On transition from local to global presence of properties of functions

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A lemma from: Klaus Weihrauch. *Computable Analysis. An Introduction*. Berlin/Heidelberg, Springer-Verlag, 2000

Lemma 4.3.5 (join of functions)

Let $f1, f2 :\subseteq \mathbb{R} \to \mathbb{R}$ be computable real functions and let $c \in \mathbb{R}$ be a computable real number. Then the function $f :\subseteq \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} f_1(x) & \text{if } x < c, \\ f_2(x) & \text{if } x > c, \\ f_1(c) & \text{if } x = c \text{ and } f_1(c) = f_2(c), \\ \mathrm{div} & \text{otherwise,} \end{cases}$$

is computable.

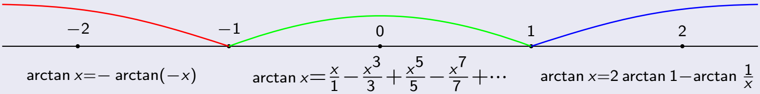
An equivalent reformulation of this lemma

Let $f :\subseteq \mathbb{R} \to \mathbb{R}$, let c be a computable real number, and let both $f \restriction [c, \infty)$, $f \restriction (-\infty, c]$ be computable. Then f is also computable.

An example and a problem related to it

Example 1

Let $\mathbf{C}$ be the class of the computable partial functions from $\mathbb{R}$ to $\mathbb{R}$, and let $f = \lambda x.\arctan x$. To prove that $f \in \mathbf{C}$, we start by consecutively proving that $f \restriction [-1,1]$, $f \restriction [1,\infty)$, $f \restriction (-\infty,-1] \in \mathbf{C}$.



We proceed then by applying the reformulated lemma as follows:

1. To $f \restriction [-1,\infty)$ with $c = 1$.
2. To f with $c = -1$.

The problem. Let X be a set, and $\mathbf{C}$ be a class of functions with domains contained in X. Give a characterization of the finite collections $\mathcal{A}$ of subsets of X such that the following holds:

$(\sigma_{\mathbf{C}}^{\mathcal{A}})$ whenever f is a function with $\mathrm{dom}(f) \subseteq \bigcup \mathcal{A}$, and $f \restriction A \in \mathbf{C}$ for any $A \in \mathcal{A}$, then $f \in \mathbf{C}$.

From local computability in $\mathbb{N}^n$ to global one

A paper at http://arxiv.org/abs/1609.04254 presents solutions of the formulated problem for certain X and $\mathbf{C}$. Results essentially contained in this paper will be indicated by $\#$ in front of them.

Definition

A set H is called to *quasi-separate* a set P from a set Q if $H \supseteq P \setminus Q$ and $H \cap (Q \setminus P) = \varnothing$.

 # Theorem 1

Let $n \in \mathbb{N}^+$, $\mathcal{A}$ be a finite collection of subsets of $\mathbb{N}^n$,

$$\mathbf{C} = \{f \mid f :\subseteq \mathbb{N}^n \to \mathbb{N},\ f \text{ is potentially partial recursive}\},$$

$$\mathbf{H} = \{H \mid H \subseteq \mathbb{N}^n,\ H \text{ is recursively enumerable}\}.$$

Then the condition $(\sigma^{\mathcal{A}}_{\mathbf{C}})$ is equivalent to the following one:

$(\alpha^{\mathcal{A}}_{\mathbf{H}})$ for any subcollection $\mathcal{K}$ of $\mathcal{A}$, some set from $\mathbf{H}$ quasi-separates $\bigcup\mathcal{K}$ from $\bigcup(\mathcal{A} \setminus \mathcal{K})$.

This equivalence remains valid after replacement of “partial recursive” and “recursively enumerable” with “recursive” or with “primitive recursive”.

A slight generalization of Theorem 1

Let $Y \subseteq \mathbb{N}$. A partial function from $\mathbb{N}^n$ to $\mathbb{N}$ will be called *potentially partial recursive to* Y if it can be extended to a partial recursive function with values in Y. One defines similarly potential recursiveness to Y and potential primitive recursiveness to Y.

