Definability in the local structure of the enumeration degrees and the ω-enumeration degrees

Mariya I. Soskova[1]

Faculty of Mathematics and Informatics
Sofia University
joint work with H. Ganchev

10.03.2011

[1] Research supported by BNSF Grant No. D002-258/18.12.08 and MC-ER Grant 239193 within the 7th European Community Framework Programme.

PART I

THE ENUMERATION DEGREES

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \& \ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e \rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \ \&\ B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

Preliminaries: The enumeration degrees

Definition

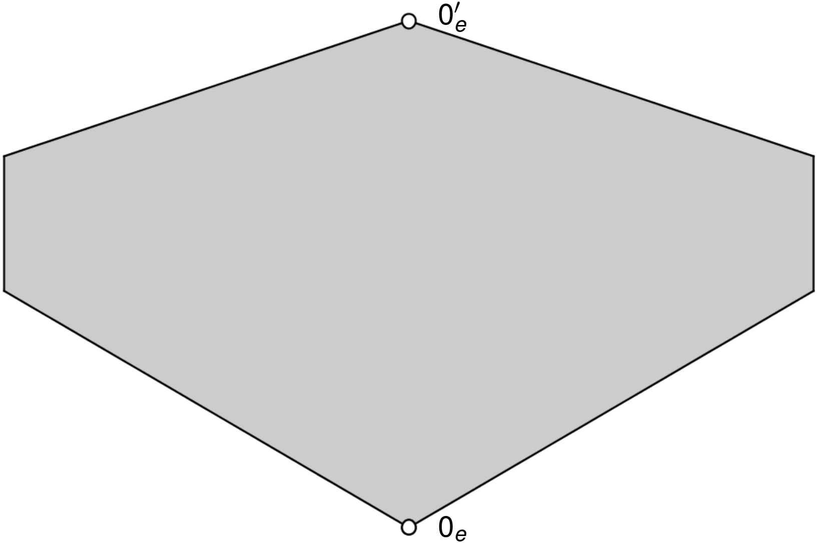
- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

The total degrees

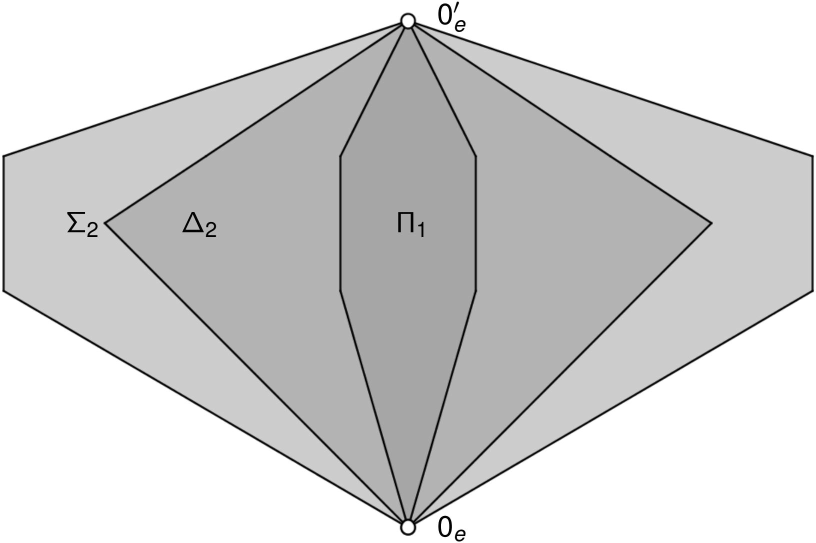
Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

