions from $\mathcal{F}$.

Example 6 *The operator K from Lemma 1 is $\mathcal{M}^2$-substitutional.*

Example 7 *If i is some of the numbers $1,\ldots,k$ then the k-ary operator F, defined by $F(f_1,\ldots,f_k) = f_i$, is $\mathcal{F}$-substitutional.*

The next propositions can be proved by induction on the construction of the operator F.

Proposition 1 *Let $\mathcal{F}$ contain the projection functions and be closed under substitution, and let F be a k-ary $\mathcal{F}$-substitutional operator. If $m \in \mathbb{N}$ and $f_1,\ldots,f_k \in \mathbb{T}_{m+1} \cap \mathcal{F}$, then $\lambda\bar{s}n.F(\lambda t.f_1(\bar{s},t),\ldots,\lambda t.f_k(\bar{s},t))(n) \in \mathcal{F}$. In particular, if $f_1,\ldots,f_k \in \mathbb{T}_1 \cap \mathcal{F}$, then $F(f_1,\ldots,f_k) \in \mathcal{F}$.*

Proposition 2 *Let F and G be k-ary $\mathcal{F}$-substitutional operators. Then so is the operator H defined by $H(f_1,\ldots,f_k)(n) = F(f_1,\ldots,f_k)(G(f_1,\ldots,f_k)(n))$.*

Proposition 3 *Let F be a k-ary $\mathcal{F}$-substitutional operator. If $l \in \mathbb{N}$, and $G_1,\ldots, G_k$ are l-ary $\mathcal{F}$-substitutional operators then so is the operator H, defined by $H(g_1,\ldots,g_l) = F(G_1(g_1,\ldots,g_l),\ldots,G_k(g_1,\ldots,g_l))$.*

Proposition 4 *Let all functions from $\mathcal{F}$ be dominated by polynomials, and let F be a k-ary $\mathcal{F}$-substitutional operator. Then, for any two-argument polynomial P, there exists a two-argument polynomial Q such that, whenever α is a non-negative real number and $f_1,\ldots,f_k$ are functions from $\mathbb{T}_1$ dominated by $\lambda n.P(\alpha,n)$, the function $F(f_1,\ldots,f_k)$ is dominated by $\lambda n.Q(\alpha,n)$.*

Remark 2 *The class of the $\mathcal{F}$-substitutional operators increases monotonically in $\mathcal{F}$ (with respect to inclusion).*

Definition 7 *Let $N \in \mathbb{N}$, and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. We will call a* 3,3-computing system *for θ any triple (F, G, H) of $3N$-ary operators such that, whenever $(\xi_1, \dots, \xi_N) \in D$ and $(f_1, g_1, h_1), \dots, (f_N, g_N, h_N)$ are triples from $\mathbb{T}_1^3$ naming $\xi_1, \dots, \xi_N$, respectively, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$, where*

$$\tilde{f} = F(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \quad \tilde{g} = G(f_1, g_1, h_1, \dots, f_N, g_N, h_N),$$
$$\tilde{h} = H(f_1, g_1, h_1, \dots, f_N, g_N, h_N),$$

names $\theta(\xi_1, \dots, \xi_N)$.

Definition 8 *Let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$. A tuple of operators will be called* $\mathcal{F}$-substitutional *if all its components are $\mathcal{F}$-substitutional. A function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, will be called* uniformly $\mathcal{F}$-computable *if there exists an $\mathcal{F}$-substitutional 3,3-computing system for θ.*

Example 8 *Let $N \in \mathbb{N}$, $i \in \{1, \dots, N\}$, and $\theta : \mathbb{R}^N \to \mathbb{R}$ be defined by the equality $\theta(\xi_1, \dots, \xi_N) = \xi_i$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Example 9 *The functions $\lambda\xi. -\xi$ and $\lambda\xi.|\xi|$ are uniformly $\mathcal{M}^2$-computable.*

Example 10 *Let $\mathcal{F}$ be a convenient class, and α be an $\mathcal{F}$-computable real number. Let $N \in \mathbb{N}$. Then the function $\theta : \mathbb{R}^N \to \mathbb{R}$, defined by $\theta(\xi_1, \dots, \xi_N) = \alpha$, is uniformly $\mathcal{F}$-computable.*

Example 11 *Let $\theta : \mathbb{R} \to [0, \infty)$ be defined by $\theta(\xi) = \sqrt{|\xi|}$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Proof. By the proof for Example 2, there exists a three-argument function $a \in \mathcal{M}^2$ such that

$$\left| \frac{a(x_1, x_2, n)}{n+1} - \sqrt{\frac{x_1}{x_2}} \right| < \frac{1}{n+1}$$

for all $x_2 \in \mathbb{N} \setminus \{0\}$ and all $x_1, n \in \mathbb{N}$. Let the ternary operators F, G, H be defined by

$$F(f,g,h)(n) = a(m_1, m_2, 2n+1), \quad G(f,g,h)(n) = 0, \quad H(f,g,h)(n) = 2n+1,$$

where $m_1 = |f((2n+2)^2 - 1) - g((2n+2)^2 - 1)|$, $m_2 = h((2n+2)^2 - 1) + 1$. These operators are $\mathcal{M}^2$-substitutional. Suppose $f, g, h \in \mathbb{T}_1$ are such that the triple (f, g, h) names a real number ξ. We will show that the corresponding triple $(F(f,g,h), G(f,g,h), H(f,g,h))$ names the number $\theta(\xi)$. Let $n \in \mathbb{N}$, and m_1, m_2 be defined as above. Then

$$|\langle F(f,g,h), G(f,g,h), H(f,g,h)\rangle(n) - \theta(\xi)| = \left| \frac{a(m_1, m_2, 2n+1)}{2n+2} - \sqrt{|\xi|} \right| \le$$
$$\left| \frac{a(m_1, m_2, 2n+1)}{2n+2} - \sqrt{\frac{m_1}{m_2}} \right| + \left| \sqrt{\frac{m_1}{m_2}} - \sqrt{|\xi|} \right| < \frac{1}{2n+2} + \sqrt{\left| \frac{m_1}{m_2} - |\xi| \right|} \le$$
$$\frac{1}{2n+2} + \sqrt{|\langle f,g,h\rangle((2n+2)^2 - 1) - \xi|} < \frac{1}{2n+2} + \frac{1}{2n+2} = \frac{1}{n+1}. \quad \square$$

Proposition 4 allows proving that the absolute value of any uniformly $\mathcal{M}^2$-computable function is bounded by some polynomial (hence the function is bounded on each bounded subset of its domain). This can be done by using the fact that any real number ξ is named by some triple $(f, g, \mathrm{id}_{\mathbb{N}})$ such that, for any $n \in \mathbb{N}$, some of the numbers $f(n)$ and $g(n)$ is 0, and therefore they both are less than $(n+1)|\xi|+1$.

Let $\mathcal{F}$ be a convenient class. Definitions 7 and 8 obviously imply that all restrictions of a uniformly $\mathcal{F}$-computable function are also uniformly $\mathcal{F}$-computable. By Remark 2, all uniformly $\mathcal{M}^2$-computable functions are uniformly $\mathcal{F}$-computable. Making use of the last statement in Proposition 1, we see that, whenever a function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, is uniformly $\mathcal{F}$-computable, and $\xi_1, \ldots, \xi_N$ are $\mathcal{F}$-computable real numbers such that $(\xi_1, \ldots, \xi_N) \in D$, the real number $\theta(\xi_1, \ldots, \xi_N)$ is also $\mathcal{F}$-computable. Proposition 3 shows that the class of the uniformly $\mathcal{F}$-computable functions is closed under substitution.

