Cupping $\Delta_2$ Enumeration Degrees to $0'$

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21.07.07
Definitions

Definition

1. A set $A$ is enumeration reducible to a set $B$ ($A \leq_e B$), if there is a c.e. set $\Phi$ such that

   \[ n \in A \iff \exists D (\langle n, [D] \rangle \in \Phi \land D \subseteq B). \]

2. $A$ is enumeration equivalent to $B$ ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$.

3. Let $d_e(A) = \{ B | A \equiv_e B \}$.

4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees.
Local Degree Structure

There is a natural embedding of the Turing degrees in the Enumeration degrees. Images of Turing degrees under this embedding are the total e-degrees.

\[ 0' \]

\[ \Sigma_2 \] e-degrees

Partial \( \Delta_2 \) e-degrees

Total \( \Delta_2 \) e-degrees

\[ \Pi_1 \] e-degrees

\[ 0_e \]
Cupping

We say that a degree $a$ is cuppable if there exists a degree $b < 0'_e$ such that $a \cup b = 0'_e$.

Cooper Sorbi and Yui proved that every nonzero $\Delta_2$ $e$-degree is cuppable by a total $\Delta_2$ $e$-degree.
Generic Sets

Definition
A set $A$ is generic if for every c.e. set $W$ there exists a finite string $\lambda \subset \chi_A$ such that:

$$
\lambda \in W \lor (\forall \mu \supseteq \lambda)(\mu \notin W).
$$

Degrees of generic sets are called generic degrees.

- Every generic enumeration degree $a$ is quasiminimal, hence partial.
- Copestake proved that generic degrees are low if and only if they are $\Delta_2$. 
Theorem 1

Theorem

Every nonzero $\Delta_2$ enumeration degree $a$ can be cupped by a $\Delta_2$ generic $e$-degree $b$, hence by a partial low degree.
Requirements

Given a nonzero $\Delta_2$ set $A$ we will construct a $\Delta_2$-set $B$ such that:

$$S : \Gamma^{A,B} = \overline{K}.$$  

$$G_i : (\exists \lambda \subset B)(\lambda \in W_i \lor \forall \mu \supseteq \lambda[\mu \notin W_i]).$$
The $S$-strategy

\[ S : \Gamma^{A,B} = \overline{K}. \]

The $S$-strategy runs at the beginning of every stage and constructs an e-operator $\Gamma$ such that:

- For every $n \in \overline{K}$ there is a valid axiom $\langle n, A \upharpoonright a_n, B \upharpoonright b_n \rangle$.
- If $n$ exits $\overline{K}$ we correct $\Gamma$ by extracting $b_n$ from $B$. 
The $G$-strategy

$$G_i : (\exists \lambda \subset B)(\lambda \in W_i \lor \forall \mu \supseteq \lambda [\mu \notin W_i]).$$

The $G$-strategy will select a threshold $k$. Choose a witness $\lambda = B \upharpoonright b_k$. Wait for $\mu \supseteq \lambda$ to enter $W$. 
Conflict

$G$ would like to preserve $\mu$ as an initial segment of $B$, meanwhile $S$ might like to change $B$ to rectify $\Gamma$. 
Solution

Extract the marker $b_k$ to prevent $S$ from injuring the restraint. Approximate $A$ up to $a_k$ threatening to prove that it is c.e. Start a new cycle.
A-retreat

If there is an $A$-change, restore $B$. Now $\mu \subseteq B$. $A$ is nonzero and $\Delta_2$ hence there will be a permanent change in $A$ eventually.
Definitions

1. A set $A$ is $n$-c.e. if there is a computable function $f$ such that for each $x$, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq n$ and $A(x) = \lim_s f(x, s)$.

2. $A$ is $\omega$-c.e. if there are two computable functions $f(x, s), g(x)$ such that for all $x$, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq g(x)$ and $\lim_s f(x, s) \downarrow = A(x)$.

3. A degree $a$ is $n$-c.e.$(\omega$-c.e.) if it contains a $n$-c.e.$(\omega$-c.e.) set.
A noncuppable c.e. degree

Cooper and Yates proved that there is a noncuppable c.e. Turing degree. Hence a 2-c.e. e-degree that cannot be cupped by any 2-c.e. e-degree.
Theorem 2

**Theorem**

*Given a nonzero $\omega$-c.e. $e$-degree $a$, there is a 3-c.e. $e$-degree $b$ such that $a \cup b = 0'_e$.***
Requirements

Given a nonzero $\omega$-c.e set $A$ we will construct a 3-c.e. set $B$ and an extra $\Pi_1$ set $E$ such that:

$$S : \Gamma^{A,B} = \overline{K}.$$  

$$N_i : E \neq \psi_i^B.$$
The $N$-strategy

\[ N_i : E \neq \Psi^B_i. \]

- Choose a threshold $k$ and a witness $x > k$.
- Wait for $x$ to enter $\Psi^B$.
- Approximate $A \upharpoonright a_k$ and extract $b_k$. Start a new cycle.
- If there is an $A$-change re-enumerate $b_k$ to restore the initial segment of $B$. 
New tricks

- Sets of markers - if $n$ has $A$-marker $a_n$ then it has a $B$-marker $B_n$ a set of size $\sum_{x < a_n} g(x)$.
- Make the approximations of the set $A$ monotone and always restore the last computation.

![Diagram showing sets $A$, $B$, and $B_k$ with intersections and unions](image)
Bibliography


