TTE computability on Moschovakis extensions

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Introduction

Two directions of Computability Theory have been the main ones at the beginning – computability on the natural numbers and computability on the real numbers. Certain fargoing but diverging generalizations of them were given later in the following two papers, respectively:

- Yiannis Moschovakis. Abstract first order computability. I. *Trans. Amer. Math. Soc.*, **138**, 427–464, 1969.
- Christoph Kreitz, Klaus Weihrauch. Theory of representations. *Theor. Comput. Sci.*, **38**, 35–53, 1985.

The present talk points to a possibility to combine the work in the frame of these two generalizations.

Moschovakis extension of a set

Let B be a set, let $B^0 = B \cup \{o\}$, where o is some element not belonging to B, and let no element of B^0 be an ordered pair.

Definition

The *Moschovakis extension* B^* of B is the closure of B^0 under formation of ordered pairs.

Thus B^* is the least set M such that $B^0 \subseteq M$ and $M \times M \subseteq M$.

Inductive definition of B^*

1. If $z \in B$ then $z \in B^*$.
2. $o \in B^*$.
3. If $x \in B^*$ and $y \in B^*$ then $(x, y) \in B^*$.

Clearly each element of the set B^* belongs to it according to only one clause of the inductive definition.
The set B^* can be regarded as an universum appropriate for modelling the execution of programs for computations in B.

Absolute prime computability

In his above-mentioned paper Moschovakis introduces several computability notions for functions in B^*, absolute prime computability being the most narrow one.

The considered functions can be not only partial, but also multivalued. In the case of single-valued partial functions, the following intuitive description of absolute prime computability can be given: a function is absolutely prime computable in some given ones if there is a deterministic program for its computation using the given functions as primitive ones.

The precise definition given by Moschovakis is rather complicated. A simple functional programming style characterization of absolute prime computability follows e.g. from results in my book "Computability in Combinatory Spaces" (Kluwer Acad. Publ., 1992). This characterization will be used later in the talk (the book will be referred to as [CCS]).

Representations

Notations:
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N}^{\mathbb{N}}$ is the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are, respectively, the sets of the integers, of the rational numbers and of the real numbers.

Definition

A *representation* of a set M is a subset δ of $\mathbb{N}^{\mathbb{N}} \times M$ such that:

1. $\forall x \in M \, \exists p \, ((p, x) \in \delta)$.
2. $\forall p \forall x \forall y ((p, x) \in \delta \,\&\, (p, y) \in \delta \Rightarrow x = y)$.

Only sets with cardinality $\leq \mathfrak{c}$ can have representations.

Example

Let α be a mapping of $\mathbb{N}$ onto $\mathbb{Q}$. Let

$$\delta = \left\{ (p, x) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{R} \;\middle|\; \forall i \in \mathbb{N} \left(|\alpha(p(i)) - x| < 2^{-i} \right) \right\}.$$

Then δ is a representation of $\mathbb{R}$. It is called *Cauchy representation of* $\mathbb{R}$ in the case when α is computable.

Equivalent representations. Representations of subsets

Definition

Let δ and $\bar{\delta}$ be representations of one and the same set. It will be said that δ *is reducible to* $\bar{\delta}$ if a computable partial mapping υ of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$ exists such that

$$\forall p \forall x((p, x) \in \delta \Rightarrow p \in \mathrm{dom}(\upsilon) \,\&\, (\upsilon(p), x) \in \bar{\delta}).$$

It will be said that δ *is equivalent to* $\bar{\delta}$ if each of the representations δ and $\bar{\delta}$ is reducible to the other one.

Example

Any two Cauchy representations of $\mathbb{R}$ are mutually equivalent.

Definition

Let δ be a representation of a set M, and let $K \subseteq M$. The intersection $\delta \cap \left(\mathbb{N}^{\mathbb{N}} \times K\right)$ is a representation of K which will be said to be *induced by* δ. The representation δ will be said to be *in accordance* with the representations of K equivalent to $\delta \cap \left(\mathbb{N}^{\mathbb{N}} \times K\right)$.