Remark 3 *If $\mathcal{F}$ is a convenient class whose elements are recursive functions, then all uniformly $\mathcal{F}$-computable functions are computable in the sense of [11].*

Proof. All $\mathcal{F}$-substitutional operators will be recursive ones. □

Remark 4 *Let $\mathcal{F}$ be a good class in the sense of the paper [10], and the function $\theta : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^N$, be uniformly in $\mathcal{F}$ in the sense of that paper. Then θ is uniformly $\mathcal{F}$-computable.*

Proof. By the assumption that θ is uniformly in $\mathcal{F}$, there exist a one-argument function $d \in \mathcal{F}$ and $3N+1$-argument functions $f, g, h \in \mathcal{F}$ such that, whenever $(\xi_1, \ldots, \xi_N) \in D$, $p_1, q_1, r_1, \ldots, p_N, q_N, r_N, m \in \mathbb{N}$, and the inequalities

$$|\xi_k| \leq m+1, \quad \left|\frac{p_k - q_k}{r_k + 1} - \xi_k\right| < \frac{1}{d(m)+1}, \quad k = 1, \ldots, N, \tag{1}$$

hold, the numbers

$$p = f(p_1, q_1, r_1, \ldots, p_N, q_N . r_N, m), \quad q = g(p_1, q_1, r_1, \ldots, p_N, q_N . r_N, m), \tag{2}$$

$$r = h(p_1, q_1, r_1, \ldots, p_N, q_N . r_N, m) \tag{3}$$

satisfy the inequality

$$\left|\frac{p-q}{r+1} - \theta(\xi_1, \ldots, \xi_N)\right| < \frac{1}{m+1}.$$

Let us define $3N$-ary operators F, G, H by setting

$$F(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = p, \quad G(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = q,$$
$$H(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = r,$$

where the numbers p, q, r are defined by means of the equalities (2–3) with

$$m = \max\left(\lceil|\langle f_1, g_1, h_1\rangle(0)|\rceil, \ldots, \lceil|\langle f_N, g_N, h_N\rangle(0)|\rceil, n\right),$$
$$p_k = f_k(d(m)), \quad q_k = g_k(d(m)), \quad r_k = h_k(d(m)) \ \text{ for } k = 1, \ldots, N.$$

These operators are $\mathcal{F}$-substitutional, and if an element $(\xi_1, \dots, \xi_N)$ of D and functions $f_1, g_1, h_1, \dots, f_N, g_N, h_N \in \mathbb{T}_1$ are given such that (f_k, g_k, h_k) names ξ_k for $k = 1, \dots, N$, then, for any $n \in \mathbb{N}$, the above numbers $m, p_1, q_1, r_1, \dots, p_N, q_N, r_N$ will satisfy the inequalities (1) and the inequality $m \geq n$, hence the corresponding numbers p, q, r will satisfy the inequality

$$\left| \frac{p - q}{r + 1} - \theta(\xi_1, \dots, \xi_N) \right| < \frac{1}{n + 1}. \quad \square$$

Definition 9 *Let $N \in \mathbb{N}$, and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. Then we will call a* 2,3-computing system *for θ any triple (F, G, H) of $2N$-ary operators F, G, H such that, whenever $(\xi_1, \dots, \xi_N) \in D$ and $(f_1, g_1, \mathrm{id}_\mathbb{N}), \dots, (f_N, g_N, \mathrm{id}_\mathbb{N})$ are triples from $\mathbb{T}_1^3$ naming $\xi_1, \dots, \xi_N$, respectively, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$, where*

$$\tilde{f} = F(f_1, g_1, \dots, f_N, g_N), \quad \tilde{g} = G(f_1, g_1, \dots, f_N, g_N),$$
$$\tilde{h} = H(f_1, g_1, \dots, f_N, g_N),$$

names $\theta(\xi_1, \dots, \xi_N)$.

Proposition 5 *Let $\mathcal{F}$ be a convenient class, and let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. Then θ is uniformly $\mathcal{F}$-computable iff there exists an $\mathcal{F}$-substitutional 2,3-computational system for θ.*

Proof. We make use of Lemma 1, Example 6 and Proposition 3 (clause 1 of Definition 6 is also used). $\square$.

Proposition 6 *Let $\mathcal{F}$ be a convenient class. Let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{N}^N$ for some $N \in \mathbb{N}$. The function θ is uniformly $\mathcal{F}$-computable iff θ is $\mathcal{F}$-computable in the sense of Definition 5.*

Proof. For the "if"-part of the proof, suppose f, g, h are such functions as in Definition 5. Let the $2N$-ary operators F, G, H be defined as follows:

$$F(f_1, g_1, \dots, f_N, g_N)(n) = f(f_1(0) \dot{-} g_1(0), \dots, f_N(0) \dot{-} g_N(0), n),$$
$$G(f_1, g_1, \dots, f_N, g_N)(n) = g(f_1(0) \dot{-} g_1(0), \dots, f_N(0) \dot{-} g_N(0), n),$$
$$H(f_1, g_1, \dots, f_N, g_N)(n) = h(f_1(0) \dot{-} g_1(0), \dots, f_N(0) \dot{-} g_N(0), n).$$

Then F, G, H are $\mathcal{F}$-substitutional, and whenever $\overline{x} \in D$, $f_1, g_1, \dots, f_N, g_N \in \mathbb{T}_1$ and the triples $(f_1, g_1, \mathrm{id}_\mathbb{N}), \dots, (f_N, g_N, \mathrm{id}_\mathbb{N})$ name $\overline{x}$, respectively, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$, defined as in the first clause of Definition 9, names $\theta(\overline{x})$ thanks to the equalities $\tilde{f}(n) = f(\overline{x}, n)$, $\tilde{g}(n) = g(\overline{x}, n)$, $\tilde{h}(n) = h(\overline{x}, n)$. For the "only if" part, suppose the function θ is uniformly $\mathcal{F}$-computable. Let (F, G, H) be an $\mathcal{F}$-substitutional 3,3-computing system for θ, and let us set

$$f(\overline{x}, n) = F(\lambda t.x_1, \lambda t.0, \lambda t.0, \dots, \lambda t.x_N, \lambda t.0, \lambda t.0)(n),$$
$$g(\overline{x}, n) = G(\lambda t.x_1, \lambda t.0, \lambda t.0, \dots, \lambda t.x_N, \lambda t.0, \lambda t.0)(n),$$
$$h(\overline{x}, n) = H(\lambda t.x_1, \lambda t.0, \lambda t.0, \dots, \lambda t.x_N, \lambda t.0, \lambda t.0)(n).$$

By Proposition 1, the functions f, g, h belong to $\mathcal{F}$. Since $(\lambda t.x, \lambda t.0, \lambda t.0)$ names x for any $x \in \mathbb{N}$, the functions f, g, h satisfy the condition from Definition 5. $\square$

Corollary 4 *If $\mathcal{F}$ is a convenient class then all functions from it are uniformly $\mathcal{F}$-computable.*

Proposition 5 facilitates giving some more examples to Definition 8.

Example 12 *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\theta(\xi_1, \xi_2) = \xi_1 + \xi_2$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Proof. We can build a 2,3-computing system (F, G, H) for θ by setting

$$F(f_1, g_1, f_2, g_2)(n) = f_1(2n+1) + f_2(2n+1),$$
$$G(f_1, g_1, f_2, g_2)(n) = g_1(2n+1) + g_2(2n+1),$$
$$H(f_1, g_1, f_2, g_2)(n) = 2n+1. \quad \square$$

Corollary 5 *The function $\lambda\xi_1\xi_2.\xi_1 - \xi_2$ is uniformly $\mathcal{M}^2$-computable.*

Example 13 *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\theta(\xi_1, \xi_2) = \xi_1\xi_2$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Proof. Let the operators F, G, H be defined by

$$F(f_1, g_1, f_2, g_2)(n) = f_1(m)f_2(m) + g_1(m)g_2(m),$$
$$G(f_1, g_1, f_2, g_2)(n) = f_1(m)g_2(m) + g_1(m)f_2(m),$$
$$H(f_1, g_1, f_2, g_2)(n) = (m+1)^2 - 1,$$

where $m = (|f_1(0) - g_1(0)| + |f_2(0) - g_2(0)| + 3)(n+1) - 1$. Then (F, G, H) is a 2,3-computing system for θ (to prove this, one makes use of the fact that, whenever $\xi_1, \xi_2 \in \mathbb{R}$, $f_1, g_1, f_2, g_2 \in \mathbb{T}_1$, $(f_1, g_1, \mathrm{id}_{\mathbb{N}})$ names ξ_1, and $(f_2, g_2, \mathrm{id}_{\mathbb{N}})$ names ξ_2, the inequalities $|\xi_i| \le |f_i(0) - g_i(0)| + 1$, $i = 1, 2$, hold). $\square$

The function $\lambda\xi.1/\xi$, is not uniformly $\mathcal{M}^2$-computable, but some functions related to it are uniformly $\mathcal{M}^2$-computable.