Theorem 1'

Let $n \in \mathbb{N}^+$, $\mathcal{A}$ be a finite collection of subsets of $\mathbb{N}^n$,

$$\mathbf{C} = \{f \,|\, f :\subseteq \mathbb{N}^n \to \mathbb{N},\ f \text{ is potentially partial recursive to} Y\},$$

$$\mathbf{H} = \{H \,|\, H \subseteq \mathbb{N}^n,\ H \text{ is recursively enumerable}\},$$

and let Y have more than one element. Then $(\sigma^{\mathcal{A}}_{\mathbf{C}}) \Leftrightarrow (\alpha^{\mathcal{A}}_{\mathbf{H}})$. This equivalence remains valid after replacement of "partial recursive" and "recursively enumerable" with "recursive" or with "primitive recursive".

Remark 1

A function with values in Y can be potentially partial recursive without being potentially partial recursive to Y. Such is, for instance, id_Y if Y is not recursively enumerable.

Similar theorems for continuity and for computability in topological spaces

Theorem 2

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces with carriers X and Y, respectively, $\mathcal{A}$ be a finite collection of subsets of X,

$$\mathbf{C} = \{f \mid f :\subseteq \mathfrak{X} \to \mathfrak{Y},\ f \text{ is continuous}\},$$

$$\mathbf{H} = \{H \mid H \text{ is an open set of } \mathfrak{X}\}.$$

Then $(\alpha^{\mathcal{A}}_{\mathbf{H}}) \Rightarrow (\sigma^{\mathcal{A}}_{\mathbf{C}})$, and if there exists an open set of $\mathfrak{Y}$ different from $\varnothing$ and Y then $(\sigma^{\mathcal{A}}_{\mathbf{C}}) \Leftrightarrow (\alpha^{\mathcal{A}}_{\mathbf{H}})$.

Theorem 3

Let $\mathfrak{X}$ be a computable topological space with carrier X, $\mathfrak{Y}$ be an effective topological space, $\mathcal{A}$ be a finite collection of subsets of X,

$$\mathbf{C} = \{f \mid f :\subseteq \mathfrak{X} \to \mathfrak{Y},\ f \text{ is TTE computable}\},$$

$$\mathbf{H} = \{H \mid H \text{ is an effectively open set of } \mathfrak{X}\}.$$

Then $(\alpha^{\mathcal{A}}_{\mathbf{H}}) \Rightarrow (\sigma^{\mathcal{A}}_{\mathbf{C}})$, and if there exist at least two different computable elements of $\mathfrak{Y}$ then $(\sigma^{\mathcal{A}}_{\mathbf{C}}) \Leftrightarrow (\alpha^{\mathcal{A}}_{\mathbf{H}})$.

Theorem 4

Let $\mathcal{A}$ be a finite collection of sets, and $\mathbf{H}$ be a class of sets such that:

1. Whenever $H_1, H_2 \in \mathbf{H}$, then $H_1 \cup H_2$, $H_1 \cap H_2 \in \mathbf{H}$.
2. $\varnothing \in \mathbf{H}$.
3. $\bigcup \mathcal{A} \subseteq X$ for some $X \in \mathbf{H}$.

Then the following condition is sufficient for the fulfillment of the condition $(\alpha^{\mathcal{A}}_{\mathbf{H}})$:

$(\beta^{\mathcal{A}}_{\mathbf{H}})$ for any two different sets A and A' from $\mathcal{A}$, some set belonging to $\mathbf{H}$ quasi-separates A from A'.

Outline of the proof. Let condition $(\beta^{\mathcal{A}}_{\mathbf{H}})$ be satisfied, and $\mathcal{K}$ be a subcollection of $\mathcal{A}$. In the case when $\mathcal{K} \neq \varnothing$ and $\mathcal{K} \neq \mathcal{A}$, let, for any $A \in \mathcal{K}$ and any $A' \in \mathcal{A} \setminus \mathcal{K}$, $H_{A,A'}$ be a set of $\mathbf{H}$ quasi-separating A from A'. Then $\bigcup \mathcal{K}$ is quasi-separated from $\bigcup(\mathcal{A} \setminus \mathcal{K})$ by the set $\bigcup_{A \in \mathcal{K}} \bigcap_{A' \in \mathcal{A} \setminus \mathcal{K}} H_{A,A'}$. The case when $\mathcal{K} = \varnothing$ or $\mathcal{K} = \mathcal{A}$ is easy.