TTE computability (type two effective computability)

Definition

Let δ be a representation of a set M. An element x of M is called *δ-computable* if $(p, x) \in \delta$ for some computable p from $\mathbb{N}^{\mathbb{N}}$.

Example

If δ is a Cauchy representation of $\mathbb{R}$ then π is δ-computable.

Definition

Let φ be a partial function from the set M_1 to the set M_2, δ_k be a representation of M_k for $k = 1, 2$. A *(δ_1, δ_2)-realization of φ* is any partial mapping γ of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$ such that

$$\forall x \in \mathrm{dom}(\varphi) \forall p((p, x) \in \delta_1 \\ \Rightarrow p \in \mathrm{dom}(\gamma) \,\&\, \gamma(p) \in \mathrm{dom}(\varphi) \,\&\, (\gamma(p), \varphi(x)) \in \delta_2).$$

The function φ is called *(δ_1, δ_2)-computable* if there is a computable (δ_1, δ_2)-realization of φ.

Example

If δ is a Cauchy representation of $\mathbb{R}$ then the function $x \mapsto \ln x$ is (δ, δ)-computable.

Definition

Let δ be a representation of a set M, and h be a partial mapping of M into $\mathbb{N}$. The mapping h will be said to be δ-*computable* if there is a computable partial mapping of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}$ such that

$$\forall x \in \mathrm{dom}(h) \forall p((p, x) \in \delta \Rightarrow p \in \mathrm{dom}(\gamma) \,\&\, \gamma(p) = h(x)).$$

Example

Let $h : [0, +\infty) \setminus \mathbb{N} \to \mathbb{N}$ be such that $h(x) = \lfloor x \rfloor$ for all x in $\mathrm{dom}(h)$. If δ is a Cauchy representation of $\mathbb{R}$ then h is δ-computable.

Acceptable representations of the Moschovakis extension

Definition

For any q and r in $\mathbb{N}^{\mathbb{N}}$, we set $\langle q, r\rangle$ to be the element p of $\mathbb{N}^{\mathbb{N}}$ such that $p(2k) = q(k)$ and $p(2k+1) = r(k)$ for all k in $\mathbb{N}$. If δ is a representation of a set M then we define a representation $\delta \times \delta$ of $M \times M$ by setting

$$\delta \times \delta = \{(\langle q, r\rangle, (x, y) \,|\, (q, x) \in \delta \,\&\, (r, y) \in \delta\}.$$

Definition

A representation δ of the Moschovakis extension B^* will be called *acceptable* if the following conditions are satisfied:

1. The representation $\delta \times \delta$ of the set $B^* \times B^*$ is equivalent to its representation induced by δ.
2. The element o and the mapping h of B^* into $\mathbb{N}$ defined by

$$h(z) = \begin{cases} 0 & \text{if } z = o, \\ 1 & \text{if } z \in B, \\ 2 & \text{otherwise} \end{cases}$$

   are δ-computable.

An equivalent to condition 1 of the previous definition

Lemma

Let the functions L and R from $B^* \times B^*$ to B^* be defined as follows: $L(z) = x$ and $R(z) = y$ whenever $z = (x, y)$. A representation δ of B^* satisfies condition 1 of the previous definition iff each of the functions L, R is (δ, δ)-computable, and the implication

$$(q, x) \in \delta \,\&\, (r, y) \in \delta \Rightarrow (q, r) \in \mathrm{dom}(\gamma) \,\&\, (\gamma(q, r), (x, y)) \in \delta$$

is identically satisfied by some computable partial mapping γ of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$.