Example 14 *Let $D = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \,\middle|\, |\xi_1\xi_2| \ge 1\right\}$, and let $\theta : D \to \mathbb{R}$ be defined by $\theta(\xi_1, \xi_2) = 1/\xi_1$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Proof. We consider the quaternary operators F, G, H defined by

$$F(f_1, g_1, f_2, g_2)(n) = (m+1)f_1(m), \quad G(f_1, g_1, f_2, g_2)(n) = (m+1)g_1(m),$$
$$H(f_1, g_1, f_2, g_2)(n) = (f_1(m) - g_1(m))^2 \dot{-} 1,$$

where $m = (n+1)l^2 + l - 1$ with $l = |f_2(0) - g_2(0)| + 1$. These operators are $\mathcal{M}^2$-substitutional. Suppose that $(\xi_1, \xi_2) \in D$, $f_1, g_1, f_2, g_2 \in \mathbb{T}_1$, and $(f_k, g_k, \mathrm{id}_{\mathbb{N}})$ names ξ_k for $k = 1, 2$. Then $|\xi_2| < l$, hence $|\xi_1| > \frac{1}{l}$. Let $n \in \mathbb{N}$, and let us set

$$\rho = \frac{f_1(m) - g_1(m)}{m+1},$$

where m is as in the definition of F, G, H. Then $|\rho - \xi_1| < \frac{1}{m+1}$, hence

$$|\rho| > |\xi_1| - \frac{1}{m+1} > \frac{1}{l} - \frac{1}{m+1} = \frac{(n+1)l}{m+1}.$$

Therefore

$$|\langle F(f_1, g_1, f_2, g_2), G(f_1, g_1, f_2, g_2), H(f_1, g_1, f_2, g_2)\rangle(n) - \theta(\xi_1)| =$$
$$\left|\frac{1}{\rho} - \frac{1}{\xi_1}\right| = \frac{|\xi_1 - \rho|}{|\rho||\xi_1|} < \frac{1}{n+1}. \quad \square$$

Corollary 6 *Let $\mathcal{F}$ be a convenient class. Let $\varphi : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, be uniformly $\mathcal{F}$-computable, and let some uniformly $\mathcal{F}$-computable function ψ defined everywhere in D satisfy $\varphi(\xi_1, \ldots, \xi_N)\psi(\xi_1, \ldots, \xi_N) \geq 1$ for all $(\xi_1, \ldots, \xi_N)$ in D. Then the function $\lambda\xi_1 \ldots, \xi_N.1/\varphi(\xi_1, \ldots, \xi_N)$ is also uniformly $\mathcal{F}$-computable (in particular, the restriction of $\lambda\xi.1/\xi$ to the complement of a neighbourhood of 0 is uniformly $\mathcal{M}^2$-computable).*

Proof. If θ is the function from Example 14, and $(\xi_1, \ldots, \xi_N) \in D$, then

$$1/\varphi(\xi_1, \ldots, \xi_N) = \theta(\varphi(\xi_1, \ldots, \xi_N), \psi(\xi_1, \ldots, \xi_N)). \quad \square$$

3 Uniform $\mathcal{M}^2$-computability of certain functions related to the logarithmic and to the exponential ones

We will strengthen here some results from [9] by showing that certain two-argument functions related to the logarithmic and to the exponential one, respectively, are uniformly $\mathcal{M}^2$-computable.

3.1 A formula for the logarithms of the positive integers

Theorem 4 *For any $x \in \mathbb{N} \setminus \{0\}$, the following equality holds:*

$$x = 2^{\lfloor \log_2 x \rfloor} \prod_{k < \lfloor \log_2 x \rfloor} \frac{\lfloor x/2^k \rfloor}{\lfloor x/2^k \rfloor - \lfloor x/2^k \rfloor \bmod 2}.$$

Proof. Let $x \in \mathbb{N} \setminus \{0\}$, and let us set $m = \lfloor \log_2 x \rfloor$. Since $\lfloor x/2^0 \rfloor = x$, $\lfloor x/2^m \rfloor = 1$, $\lfloor x/2^{k+1} \rfloor \geq 1$ for any $k < m$, and $\lfloor x/2^k \rfloor = 2\lfloor x/2^{k+1} \rfloor + \lfloor x/2^k \rfloor \bmod 2$ for any $k \in \mathbb{N}$, we have

$$x = \prod_{k<m} \frac{\lfloor x/2^k \rfloor}{\lfloor x/2^{k+1} \rfloor} = 2^m \prod_{k<m} \frac{\lfloor x/2^k \rfloor}{\lfloor x/2^k \rfloor - \lfloor x/2^k \rfloor \bmod 2}. \quad \square$$

Corollary 7 *For any $x \in \mathbb{N} \setminus \{0\}$, the following equality holds:*

$$\ln x = \lfloor \log_2 x \rfloor \ln 2 - \sum_{k < \lfloor \log_2 x \rfloor} \left(\lfloor x/2^k \rfloor \bmod 2\right) \ln\left(1 - \frac{1}{\lfloor x/2^k \rfloor}\right).$$

3.2 $\mathcal{M}^2$-computability of the logarithmic function on the positive integers

Theorem 5 *The function $\Lambda : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ defined by $\Lambda(x) = \ln x$ is $\mathcal{M}^2$-computable.*

Proof. The statement follows from Lemma 5, since, by Corollary 7,

$$\Lambda(x) = \lfloor \log_2 x \rfloor \sigma(2) + \sum_{k<\lfloor \log_2 x \rfloor} \left(\lfloor x/2^k \rfloor \bmod 2 \right) \sigma\left(\lfloor x/2^k \rfloor \right) = \sum_{k \le \log_2 x} \theta(x,k),$$

where σ is as the function from Example 5, and θ is defined by the equalities

$$\theta(x,0) = \lfloor \log_2 x \rfloor \sigma(2) \quad \theta(x,k+1) = \left(\lfloor x/2^k \rfloor \bmod 2 \right) \sigma\left(\lfloor x/2^k \rfloor \right) \quad \square$$

Corollary 8 *There exist three-argument functions $a, b \in \mathcal{M}^2$ such that*

$$\left| \frac{a(x,y,n) - b(x,y,n)}{n+1} - \ln \frac{x}{y} \right| < \frac{1}{n+1}$$

for all $x, y \in \mathbb{N} \setminus \{0\}$ and all $n \in \mathbb{N}$.

Proof. By the equality $\ln \frac{x}{y} = \ln x - \ln y$ and Lemma 4. $\square$

3.3 Uniform $\mathcal{M}^2$-computability of a function related to the logarithmic one

Theorem 6 *Let $D = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \,\middle|\, \xi_1 > 0, \xi_1 \xi_2 \ge 1 \right\}$. Let $\theta : D \to \mathbb{R}$ be defined by $\theta(\xi_1, \xi_2) = \ln \xi_1$. Then θ is uniformly $\mathcal{M}^2$-computable.*

Proof. We define quaternary operators F, G, H by setting

$$F(f_1, g_1, f_2, g_2)(n) = a(p, q, 2n+1), \quad G(f_1, g_1, f_2, g_2)(n) = b(p, q, 2n+1),$$
$$H(f_1, g_1, f_2, g_2)(n) = 2n+1,$$

where a and b are as in Corollary 8, $q = 3(n+1)l$ with $l = (f_2(0) \dot{-} g_2(0)) + 1$, $p = f_1(q-1) \dot{-} g_1(q-1)$. The operators F, G, H are $\mathcal{M}^2$-substitutional. Suppose now that $(\xi_1, \xi_2) \in D$, $f_1, g_1, f_2, g_2 \in \mathbb{T}_1$, and $(f_k, g_k, \mathrm{id}_{\mathbb{N}})$ names ξ_k for $k = 1, 2$. The inequalities $\xi_1 > 0$, $\xi_1 \xi_2 \ge 1$, $\xi_2 < l$ imply that $\xi_1 > \frac{1}{l}$. Let $n \in \mathbb{N}$. Then

$$\left| \frac{p}{q} - \xi_1 \right| \le \left| \frac{f_1(q-1) - g_1(q-1)}{q} - \xi_1 \right| < \frac{1}{q} = \frac{1}{3(n+1)l} \le \frac{1}{3l},$$

hence $\frac{p}{q} > \xi_1 - \frac{1}{3l} > \frac{2}{3l}$ (thus both p and q are greater than $\frac{2}{3l}$). Therefore

$$|\langle F(f_1, g_1, f_2, g_2), G(f_1, g_1, f_2, g_2), H(f_1, g_1, f_2, g_2) \rangle(n) - \theta(\xi_1, \xi_2)| \le$$
$$\left| \frac{a(p,q,2n+1) - b(p,q,2n+1)}{2n+1} - \ln \frac{p}{q} \right| + \left| \ln \frac{p}{q} - \ln \xi_1 \right| < \frac{1}{2n+2} + \frac{3l}{2} \left| \frac{p}{q} - \xi_1 \right| <$$
$$\frac{1}{2n+2} + \frac{1}{2(n+1)} = \frac{1}{n+1}. \quad \square$$