Collections of sets from **H** or of complements of such ones

Corollary

Under the assumptions of Theorem 4, the condition $(\alpha^{\mathcal{A}}_{\mathbf{H}})$ is surely satisfied if $\mathcal{A} \subseteq \mathbf{H}$ or $\mathcal{A} \subseteq \{X \setminus H \mid H \in \mathbf{H}\}$. Therefore:

- # If $n \in \mathbb{N}^+$ and $\mathcal{A}$ is a finite collection of recursively enumerable subsets of $\mathbb{N}^n$ or of co-recursively enumerable subsets of $\mathbb{N}^n$ then condition $(\sigma^{\mathcal{A}}_{\mathbf{C}})$ is surely satisfied for
$$\mathbf{C} = \{f \mid f :\subseteq \mathbb{N}^n \to \mathbb{N},\ f \text{ is potentially partial recursive}\}.$$
- # If $\mathfrak{X}$ is a computable topological space, $\mathfrak{Y}$ is an effective topological space, and $\mathcal{A}$ is a finite collection of effectively open subsets of $\mathfrak{X}$ or a finite collection of effectively closed subsets of $\mathfrak{X}$ then condition $(\sigma^{\mathcal{A}}_{\mathbf{C}})$ is surely satisfied for
$$\mathbf{C} = \{f \mid f :\subseteq \mathfrak{X} \to \mathfrak{Y},\ f \text{ is TTE computable}\}.$$

Remark 2

Let $\mathcal{A}$ be a finite collection of sets, $X \supseteq \bigcup \mathcal{A}$, **H** be a class of sets, and $\mathbf{H}' = \{X \setminus H \mid H \in \mathbf{H}\}$. Then $(\alpha^{\mathcal{A}}_{\mathbf{H}}) \Leftrightarrow (\alpha^{\mathcal{A}}_{\mathbf{H}'})$ and $(\beta^{\mathcal{A}}_{\mathbf{H}}) \Leftrightarrow (\beta^{\mathcal{A}}_{\mathbf{H}'})$.

Some instructive counterexamples to $(\alpha^{\mathcal{A}}_{\mathbf{H}}) \Rightarrow (\beta^{\mathcal{A}}_{\mathbf{H}})$

Let X be a set, $\mathbf{H}$ be a class of subsets of X such that $\varnothing \in \mathbf{H}$, $X \in \mathbf{H}$.

Example 2

Suppose $A \subset X$ and $A \notin \mathbf{H}$. Let $\mathcal{A} = \{A, X \setminus A, X\}$. Then $(\alpha^{\mathcal{A}}_{\mathbf{H}})$ is satisfied: if $\mathcal{K} \subseteq \mathcal{A}$ then $\bigcup\mathcal{K}$ is quasi-separated from $\bigcup(\mathcal{A} \setminus \mathcal{K})$ by X in the case of $X \in \mathcal{K}$ and by $\varnothing$ otherwise. However, $(\beta^{\mathcal{A}}_{\mathbf{H}})$ fails, since no set from $\mathbf{H}$ quasi-separates A from $X \setminus A$.

Example 3

Suppose $E_0, E'_0, E_1, E'_1, E_2, E'_2$ are pairwise disjoint subsets of X not belonging to $\mathbf{H}$ such that $H \cap (E_i \cup E'_i) \in \mathbf{H}$ for any $H \in \mathbf{H}$ and any $i \in \{0, 1, 2\}$. Let $\mathcal{A} = \{A_0, A_1, A_2\}$, where

$$A_0 = E_0 \cup E'_0 \cup E'_1 \cup E_2, \;\; A_1 = E_1 \cup E'_1 \cup E'_2 \cup E_0, \;\; A_2 = E_2 \cup E'_2 \cup E'_0 \cup E_1.$$

Condition $(\alpha^{\mathcal{A}}_{\mathbf{H}})$ is satisfied thanks to the inclusions

$$A_0 \subseteq A_1 \cup A_2, \;\; A_1 \subseteq A_0 \cup A_2, \;\; A_2 \subseteq A_0 \cup A_1.$$

However, for any $i, j \in \{0, 1, 2\}$, no set from $\mathbf{H}$ quasi-separates A_i from A_j if $i \neq j$. E. g. if a set H quasi-separates A_0 from A_1 then $H \supseteq E_2$, $H \cap E'_2 = \varnothing$, hence $H \cap (E_2 \cup E'_2) = E_2 \notin \mathbf{H}$, thus $H \notin \mathbf{H}$.

Thank you for your attention!