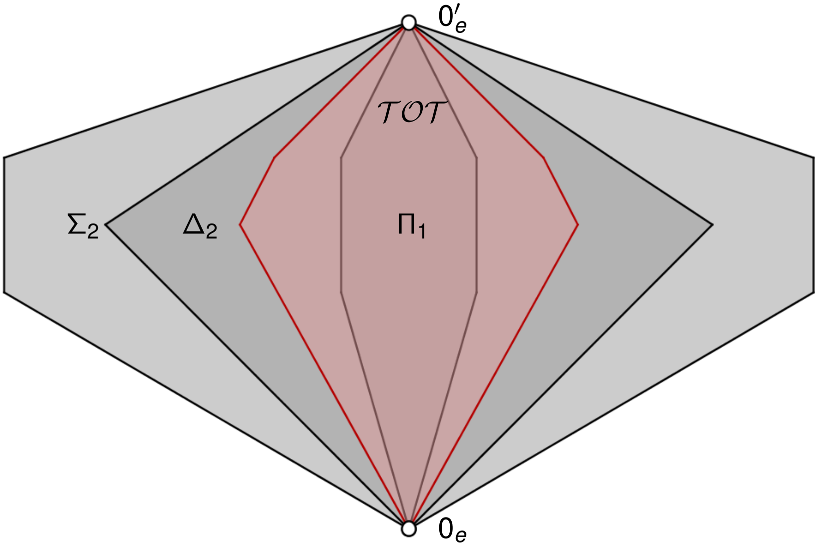
The sub structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.



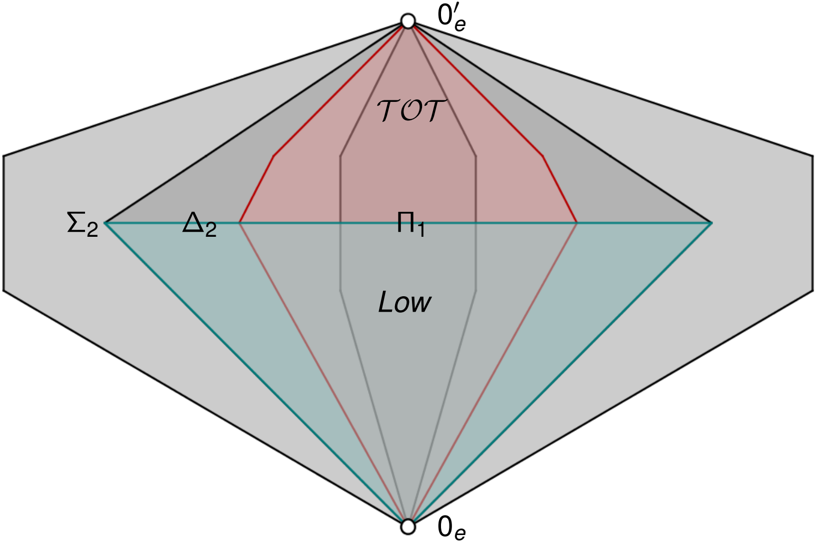
The local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$ consists of all degrees below $\mathbf{0}'_e$.



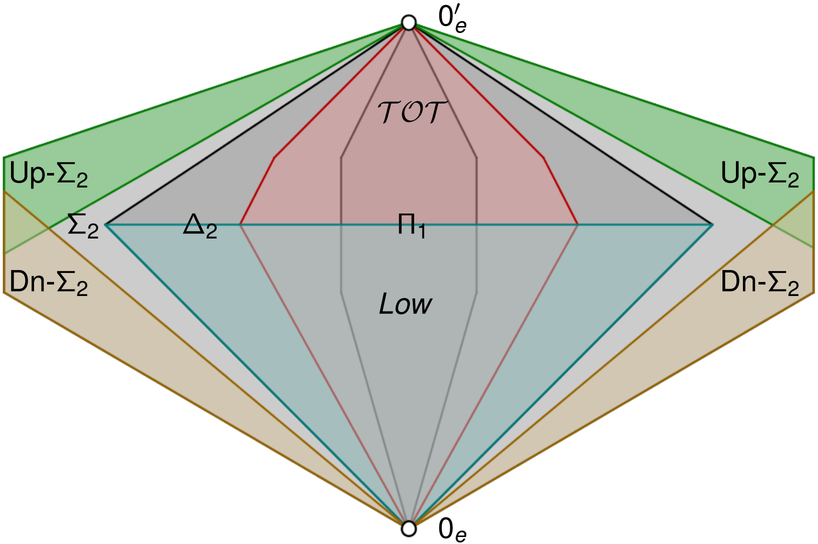
With respect to the arithmetic hierarchy the degrees can be partitioned into three classes.