Corollary 9 *Let $\mathcal{F}$ be a convenient class. Then*

1. *For any positive $\xi \in \mathbb{R}_{\mathcal{F}}$, the number $\ln \xi$ is also $\mathcal{F}$-computable.*
2. *If $\varphi : D \to (0, \infty)$, where $D \subseteq \mathbb{R}^N$, is a uniformly $\mathcal{F}$-computable function, such that some uniformly $\mathcal{F}$-computable function ψ defined everywhere in D satisfies the inequality $\varphi(\xi_1, \ldots, \xi_N)\psi(\xi_1, \ldots, \xi_N) \geq 1$ for any element $(\xi_1, \ldots, \xi_N)$ of D, then the function $\lambda \xi_1 \ldots, \xi_N . \ln \varphi(\xi_1, \ldots, \xi_N)$ is also uniformly $\mathcal{F}$-computable (in particular, the restriction of $\lambda \xi . \ln \xi$ to the numbers, greater than a positive one, is uniformly $\mathcal{M}^2$-computable).*

Proof. Let θ be the function from the above theorem. Then, for any positive real number ξ, we have $\ln \xi = \theta(\xi, k)$ if k is a natural number satisfying the inequality $\xi k \geq 1$. Under the assumptions of item 2 of the corollary, we have the equality $\ln \varphi(\xi_1, \ldots, \xi_N) = \theta(\varphi(\xi_1, \ldots, \xi_N), \psi(\xi_1, \ldots, \xi_N))$. $\square$

3.4 Uniform $\mathcal{M}^2$-computability of a function related to the exponential one

Let the function $\theta : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\theta(\xi, \eta) = \min(\exp(\xi), \eta)$. We will prove that θ is uniformly $\mathcal{M}^2$-computable.

Lemma 7 *There exists a five-argument function $c \in \mathcal{M}^2$ such that*

$$\left| \frac{c(r,s,t,u,n)}{n+1} - \theta\left(\frac{r-s}{t+1}, \frac{u+1}{n+1}\right) \right| \leq \frac{1}{n+1} \tag{4}$$

for all $r, s, t, u, n \in \mathbb{N}$.

Proof. Let a and b be such functions as in Corollary 8. For any $u, n, k \in \mathbb{N}$ we set

$$\xi_{n,k} = \frac{k+1}{n+1}, \quad \eta_{u,n,k} = \frac{a(k+1, n+1, 2u+1) - b(k+1, n+1, 2u+1)}{2u+2},$$

thus we will have $|\eta_{u,n,k} - \ln \xi_{n,k}| < \frac{1}{2u+2}$. Given any $r, s, t, u, n \in \mathbb{N}$, we set

$$c(r,s,t,u,n) = \min\left\{ k \;\middle|\; k = u \;\vee\; \eta_{u,n,k} \geq \frac{r-s}{t+1} + \frac{1}{2u+2} \right\}.$$

Then $c \in \mathcal{M}^2$, and we will show that the inequality (4) holds.

Let us set $j = c(r,s,t,u,n)$ for short. Of course

$$\frac{c(r,s,t,u,n)}{n+1} = \frac{j}{n+1} = \xi_{n,j} - \frac{1}{n+1},$$
$$\theta\left(\frac{r-s}{t+1}, \frac{u+1}{n+1}\right) = \min\left(\exp\left(\frac{r-s}{t+1}\right), \frac{u+1}{n+1}\right).$$

We will prove that
$$\xi_{n,j} \geq \theta\left(\frac{r-s}{t+1}, \frac{u+1}{n+1}\right).$$
This is clear in the case of $j = u$, and otherwise it follows from the fact that
$$\ln \xi_{n,j} > \eta_{u,n,j} - \frac{1}{2u+2} \geq \frac{r-s}{t+1},$$
hence
$$\xi_{n,j} > \exp\left(\frac{r-s}{t+1}\right).$$
Thus surely
$$\frac{j}{n+1} \geq \theta\left(\frac{r-s}{t+1}, \frac{u+1}{n+1}\right) - \frac{1}{n+1}.$$
To complete the proof, we have to prove that
$$\frac{j}{n+1} \leq \theta\left(\frac{r-s}{t+1}, \frac{u+1}{n+1}\right) + \frac{1}{n+1},$$
This inequality is obvious in the case of $j \leq 1$. Suppose now that $j > 1$. Then the inequality is equivalent to
$$\xi_{n,j-2} \leq \min\left(\exp\left(\frac{r-s}{t+1}\right), \frac{u+1}{n+1}\right),$$
and we may reason as follows. By the minimality of j, we have
$$j \leq u, \quad \eta_{u,n,j-1} < \frac{r-s}{t+1} + \frac{1}{2u+2},$$
Thus
$$\xi_{n,j-2} < \xi_{n,j-1} < \frac{u+1}{n+1}$$
and
$$\begin{aligned}\ln \xi_{n,j-2} < \ln \xi_{n,j-1} - \frac{n+1}{u+1}(\xi_{n,j-1} - \xi_{n,j-2}) = \ln \xi_{n,j-1} - \frac{1}{u+1} \\ < \eta_{u,n,j-1} - \frac{1}{2u+2} < \frac{r-s}{t+1},\end{aligned}$$
therefore
$$\xi_{n,j-2} < \exp\frac{r-s}{t+1}. \quad \square$$

Lemma 8 *For any $\xi_1, \xi_2 \in \mathbb{R}$ and any positive real number η, we have*
$$|\theta(\xi_1, \eta) - \theta(\xi_2, \eta)| \leq \eta|\xi_1 - \xi_2|.$$

Proof. Let us set $\xi = \ln \eta$. Then
$|\theta(\xi_1, \eta) - \theta(\xi_2, \eta)| = |\min(\exp(\xi_1), \exp(\xi)) - \min(\exp(\xi_2), \exp(\xi))| =$
$|\exp(\min(\xi_1, \xi)) - \exp(\min(\xi_2, \xi))| \leq \exp(\xi)|\min(\xi_1, \xi) - \min(\xi_2, \xi)| \leq \eta|\xi_1 - \xi_2|.$ $\square$

Theorem 7 *The function θ is uniformly $\mathcal{M}^2$-computable.*

Proof. Let the function c be as in Lemma 7. We set

$$C(f_1,g_1,f_2,g_2)(n) = c(r,s,t,u,3n+2),$$

where

$$t = ((f_2(0) \dot{-} g_2(0)) + 2)(3n+3) - 1, \quad r = f_1(t), \quad s = g_1(t),$$
$$u = f_2(3n+2) \dot{-} (g_2(3n+2)+1).$$

Then we define quaternary operators F, G, H as follows:

$$F(f_1,g_1,f_2,g_2)(n) = \begin{cases} C(f_1,g_1,f_2,g_2)(n) & \text{if } f_2(3n+2) > g_2(3n+2), \\ f_2(3n+2) & \text{otherwise,} \end{cases}$$

$$G(f_1,g_1,f_2,g_2)(n) = \begin{cases} 0 & \text{if } f_2(3n+2) > g_2(3n+2), \\ g_2(3n+2) & \text{otherwise,} \end{cases}$$

$$H(f_1,g_1,f_2,g_2)(n) = 3n+2.$$

The operators F, G, H are $\mathcal{M}^2$-substitutional. Suppose now that $(\xi_1,\xi_2) \in \mathbb{R}$, $f_1,g_1,f_2,g_2 \in \mathbb{T}_1$, $(f_1,g_1,\mathrm{id}_{\mathbb{N}})$ names ξ_1 and $(f_2,g_2,\mathrm{id}_{\mathbb{N}})$ names ξ_2.