Proof. Let δ be a representation of B^* and $\bar{\delta}$ be the representation of $B^* \times B^*$ induced by δ. The equivalence of $\delta \times \delta$ to $\bar{\delta}$ is equipollent to the existence of computable partial mappings υ and $\bar{\upsilon}$ of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$ which identically satisfy the implications

$$(\langle q, r\rangle, (x, y)) \in \delta \Rightarrow \langle q, r\rangle \in \mathrm{dom}(\upsilon) \,\&\, (q, x) \in \delta \,\&\, (r, y) \in \delta,$$

$$(q, x) \in \delta \,\&\, (r, y) \in \delta \Rightarrow \langle q, r\rangle \in \mathrm{dom}(\bar{\upsilon}) \,\&\, (\bar{\upsilon}(\langle q, r\rangle), (x, y)) \in \delta. \quad \square$$

On the existence of acceptable representations of B^*

Theorem 1

Let $\dot{\delta}$ be a representation of B. Then there is an acceptable representation of B^* which is in accordance with $\dot{\delta}$. All such representations of B^* are mutually equivalent.

Proof of the first statement. We choose a computable injective mapping J of $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$ such that $2J(i,j) \geq \max(i,j)$ for all i,j in $\mathbb{N}$, and then we define inductively a subset δ of $\mathbb{N}^{\mathbb{N}} \times B^*$ as follows:

1. $(\lambda k.0, o) \in \delta$.
2. If $(p,x) \in \dot{\delta}$ then $(\lambda k.2p(k)+2, x) \in \delta$.
3. If $(q,x) \in \delta$ and $(r,y) \in \delta$ then $$(\lambda k.2J(q(k),r(k))+1,(x,y)) \in \delta.$$

The set δ is a representation of B^*. It satisfies condition 1 in the definition of acceptability thanks to the equivalence

$$(\langle q,r\rangle,(x,y)) \in \delta \times \delta \Leftrightarrow (\lambda k.2J(q(k),r(k))+1,(x,y)) \in \delta.$$

It is easy to see that the second condition is also satisfied and that δ is in accordance with $\dot{\delta}$. $\square$

Some functions in B^* and some operations on such functions

At first, we will restrict ourselves to single-valued unary functions. The functions $\hat{L}$ and $\hat{R}$ from B^* to B^* will be defined as follows:

$$\text{if } z = (x, y) \text{ then } \hat{L}(z) = x \text{ and } \hat{R}(z) = y,$$

$$\hat{L}(z) = \hat{R}(z) = (o, o) \text{ for } z \in B, \quad \hat{L}(o) = \hat{R}(o) = o$$

(Moschovakis denotes them by π and δ). Let φ and ψ be single-valued unary partial functions in B^*. As usual, the *composition* $\varphi\psi$ is the function $z \mapsto \varphi(\psi(z))$ with $\{z \in \mathrm{dom}(\psi) \mid \psi(z) \in \mathrm{dom}(\varphi)\}$ as its domain. The *juxtaposition* of φ and ψ is $z \mapsto (\varphi(z), \psi(z))$ with domain $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$. The *iteration of* φ *controlled by* ψ can be briefly described as the action of the Pascal-style statement

$$\text{while } \psi(z) \in B^* \times B^* \text{ do } z := \varphi(z).$$

Example

If ι is the iteration of $\hat{L}$ controlled by $\hat{L}$ then $\hat{L}^3\iota(z) = o$ for all z in B^* (of course $\hat{L}^3$ is the composition $\hat{L}\hat{L}\hat{L}$).

Absolute prime computability of unary functions

Lemma

Let $\varphi_1, \ldots, \varphi_l$ be unary functions in B^*. A unary function in B^* is absolutely prime computable in $\varphi_1, \ldots, \varphi_l$ iff it can be obtained from the functions $\hat{L}, \hat{R}, \varphi_1, \ldots, \varphi_l$ by means of finitely many applications of the operations composition, juxtaposition and iteration.

Proof. Follows from Theorem 1 on pp. 192–193 in [CCS] and the example on the previous slide. □

Remark

The above theorem actually holds in the general case of multivalued partial functions if one adopts the usual definition of composition of such functions (corresponding to the usual composition of relations) and the natural definitions in the same spirit of juxtaposition and iteration.