The total degrees below $\mathbf{0}'_e$ are images of the Turing degrees below $\mathbf{0}'$. Every total degree is Δ^0_2, but not all Δ^0_2 are total.



A degree is low if its jump is as low as possible: $\mathbf{0}'_e$. Every low degree is Δ^0_2.



The upwards properly Σ^0_2 have no incomplete Δ^0_2 above them. The downwards properly Σ^0_2 have no nonzero Δ^0_2 below them.

$\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A, B and U be sets of natural numbers. The pair (A, B) is a $\mathcal{K}$-pair over U if there exists a set $W \leq_e U$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

A and B are a $\mathcal{K}$-pair if they are a $\mathcal{K}$-pair over $\emptyset$.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair (over $\emptyset$) for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

An order theoretic characterization of $\mathcal{K}$-pairs

Kalimullin has proved that the property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in $\mathcal{D}_e$.

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair over U if and only if the degrees $\mathbf{a} = d_e(A)$, $\mathbf{b} = d_e(B)$ and $\mathbf{u} = d_e(U)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}, \mathbf{u}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{u} \vee \mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{u} \vee \mathbf{b} \vee \mathbf{x}) = \mathbf{u} \vee \mathbf{x})$$

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$-pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

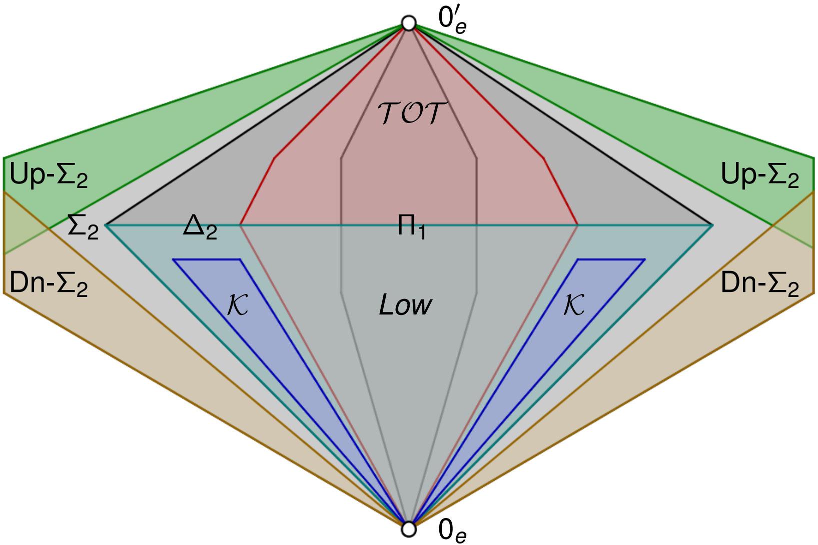
1. The enumeration degrees of the elements of a $\mathcal{K}$-pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$-pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.

Properties of $\mathcal{K}$-pairs in the local structure

1. The enumeration degrees of the elements of a $\mathcal{K}$-pair are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
2. The enumeration degrees of the elements of a $\mathcal{K}$ pair are low.
3. Every Δ^0_2 degree bounds a $\mathcal{K}$-pair.
4. The class of the enumeration degrees of sets that form a $\mathcal{K}$-pair with a fixed set A is an ideal.