Let $n \in \mathbb{N}$. If $f_2(3n+2) > g_2(3n+2)$ then we get

$$|\langle F(f_1,g_1,f_2,g_2), G(f_1,g_1,f_2,g_2), H(f_1,g_1,f_2,g_2)\rangle(n) - \theta(\xi_1,\xi_2)| =$$
$$\left|\frac{c(r,s,t,u,3n+2)}{3n+3} - \theta(\xi_1,\xi_2)\right| \le$$
$$\left|\frac{c(r,s,t,u,3n+2)}{3n+3} - \theta\left(\frac{r-s}{t+1}, \frac{u+1}{3n+3}\right)\right| + \left|\theta\left(\frac{r-s}{t+1}, \frac{u+1}{3n+3}\right) - \theta(\xi_1,\xi_2)\right| \le$$
$$\frac{1}{3n+3} + \left|\theta\left(\frac{r-s}{t+1}, \frac{u+1}{3n+3}\right) - \theta\left(\xi_1, \frac{u+1}{3n+3}\right)\right| + \left|\theta\left(\xi_1, \frac{u+1}{3n+3}\right) - \theta(\xi_1,\xi_2)\right| \le$$
$$\frac{1}{3n+3} + \frac{u+1}{3n+3}\left|\frac{r-s}{t+1} - \xi_1\right| + \left|\frac{u+1}{3n+3} - \xi_2\right| < \frac{1}{3n+3} + \frac{u+1}{3n+3}\frac{1}{t+1} + \frac{1}{3n+3} < \frac{1}{n+1},$$

making use of Lemma 8 and of the fact that

$$\frac{u+1}{3n+3} < \xi_2 + \frac{1}{3n+3} < f_2(0) - g_2(0) + 2 \le (f_2(0) \dot{-} g_2(0)) + 2 = \frac{t+1}{3n+3}.$$

Suppose now that $f_2(3n+2) \le g_2(3n+2)$. Then

$$\langle F(f_1,g_1,f_2,g_2), G(f_1,g_1,f_2,g_2), H(f_1,g_1,f_2,g_2)\rangle(n) = \langle f_2,g_2,\mathrm{id}_{\mathbb{N}}\rangle(3n+2),$$

and

$$|\langle f_2,g_2,\mathrm{id}_{\mathbb{N}}\rangle(3n+2) - \theta(\xi_1,\xi_2)| < \frac{1}{3n+3} < \frac{1}{n+1},$$

since we have the inequality

$$\langle f_2,g_2,\mathrm{id}_{\mathbb{N}}\rangle(3n+2) - \frac{1}{3n+3} < \theta(\xi_1,\xi_2)$$

(because its left-hand side is less than each of the numbers $\exp(\xi_1)$ and ξ_2 in this case), whereas always

$$\theta(\xi_1,\xi_2) \le \xi_2 < \langle f_2, g_2, \mathrm{id}_{\mathbb{N}}\rangle(3n+2) + \frac{1}{3n+3}\,. \quad \square$$

Corollary 10 *Let $\mathcal{F}$ be a convenient class. Then*

1. *For any $\xi \in \mathbb{R}_{\mathcal{F}}$ the number $\exp(\xi)$ is also $\mathcal{F}$-computable.*
2. *If $\varphi : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, is a uniformly $\mathcal{F}$-computable function, such that some uniformly $\mathcal{F}$-computable function ψ defined everywhere in D satisfies $\exp(\varphi(\xi_1,\dots,\xi_N)) \le \psi(\xi_1,\dots,\xi_N)$ for all $(\xi_1,\dots,\xi_N)$ in D, then the function $\lambda\xi_1\dots,\xi_N.\exp(\varphi(\xi_1,\dots,\xi_N))$ is also uniformly $\mathcal{F}$-computable (in particular, the restriction of the exponential function to the numbers less than a fixed one is uniformly $\mathcal{M}^2$-computable).*

Proof. Let θ be the function from the above theorem. Then, for any real number ξ, we have $\exp(\xi) = \theta(\xi, k)$ if k is a natural number satisfying the inequality $k \ge \exp(\xi)$. Under the assumptions of item 2 of the corollary, we have the equality $\exp(\varphi(\xi_1,\dots,\xi_N)) = \theta(\varphi(\xi_1,\dots,\xi_N), \psi(\xi_1,\dots,\xi_N))$. $\square$

4 The function arctan is uniformly $\mathcal{M}^2$-computable

The proof of the uniform $\mathcal{M}^2$-computability of the function arctan will be based on the fact that $\arctan\frac{y}{x}$, where $x \in \mathbb{N}\setminus\{0\}$, $y \in \mathbb{N}$, is $\mathcal{M}^2$-computable as a function of x and y. This fact will be proved by using the interconnection between complex logarithms and the arctan function. As well-known, the domain of the main branch of the complex logarithmic function consists of all complex numbers except the zero and the negative reals. For any ζ in this set, let $\ln\zeta$ denote the corresponding value of the branch in question, i.e. $\ln\zeta$ be the unique complex number ζ' which satisfies the conditions $\exp(\zeta') = \zeta$, $|\mathrm{Im}(\zeta')| < \pi$. Then, in particular, $\ln(\xi+\eta i) = \frac{1}{2}\ln(\xi^2+\eta^2) + i\arctan\frac{\eta}{\xi}$ for any positive real number ξ and any $\eta \in \mathbb{R}$. Thanks to this equality, any computational algorithm for the logarithms of the Gaussian integers in the first quadrant yields also a computational algorithm for the values of the arctan function on the positive rational numbers. Of course, we will look for an algorithm that can be performed by means of functions of the class $\mathcal{M}^2$.

4.1 A formula for logarithms of Gaussian integers in the first quadrant

For any natural number t, let t° denote the greatest even number not exceeding t, i.e. $t^\circ = t - t \bmod 2$. If $z = x + yi$, where $x, y \in \mathbb{N}$, we set $z^\circ = x^\circ + y^\circ i$. We set also $\lfloor x + yi\rfloor = \lfloor x\rfloor + \lfloor y\rfloor i$ for all $x, y \in \mathbb{R}$.

Theorem 8 *Let $z = x + yi \neq 0$, where $x, y \in \mathbb{N}$, and let $m = \lfloor \log_2 \max(x,y) \rfloor$. Then*

$$z = 2^m \lfloor z/2^m \rfloor \prod_{k<m} \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}. \tag{5}$$

Proof. For any $k < m$, we have $\lfloor z/2^{k+1} \rfloor \neq 0$ thanks to the inequalities $2^{k+1} \leq 2^m \leq \max(x,y)$. Therefore

$$z = \lfloor z/2^m \rfloor \prod_{k<m} \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^{k+1} \rfloor}.$$

For any $l, k \in \mathbb{N}$, the equality $\lfloor l/2^k \rfloor = 2 \lfloor l/2^{k+1} \rfloor + \lfloor l/2^k \rfloor \bmod 2$ holds, hence $\lfloor l/2^{k+1} \rfloor = \frac{1}{2} \lfloor l/2^k \rfloor^\circ$. Therefore

$$\prod_{k<m} \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^{k+1} \rfloor} = \prod_{k<m} \frac{\lfloor z/2^k \rfloor}{\frac{1}{2} \lfloor z/2^k \rfloor^\circ} = 2^m \prod_{k<m} \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}. \quad \square$$

Note. Under the assumptions of Theorem 8, $\lfloor z/2^m \rfloor = 1$ in the case of $2^m > y$, $\lfloor z/2^m \rfloor = i$ in the case of $2^m > x$, and $\lfloor z/2^m \rfloor = 1 + i$ otherwise (this follows from the inequalities $2^m \leq \max(x,y) < 2^{m+1}$).

By formal application of the equality $\ln(\zeta_1 \zeta_2) = \ln \zeta_1 + \ln \zeta_2$ to the equality (5) we could get the following one:

$$\ln z = m \ln 2 + \ln \lfloor z/2^m \rfloor + \sum_{k<m} \ln \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}. \tag{6}$$

Both sides of this equality make sense, but its validity would not be certain without some additional reasoning, since unfortunately $\ln(\zeta_1 \zeta_2) = \ln \zeta_1 + \ln \zeta_2$ does not always hold (one could take $\zeta_1 = \zeta_2 = -1 + i$ as a counter-example).

