Co-total enumeration degrees and the skip operaotor

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The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

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- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.

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- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

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A set A is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The set of total degrees is denoted by $\mathcal{TOT}$.

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Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

Motivation from symbolic dynamics by Emmanuel Jeandel

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3. $L_X \leq_e \overline{L_X}$.

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total $\Rightarrow$ graph co-total $\Rightarrow$ co-total $\Rightarrow$ Solon co-total.

Examples of co-total enumeration degrees: Σ^0_2 enumeration degrees

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Corollary

Graph co-total does not imply *total*.

Examples of co-total enumeration degrees: Joins of nontrivial $\mathcal{K}$-pairs

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A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

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Proof: $A \oplus B \leq_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

Examples of co-total enumeration degrees: Continuous degrees

Definition (J. Miller)

An e-degree is continuous if it contains a set of the form $A = \bigoplus_{i<\omega}(\{q \mid q < \alpha_i\} \oplus \{q \mid q > \alpha_i\}$, where $\{\alpha_i\}_{i<\omega}$ is a sequence of real numbers.

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Proposition

Continuous degrees are co-total.

Proof: On the board.

Unique correct axiom

Theorem

An e-degree $\mathbf{a}$ is graph co-total if and only if $\mathbf{a}$ contains a co-total set A, such that for some enumeration operator Γ, we have that $A = \Gamma(\overline{A})$ and for every $n \in A$ there is a unique axiom $\langle n, D \rangle \in \Gamma$ such that $D \subseteq A$.

Exact covers

Definition

Let $G = (\mathbb{N}, E)$ be a graph. $S \subseteq \mathbb{N}$ is an *exact cover* for G if:

1. If $i \neq j$ are in S then $(i, j) \notin E$.
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There is an exact cover S of $\omega^{<\omega}$, such that $\overline{S}$ does not have graph co-total degree.

Proof: Infinite injury relative to $\emptyset'$.

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Co-total does not imply *graph co-total*.

The skip operator

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Solon co-total does not imply *co-total*.

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1. If A is Σ^0_2 and nonlow then $\{L_{K_A}, R_{K_A}\}$ is a nontrivial $\mathcal{K}$-pair.

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2. If A and B are non-arithmetical then their skips form a double helix.

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Proposition

If G is generic relative to a total set X then $(G \oplus X)^\circ = \overline{G} \oplus X'$.

1. If G is arithmetically generic then the skips of G and $\overline{G}$ form a double helix.

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By Knaster-Tarski's fixed point theorem:

Proposition

There are sets A and B such that $B = A^\circ$ and $A = B^\circ$.

A fixed point for the double skip

The enumeration operator is monotone: if $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$.

$$A \subseteq B \Rightarrow K_A \subseteq K_B \Rightarrow \overline{K_A} \supseteq \overline{K_B} \Rightarrow K_{\overline{K_A}} \supseteq K_{\overline{K_B}} \Rightarrow$$

$$\overline{K_{\overline{K_A}}} \subseteq \overline{K_{\overline{K_B}}}$$

By Knaster-Tarski's fixed point theorem:

Proposition

There are sets A and B such that $B = A^\circ$ and $A = B^\circ$. The sets A and B are above all hyperarithmetical sets.

Sack's intermediate skip

Theorem

If $S \geq_e X'$ is $\Pi^0_2(X')$ then there is a non-total $\Pi^0_1(X')$ set $A > X$ such that $A^\circ \equiv_e S$.

Proof: Infinite injury relative to X'.

Sack's intermediate skip

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Let P^X be a computable in X listing of all $\Pi^0_2(X)$ and above X sets.

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Let P^X be a computable in X listing of all $\Pi^0_2(X)$ and above X sets.

We have a computable function f such that $(P^X_{f(e)})^\circ \equiv_e P^{X'}_e$, $P^X_{f(e)} > X$ and $P^X_{f(e)}$ is not total.

Sack's intermediate skip

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Let e be a fixed point such that $P^X_e = P^X_{f(e)}$ and consider the sequence

$$P^{\emptyset}_e, P^{\emptyset'}_e, P^{\emptyset''}_e, \ldots$$

Corollary

There is an arithmetical set A, such that for every n, $A^{\textcircled{n}}$ is non-total.