Preservation of TTE computability under absolute prime computability

Theorem 2

Let δ be an acceptable representation of B^*, and $\varphi_1, \dots, \varphi_l$ be (δ, δ)-computable unary partial functions in B^*. Then all unary partial functions in B^* absolutely prime computable in $\varphi_1, \dots, \varphi_l$ are also (δ, δ)-computable.

Outline of the proof. One makes use of the preservation of (δ, δ)-computability of unary functions in B^* under composition, as well as of the validity of the next statements which can be proved by using the acceptability of δ:

1. The functions $\hat{L}$ and $\hat{R}$ are (δ, δ)-computable.
2. If φ and ψ are (δ, δ)-computable unary partial functions in B^* then the juxtaposition of φ and ψ, as well as the iteration of φ controlled by ψ are also (δ, δ)-computable. □

TTE computability of least fixed points

Corollary

Let δ be an acceptable representation of B^*, and $\varphi_1, \ldots, \varphi_l$ be (δ, δ)-computable unary partial functions in B^*. Let Γ be a mapping of the set of the unary partial functions in B^* into itself such that, for any such function θ, the corresponding function $\Gamma(\theta)$ can be obtained from the functions $\theta, \varphi_1, \ldots, \varphi_l$ by means of finitely many applications of the operations composition, juxtaposition and iteration in a uniform way. Then Γ has a least fixed point, and it is (δ, δ)-computable.

Proof. By application of the First Recursion Theorem for iterative combinatory spaces (Theorem 1 on p. 170 in [CCS]). □

About branching

Remark

A ternary operation of branching in the set of the unary partial functions in B^* can be defined in the following natural way: for any three partial functions χ, φ, ψ in B^*, the result of the operation, briefly described, is the action of the following Pascal-style statement:

$$\text{if } \chi(z) \in B^* \times B^* \text{ then } z := \varphi(z) \text{ else } z := \psi(z).$$

This operation turns out to be uniformly reducible to the operations composition, juxtaposition and iteration using $\chi, \varphi, \psi, \hat{L}, \hat{R}$ as primitive objects. Therefore one can add it to the list of operations permitted in obtaining $\Gamma(\theta)$ on the previous slide.

An application

Example

Let $\dot{\delta}$ be a Cauchy representations of $\mathbb{R}$, and φ be a $(\dot{\delta}, \dot{\delta})$-computable real function whose domain is the open interval $(0, 1)$. It is easily seen that a real-valued extension τ of φ to $\mathbb{R} \setminus \mathbb{Z}$ exists such that $\tau(x + 1) = x\tau(x)$ for all x in $\mathbb{R} \setminus \mathbb{Z}$. The function τ can be shown to be also $(\dot{\delta}, \dot{\delta})$-computable. To prove this, one could consider an acceptable representation δ of $\mathbb{R}^*$ which is in accordance with $\dot{\delta}$ and establish the (δ, δ)-computability of τ. The last can be done by applying the corollary from Theorem 2 and the remark on the previous slide to the mapping Γ defined as follows:

$$\Gamma(\theta)(z) = \begin{cases} \varphi(z) & \text{if } z \in \mathbb{R} \text{ and } 0 < z < 1, \\ (z-1)\theta(z-1) & \text{if } z \in \mathbb{R} \text{ and } z > 1, \\ \dfrac{\theta(z+1)}{z} & \text{if } z \in \mathbb{R} \text{ and } z < 0. \end{cases}$$

($\Gamma(\theta)(z)$ being undefined in all remaining cases).