Lemma 9 *Let ζ_1 and ζ_2 be numbers from the domain of the main branch of the complex logarithmic function, and let ζ_2 and $\zeta_1 \zeta_2$ have non-negative real parts. Then $\ln(\zeta_1 \zeta_2) = \ln \zeta_1 + \ln \zeta_2$.*

Proof. Let $\zeta_1 \zeta_2 = \zeta$. Since $\zeta \neq 0$ and $\mathrm{Re}(\zeta) \geq 0$, $\ln \zeta$ is also defined. Let $\ln \zeta_1 = \zeta_1'$, $\ln \zeta_2 = \zeta_2'$, $\ln \zeta = \zeta'$. Then $\exp(\zeta_1') = \zeta_1$, $|\mathrm{Im}(\zeta_1')| < \pi$, $\exp(\zeta_2') = \zeta_2$, $|\mathrm{Im}(\zeta_2')| \leq \frac{\pi}{2}$, $\exp(\zeta') = \zeta$, $|\mathrm{Im}(\zeta')| \leq \frac{\pi}{2}$. We get from here that

$$\exp(\zeta_1' + \zeta_2') = \exp(\zeta'), \quad |\mathrm{Im}(\zeta_1' + \zeta_2') - \mathrm{Im}(\zeta')| < \pi + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi,$$

hence $\zeta_1' + \zeta_2' = \zeta'$. $\square$

Lemma 10 *If ζ_1 is a positive real number then $\ln(\zeta_1 \zeta_2) = \ln \zeta_1 + \ln \zeta_2$ for any ζ_2 from the domain of the main branch of the complex logarithmic function.*

Proof. Rather easy. $\square$

Theorem 9 *Let $z = x + yi \neq 0$, where $x, y \in \mathbb{N}$, and let $m = \lfloor \log_2 \max(x, y) \rfloor$. Then the equality (6) holds.*

Proof. By means of induction downwards from m to 0, we will show that

$$\ln\lfloor z/2^l \rfloor = (m - l)\ln 2 + \ln\lfloor z/2^m \rfloor + \sum_{l \le k < m} \ln \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}. \tag{7}$$

for any natural number $l \le m$, and of course (6) is the case $l = 0$ of (7). The equality (7) is trivially valid if $l = m$. Suppose now that $l < m$ and

$$\ln\lfloor z/2^{l+1} \rfloor = (m - l - 1)\ln 2 + \ln\lfloor z/2^m \rfloor + \sum_{l+1 \le k < m} \ln \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}.$$

Making use of the equality

$$\lfloor z/2^l \rfloor = \frac{\lfloor z/2^l \rfloor}{\lfloor z/2^{l+1} \rfloor} \lfloor z/2^{l+1} \rfloor,$$

of the inequalities $\mathrm{Re}(\lfloor z/2^{l+1} \rfloor) \ge 0$, $\mathrm{Re}(\lfloor z/2^l \rfloor) \ge 0$ and of Lemma 9, we get

$$\ln\lfloor z/2^l \rfloor = \ln \frac{\lfloor z/2^l \rfloor}{\lfloor z/2^{l+1} \rfloor} + \ln\lfloor z/2^{l+1} \rfloor.$$

Since the equality

$$\frac{\lfloor z/2^l \rfloor}{\lfloor z/2^{l+1} \rfloor} = 2\frac{\lfloor z/2^l \rfloor}{\lfloor z/2^l \rfloor^\circ}$$

and Lemma 10 yield

$$\ln \frac{\lfloor z/2^l \rfloor}{\lfloor z/2^{l+1} \rfloor} = \ln 2 + \ln \frac{\lfloor z/2^l \rfloor}{\lfloor z/2^l \rfloor^\circ},$$

we thus get (7) for the considered l. □

Corollary 11 *Let $z = x + yi$, where $x, y \in \mathbb{N}$, $x \neq 0$, and let $m = \lfloor \log_2 \max(x, y) \rfloor$. Then*

$$\arctan \frac{y}{x} = \mathrm{Im}\left(\ln\lfloor z/2^m \rfloor\right) + \sum_{k < m} \mathrm{Im}\left(\ln \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}\right).$$

Remark 5 *Under the assumptions of Theorem 9, either $\lfloor z/2^m \rfloor = 1$, $\ln\lfloor z/2^m \rfloor = 0$, or $\lfloor z/2^m \rfloor = i$, $\ln\lfloor z/2^m \rfloor = \frac{\pi}{2}i$, or $\lfloor z/2^m \rfloor = 1 + i$, $\ln\lfloor z/2^m \rfloor = \frac{1}{2}\ln 2 + \frac{\pi}{4}i$.*

Let us define a partial function ι from $\mathbb{N}^2$ to $\mathbb{R}$ by means of the equality

$$\iota(s, t) = \mathrm{Im}\left(\ln \frac{s + ti}{s^\circ + t^\circ i}\right)$$

($\mathrm{dom}(\iota)$ consists of the pairs $(s, t) \in \mathbb{N}^2$ such that $\max(s, t) \ge 2$). Then, under the assumptions of the above corollary,

$$\mathrm{Im}\left(\ln \frac{\lfloor z/2^k \rfloor}{\lfloor z/2^k \rfloor^\circ}\right) = \iota\left(\lfloor x/2^k \rfloor, \lfloor y/2^k \rfloor\right).$$

Therefore it would be important to prove that ι is $\mathcal{M}^2$-computable. This will be done in the next section.

4.2 $\mathcal{M}^2$-computability of the function ι

Let $(s,t) \in \mathrm{dom}(\iota)$, and let $z = s + ti$, $\delta(z) = s \bmod 2 + (t \bmod 2)i$. Since $|\delta(z)| \leq \sqrt{2} < 2 \leq |z| < |2z|$, the following transformation can be done:

$$\ln \frac{z}{z^\circ} = \ln \frac{z}{z - \delta(z)} = \ln \frac{1 + \frac{\delta(z)}{2z-\delta(z)}}{1 - \frac{\delta(z)}{2z-\delta(z)}}.$$

If at least one of the numbers s and t is odd, then

$$\delta(z) \neq 0, \quad \frac{\delta(z)}{2z - \delta(z)} = \frac{1}{2z/\delta(z) - 1},$$

and

$$2z/\delta(z) - 1 = \begin{cases} 2s - 1 + 2t\,i & \text{if } s \text{ is odd and } t \text{ is even,} \\ 2t - 1 - 2s\,i & \text{if } s \text{ is even and } t \text{ is odd,} \\ t + s - 1 + (t - s)\,i & \text{if both } s \text{ and } t \text{ are odd.} \end{cases} \tag{8}$$

Since $\max(s,t) \geq 2$, it is easy to see that $|2z/\delta(z) - 1| \geq \sqrt{17}$ in the first two of the above three cases, and $|2z/\delta(z) - 1| \geq \sqrt{13}$ in the third one. Taking into account that $\delta(z) = 0$ if both s and t are even, we conclude that always

$$\left| \frac{\delta(z)}{2z - \delta(z)} \right| \leq \frac{1}{\sqrt{13}} < 1,$$

hence $1 + \frac{\delta(z)}{2z-\delta(z)}$ and $1 - \frac{\delta(z)}{2z-\delta(z)}$ have positive real parts. From here and Lemma 7, we conclude that

$$\ln \frac{1 + \frac{\delta(z)}{2z-\delta(z)}}{1 - \frac{\delta(z)}{2z-\delta(z)}} = \ln\left(1 + \frac{\delta(z)}{2z - \delta(z)}\right) - \ln\left(1 - \frac{\delta(z)}{2z - \delta(z)}\right),$$

and therefore

$$\ln \frac{z}{z^\circ} = 2 \sum_{j=0}^{\infty} \frac{1}{2j+1} \left(\frac{\delta(z)}{2z - \delta(z)} \right)^{2j+1}.$$

Hence $\iota(s,t) = 2 \sum_{j=0}^{\infty} \theta(s,t,j)$, where

$$\theta(s,t,j) = \frac{1}{2j+1} \operatorname{Im}\left(\left(\frac{\delta(z)}{2z - \delta(z)} \right)^{2j+1} \right),$$

We will use the above representation of the function ι for applying Lemma 6. Since $|\theta(s,t,j)| \leq (1/\sqrt{13})^{2j+1}$, it would be sufficient to prove the $\mathcal{M}^2$-computability of the partial function θ, and actually it would be enough to prove that the partial function

$$\chi(s,t,j) = \operatorname{Im}\left(\left(\frac{\delta(z)}{2z - \delta(z)} \right)^{2j+1} \right)$$

is $\mathcal{M}^2$-computable.

If both numbers s and t are even then $\chi(s,t,j) = 0$ else

$$\chi(s,t,j) = \mathrm{Im}\left(\frac{1}{(2z/\delta(z)-1)^{2j+1}}\right).$$

Having in mind the equality (8), let us consider now the function $\psi : \mathbb{N}^4 \to \mathbb{R}$ defined by

$$\psi(u,v,w,k) = \mathrm{Im}\left(\frac{1}{(u+1+(v-w)i)^k}\right).$$

Then, whenever $(s,t,j) \in \mathrm{dom}(\chi)$,

$$\chi(s,t,j) = \begin{cases} \psi(0,0,0,0) & \text{if both } s \text{ and } t \text{ are even,} \\ \psi(2s \dot{-} 2, 2t, 0, 2j+1) & \text{if } s \text{ is odd and } t \text{ is even,} \\ \psi(2t \dot{-} 2, 0, 2s, 2j+1) & \text{if } s \text{ is even and } t \text{ is odd,} \\ \psi((t+s) \dot{-} 2, t, s, 2j+1) & \text{if both } s \text{ and } t \text{ are odd.} \end{cases}$$

Therefore we can be sure that ι is $\mathcal{M}^2$-computable if we succeed to prove the $\mathcal{M}^2$-computability of the function ψ.

Since

$$\frac{1}{(u+1+(v-w)i)^k} = \frac{(u+1-(v-w)i)^k}{((u+1)^2+(v-w)^2)^k},$$

we have the equality

$$\psi(u,v,w,k) = (v-w)\frac{f_0(u,v,w,k) - g_0(u,v,w,k)}{((u+1)^2+(v-w)^2)^k},$$

where

$$f_0(u,v,w,k) = \sum_{j \le \frac{k-3}{4}} \binom{k}{4j+3}(u+1)^{k-4j-3}(v-w)^{4j+2},$$

$$g_0(u,v,w,k) = \sum_{j \le \frac{k-1}{4}} \binom{k}{4j+1}(u+1)^{k-4j-1}(v-w)^{4j}.$$

Let us set

$$\begin{aligned} f(u,v,w,k,n) &= (v \dot{-} w) f_0(u,v,w,k) + (w \dot{-} v) g_0(u,v,w,k), \\ g(u,v,w,k,n) &= (w \dot{-} v) f_0(u,v,w,k) + (v \dot{-} w) g_0(u,v,w,k), \\ h(u,v,w,k,n) &= \left((u+1)^2 + (v-w)^2\right)^k - 1 \end{aligned}$$

if $((u+1)^2+(v-w)^2)^k \le (n+1)^2$, and

$$f(u,v,w,k,n) = g(u,v,w,k,n) = h(u,v,w,k,n) = 0$$

otherwise. Then $f, g, h : \mathbb{N}^5 \to \mathbb{N}$ and, since

$$|\psi(u,v,w,k)| \le \left|\frac{1}{((u+1+(v-w)i)^k}\right| = \frac{1}{\sqrt{\left((u+1)^2+(v-w)^2\right)^k}},$$

we have

$$\left| \frac{f(u,v,w,k,n) - g(u,v,w,k,n)}{h(u,v,w,k,n) + 1} - \psi(u,v,w,k) \right| < \frac{1}{n+1}$$

for all $u, v, w, k, n \in \mathbb{N}$. It remains to show that $f, g, h \in \mathcal{M}^2$.

If u, v, w, k, n satisfy the inequalities $((u+1)^2 + (v-w)^2)^k \le (n+1)^2$, $v - w \ne 0$, then $k \le 2\log_2(n+1)$, therefore the summation for computing the corresponding values of $f_0(u,v,w,k)$ and $g_0(u,v,w,k)$ will be logarithmically bounded for such u, v, w, k, n. This, together with results from [1, 2, 5], allows proving that each of the functions f, g, h has a Δ_0 definable graph. On the other hand, for any $u, v, w, k, n \in \mathbb{N}$ the corresponding values of these functions are less than $(n+1)^2$. This is obvious for the function h, and to get the same conclusion for f and g, we may use the fact that

$$|v-w| f_0(u,v,w,k) + |v-w| g_0(u,v,w,k) < (u+1+|v-w|)^k \le ((u+1)^2 + (v-w)^2)^k.$$

Thus each of the functions f, g, h is dominated by a polynomial and therefore belongs to $\mathcal{M}^2$.

4.3 The main result

Theorem 10 *The function $\theta(\xi) = \arctan \xi$ is uniformly $\mathcal{M}^2$-computable.*

Proof. After we proved the $\mathcal{M}^2$-computability of the function ι, we can use Corollary 11 and Remark 5 to show that $\arctan \frac{y}{x}$, where $x \in \mathbb{N} \setminus \{0\}$, $y \in \mathbb{N}$, is $\mathcal{M}^2$-computable as a function of x and y. This can be done be writing the equality from Corollary 11 in the form

$$\arctan \frac{y}{x} = \sum_{k \le \log_2 \max(x,y)} \upsilon\big((\lfloor x/2^k \rfloor, \lfloor y/2^k \rfloor\big),$$

where the function $\upsilon : \mathbb{N}^2 \to \mathbb{R}$ is defined as follows:

$$\upsilon(s,t) = \begin{cases} \iota(s,t) & \text{if } \max(s,t) \ge 2, \\ (2t \dot{-} s)\frac{\pi}{4} & \text{otherwise.} \end{cases}$$

By the $\mathcal{M}^2$-computability of $\lambda xy. \arctan \frac{y}{x}$ and Lemma 4, there exist three-argument functions $a, b \in \mathcal{M}^2$ such that $(\lambda n.a(x,y,n), \lambda n.b(x,y,n), \mathrm{id}_{\mathbb{N}})$ names $\arctan \frac{y}{x}$ for all $x \in \mathbb{N} \setminus \{0\}$ and all $y \in \mathbb{N}$, and we may assume that $b(x,y,n) = 0$ by taking $a(x,y,n) \dot{-} b(x,y,n)$ as a new $a(x,y,n)$. Then $a(x,0,n) = 0$ for any $x \in \mathbb{N} \setminus \{0\}$ due to the inequality $|a(x,0,n)| < 1$, and therefore

$$\left| \frac{a(x, r \dot{-} s, n) - a(x, s \dot{-} r, n)}{n+1} - \arctan \frac{r-s}{x} \right| < \frac{1}{n+1}$$

for all $x \in \mathbb{N} \setminus \{0\}$ and all $r, s, n \in \mathbb{N}$. Let the binary operators F, G, H be defined as follows:

$$\begin{aligned} F(f,g)(n) &= a(2n+2, f(2n+1) \dot{-} g(2n+1), 2n+1), \\ G(f,g)(n) &= a(2n+2, g(2n+1) \dot{-} f(2n+1), 2n+1), \\ H(f,g)(n) &= 2n+1. \end{aligned}$$

These operators are $\mathcal{M}^2$-substitutional. Suppose now that $\xi \in \mathbb{R}$, $f, g \in \mathbb{T}_1$, and $(f, g, \mathrm{id}_\mathbb{N})$ names ξ. Let $n \in \mathbb{N}$. Then

$$|\langle F(f,g), G(f,g), H(f,g)\rangle(n) - \theta(\xi)| \le$$
$$|\langle F(f,g), G(f,g), H(f,g)\rangle(n) - \arctan\langle f, g, \mathrm{id}_\mathbb{N}\rangle(2n+1)| +$$
$$|\arctan\langle f, g, \mathrm{id}_\mathbb{N}\rangle(2n+1) - \arctan\xi| \le \frac{1}{2n+2} + |\langle f, g, \mathrm{id}_\mathbb{N}\rangle(2n+1) - \xi| < \frac{1}{n+1} \quad \square$$

Corollary 12 *The functions* arcsin *and* arccos *are also uniformly $\mathcal{M}^2$-computable.*

Proof. This follows from the equalities

$$\arcsin\xi = 2\arctan\frac{\xi}{1+\sqrt{1-\xi^2}}, \quad \arccos\xi = 2\arctan 1 - \arcsin\xi$$

by Examples 11, 12, 13 and Corollaries 5, 6. $\square$

Corollary 13 *Let $\mathcal{F}$ be a convenient class. Then $\arctan\xi \in \mathbb{R}_\mathcal{F}$ for any $\xi \in \mathbb{R}_\mathcal{F}$, and $\arcsin\xi$, $\arccos\xi \in \mathbb{R}_\mathcal{F}$ for any $\xi \in \mathbb{R}_\mathcal{F}$ with $|\xi| \le 1$.*

5 The sine and cosine functions are uniformly $\mathcal{M}^2$-computable

We will firstly prove the following statement.

Lemma 11 *There exists a three-argument function $c \in \mathcal{M}^2$ such that*

$$\left|\frac{c(s,t,n)-(n+1)}{n+1} - \cos\frac{s}{t+1}\right| \le \frac{1}{n+1} \tag{9}$$

for all $s, t \in \mathbb{N}$ satisfying the inequality $s < (t+1)\pi$ and all $n \in \mathbb{N}$.