Some details of the last step

After appropriately choosing the (δ, δ)-computable partial functions $\varphi_1, \ldots, \varphi_5$ in $\mathbb{R}^*$, $\Gamma(\theta)$ can be obtained from $\theta, \varphi_1, \ldots, \varphi_5, \varphi$ by means of finitely many applications of the operations composition, juxtaposition and branching in a uniform way. Indeed, let:

$$\mathrm{dom}(\varphi_1) = \mathbb{R} \setminus \{0\},\ \varphi_1(z) = \begin{cases} o & \text{if } z < 0, \\ (o, o) & \text{if } z > 0, \end{cases}$$

$$\mathrm{dom}(\varphi_2) = \mathrm{dom}(\varphi_3) = \mathbb{R},\ \varphi_2(z) = z + 1,\ \varphi_3(z) = z - 1,$$

$$\mathrm{dom}(\varphi_4) = \mathbb{R} \setminus \{0\},\ \varphi_4(z) = \frac{1}{z},\ \mathrm{dom}(\varphi_5) = \mathbb{R} \times \mathbb{R},\ \varphi_5((x, y)) = xy.$$

Then, for any partial function θ in $\mathbb{R}^*$, the function $\Gamma(\theta)$ can be described as the action of the following Pascal-style statement:

if $\varphi_1\varphi_3(z) \in \mathbb{R}^* \times \mathbb{R}^*$
then begin $z := (\varphi_3(z), \theta\varphi_3(z)); z := \varphi_5(z)$ end
else if $\varphi_1(z) \in \mathbb{R}^* \times \mathbb{R}^*$
    then $z := \varphi(z)$
    else begin $z := (\varphi_4(z), \theta\varphi_2(z)); z := \varphi_5(z)$ end.

TTE computability of partial multivalued functions

A partial multivalued function from a set M_1 to a set M_2 is actually a a subset of $M_1 \times M_2$. If φ is such one then

$$\mathrm{dom}(\varphi) = \{x \,|\, \exists y((x, y) \in \varphi)\},$$

and the values of φ at an element x of $\mathrm{dom}(\varphi)$ are the elements y such that $(x, y) \in \varphi$.

Definition

Let φ be a partial multivalued function from the set M_1 to the set M_2, δ_k be a representation of M_k for $k = 1, 2$. A (δ_1, δ_2)-*realization of* φ is any partial mapping γ of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}^{\mathbb{N}}$ such that

$$\forall x \in \mathrm{dom}(\varphi) \forall p((p, x) \in \delta_1$$
$$\Rightarrow p \in \mathrm{dom}(\gamma) \,\&\, \exists y((\gamma(p), y) \in \delta_2 \,\&\, (x, y) \in \varphi)).$$

The function φ is called (δ_1, δ_2)-*computable* if there is a computable (δ_1, δ_2)-realization of φ.

Example

If δ is a Cauchy representation of $\mathbb{R}$ then the multivalued function $((-\infty, 1) \times \{0\}) \cup ((0, \infty) \times \{1\})$ is (δ, δ)-computable.

On the preservation of TTE computability in the multivalued case

The usual composition of relations is

$$\varphi\psi = \{(x,z) \mid \exists y((x,y) \in \psi \,\&\, (y,z) \in \varphi)\}.$$

In general, it does not preserve TTE computability in the multivalued case, hence the same holds for absolute prime computability. It is known that another composition is appropriate for this case, namely the following one:

$$\varphi\psi = \{(x,z) \mid \exists y((x,y) \in \psi \,\&\, (y,z) \in \varphi) \\ \&\, \forall y((x,y) \in \psi \Rightarrow y \in \mathrm{dom}(\varphi))\}.$$

(clearly it coincides with the usual one in the single-valued case).

TTE computability on B^* in the multivalued case

Suppose we consider TTE computability on the set B^* with respect to acceptable representations of it. The straightforward generalization of juxtaposition to partial multivalued functions in B^* creates no problems concerning the preservation of TTE computability. However, the situation for iteration is similar to the one for composition, and in fact essentially more complicated. Fortunately an appropriate modification of the iteration notion is possible such that TTE computability is preserved by the modified iteration (the same holds also for the branching operation). Moreover, the First Recursion Theorem for iterative combinatory spaces turns out to be still applicable after these modifications.

Thank you for your attention!