Proof. By Example 2, Proposition 6 and the $\mathcal{M}^2$-computability of the function arccos, the value of

$$\arccos\frac{p-q}{r+1},$$

where $p, q, r \in \mathbb{N}$, $|p-q| \le r+1$, is an $\mathcal{M}^2$-computable function of p, q, r. Therefore (by Lemma 4 and the non-negativeness of this value) there exists a four-argument function $a \in \mathcal{M}^2$ such that

$$\left|\frac{a(p,q,r,n)}{n+1} - \arccos\frac{p-q}{r+1}\right| < \frac{1}{n+1}$$

for any $p, q, r \in \mathbb{N}$ with $|p-q| \le r+1$ and all $n \in \mathbb{N}$.

For any $n, k \in \mathbb{N}$, let us set

$$\xi_{n,k} = \frac{k-n}{n+1}, \quad \eta_{n,k} = \frac{a(k,n,n,2n+1)}{2n+2}.$$

Then $-1 < \xi_{n,k} \le 1$, $|\eta_{n,k} - \arccos \xi_{n,k}| \le \frac{1}{2n+2}$, whenever $k \le 2n+1$. Given any $s, t, n \in \mathbb{N}$, we set

$$c(s,t,n) = \min \left\{ k \;\middle|\; k = 2n+1 \;\vee\; \eta_{n,k} + \frac{1}{2n+2} \le \frac{s}{t+1} \right\}.$$

Then $c \in \mathcal{M}^2$, and we will show that (9) holds, whenever $s < (t+1)\pi$.

Let the natural numbers s and t satisfy the inequality $s < (t+1)\pi$. Of course

$$\frac{c(s,t,n) - (n+1)}{n+1} = \xi_{n,c(s,t,n)} - \frac{1}{n+1}.$$

We shall prove that

$$\xi_{n,c(s,t,n)} \ge \cos \frac{s}{t+1}.$$

This is clear if $c(s,t,n) = 2n+1$, since then $\xi_{n,c(s,t,n)} = 1$, and otherwise it follows from the fact that

$$\arccos \xi_{n,c(s,t,n)} \le \eta_{n,c(s,t,n)} + \frac{1}{2n+2} \le \frac{s}{t+1}.$$

Thus surely

$$\frac{c(s,t,n) - (n+1)}{n+1} \ge \cos \frac{s}{t+1} - \frac{1}{n+1}.$$

To complete the proof, we have to prove that

$$\frac{c(s,t,n) - (n+1)}{n+1} \le \cos \frac{s}{t+1} + \frac{1}{n+1},$$

i.e. that

$$\xi_{n,c(s,t,n)} - \frac{2}{n+1} \le \cos \frac{s}{t+1}.$$

This inequality is obvious in the case of $c(s,t,n) \le 1$, since then the left-hand side does not exceed -1. Suppose now that $c(s,t,n) > 1$. Then we may reason as follows. By the minimality of $c(s,t,n)$, we have

$$\eta_{n,c(s,t,n)-1} + \frac{1}{2n+2} > \frac{s}{t+1},$$

hence

$$\begin{aligned}
\frac{s}{t+1} &< \arccos \xi_{n,c(s,t,n)-1} + \frac{1}{2n+2} + \frac{1}{2n+2} = \arccos \xi_{n,c(s,t,n)-1} + \frac{1}{n+1} \\
&= \arccos \xi_{n,c(s,t,n)-1} + \big(\xi_{n,c(s,t,n)-1} - \xi_{n,c(s,t,n)-2}\big) \\
&\le \arccos \xi_{n,c(s,t,n)-1} + \big(\arccos \xi_{n,c(s,t,n)-2} - \arccos \xi_{n,c(s,t,n)-1}\big) \\
&= \arccos \xi_{n,c(s,t,n)-2}.
\end{aligned}$$

Therefore

$$\cos \frac{s}{t+1} > \xi_{n,c(s,t,n)-2} = \xi_{n,c(s,t,n)} - \frac{2}{n+1}. \quad \square$$

Lemma 12 *Let θ_0 be the restriction of the cosine function to the interval $[0,\pi]$. Then θ_0 is uniformly $\mathcal{M}^2$-computable.*

Proof. Let the binary operators F, G, H be defined as follows:

$$F(f,g)(n) = c(s,t,2n+1), \quad G(f,g)(n) = 2n+2, \quad H(f,g)(n) = 2n+1,$$

where $t = 4n+3$, $s = f(t) \dot{-} (g(t)+1)$, and the function c is as in the above lemma. These operators are $\mathcal{M}^2$-substitutional. Suppose now that $\xi \in [0,\pi]$, $f, g \in \mathbb{T}_1$ and the triple $(f, g, \mathrm{id}_{\mathbb{N}})$ names ξ. Let $n \in \mathbb{N}$. Then

$$|\langle F(f,g), G(f,g), H(f,g)\rangle(n) - \theta_0(\xi)| \le \left|\frac{c(s,t,2n+1)-(2n+2)}{2n+2} - \cos\frac{s}{t+1}\right| +$$
$$\left|\cos\frac{s}{t+1} - \cos\xi\right| \le \left|\frac{c(s,t,2n+1)-(2n+2)}{2n+2} - \cos\frac{s}{t+1}\right| + \left|\frac{s}{t+1} - \xi\right|.$$

The inequalities

$$\xi - \frac{1}{t+1} < \langle f, g, \mathrm{id}_{\mathbb{N}}\rangle(t) < \xi + \frac{1}{t+1}$$

imply

$$\xi - \frac{2}{t+1} < \frac{f(t)-(g(t)+1)}{t+1} < \xi.$$

hence

$$\frac{s}{t+1} < \pi, \quad \xi - \frac{2}{t+1} < \frac{s}{t+1} \le \xi.$$

Therefore

$$\left|\frac{c(s,t,2n+1)-(2n+2)}{2n+2} - \cos\frac{s}{t+1}\right| \le \frac{1}{2n+2}, \quad \left|\frac{s}{t+1} - \xi\right| < \frac{2}{t+1} = \frac{1}{2n+2}. \quad \square$$

Lemma 13 *Let $\theta_1 : \mathbb{R} \to [0,\pi]$ be defined by $\theta_1(\xi) = \arccos(\cos\xi)$. Then θ_1 is uniformly $\mathcal{M}^2$-computable.*

Proof. For any $\xi \in \mathbb{R}$, we have $\theta_1(\xi) = 2\pi \operatorname{dist}\left(\frac{\xi}{2\pi}, \mathbb{Z}\right) = 2\pi \operatorname{dist}\left(\left|\frac{\xi}{2\pi}\right|, \mathbb{N}\right)$, thus it would be sufficient to prove that the function $\lambda\xi.\operatorname{dist}(|\xi|, \mathbb{N})$ is uniformly $\mathcal{M}^2$-computable. Let the binary operators F, G, H be defined as follows:

$$F(f,g)(n) = \left|m - (n+1)\left\lfloor\frac{m}{n+1} + \frac{1}{2}\right\rfloor\right|, \quad G(f,g)(n) = 0, \quad H(f,g)(n) = n,$$

where $m = |f(n) - g(n)|$. They are $\mathcal{M}^2$-substitutional. Suppose now that $\xi \in \mathbb{R}$, $f, g \in \mathbb{T}_1$ and the triple $(f, g, \mathrm{id}_{\mathbb{N}})$ names ξ. Let $n \in \mathbb{N}$. Then

$$|\langle F(f,g), G(f,g), H(f,g)\rangle(n) - \operatorname{dist}(|\xi|, \mathbb{N})| = \left|\operatorname{dist}\left(\frac{m}{n+1}, \mathbb{N}\right) - \operatorname{dist}(|\xi|, \mathbb{N})\right| \le$$
$$\left|\frac{m}{n+1} - |\xi|\right| \le |\langle f, g, \mathrm{id}_{\mathbb{N}}\rangle(n) - \xi| < \frac{1}{n+1}. \quad \square$$

Theorem 11 *The sine and cosine functions are uniformly $\mathcal{M}^2$-computable.*

Proof. We may use the equality $\cos\xi = \theta_0(\theta_1(\xi))$, where θ_0 and θ_1 are as in the above lemmas, and the equality $\sin\xi = \cos(2\arctan 1 - \xi)$. □

Corollary 14 *Let $\mathcal{F}$ be a convenient class, and let $\xi \in \mathbb{R}_{\mathcal{F}}$. Then $\sin\xi$ and $\cos\xi$ also belong to $\mathbb{R}_{\mathcal{F}}$ (hence if $\cos\xi \neq 0$ then $\tan\xi \in \mathbb{R}_{\mathcal{F}}$ too).*

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