The $\mathcal{K}$-pairs in the local structure $\mathcal{G}_e$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?
Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a},\mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}) \;\;\&\;\; \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}))?$$

Theorem (G, S)

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A),\mathbf{d}_e(B))$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?
Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a},\mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}) \ \ \& \ \ \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}))?$$

Theorem (G, S)

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A),\mathbf{d}_e(B))$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees? Could there be a fake $\mathcal{K}$-pair $\{\mathbf{a},\mathbf{b}\}$, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}) \;\; \& \;\; \neg(\mathcal{D}_e \models \mathcal{K}(\mathbf{a},\mathbf{b}))?$$

Theorem (G, S)

There is a first order formula $\mathcal{LK}$, such that for any Σ^0_2 sets A and B, $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{LK}(\mathbf{d}_e(A), \mathbf{d}_e(B))$.

Cupping properties

Definition

A Σ_2^0 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ_2^0 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

Theorem (G, S)

If $\mathbf{u}$ and $\mathbf{v}$ are Σ_2^0 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then $\mathbf{u}$ is low-cuppable or $\mathbf{v}$ is low-cuppable.

Theorem (G, S)

For every nonzero Δ_2^0 degree $\mathbf{b}$ there is a nontrivial $\mathcal{K}$-pair, $(\mathbf{c}, \mathbf{d})$, such that

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Cupping properties

Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

Theorem (G, S)

If $\mathbf{u}$ and $\mathbf{v}$ are Σ^0_2 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then $\mathbf{u}$ is low-cuppable or $\mathbf{v}$ is low-cuppable.

Theorem (G, S)

For every nonzero Δ^0_2 degree $\mathbf{b}$ there is a nontrivial $\mathcal{K}$-pair, $(\mathbf{c}, \mathbf{d})$, such that

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Cupping properties

Definition

A Σ_2^0 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ_2^0 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

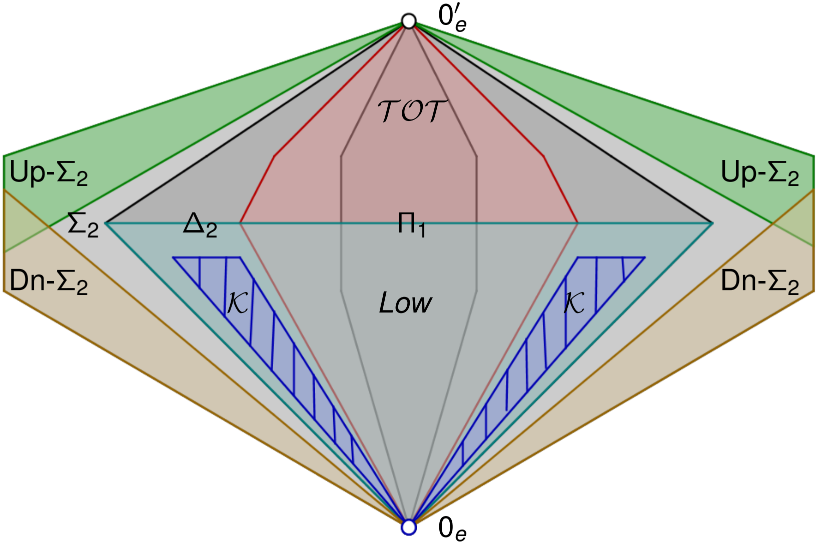
Theorem (G, S)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ_2^0 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}_e'$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Theorem (G, S)

For every nonzero Δ_2^0 *degree* $\mathbf{b}$ *there is a nontrivial* $\mathcal{K}$*-pair,* $(\mathbf{c}, \mathbf{d})$*, such that*

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e'.$$



The first example of a definable class of degrees in the local structure: $\mathcal{K}$-pairs.

An easy consequence

If **a** bounds a nonzero Δ^0_2 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ^0_2 degree, then it bounds no $\mathcal{K}$-pair.

Corollary

The class of downwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{DP}\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{x} \ \& \ \mathbf{c} \leq \mathbf{x}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

An easy consequence

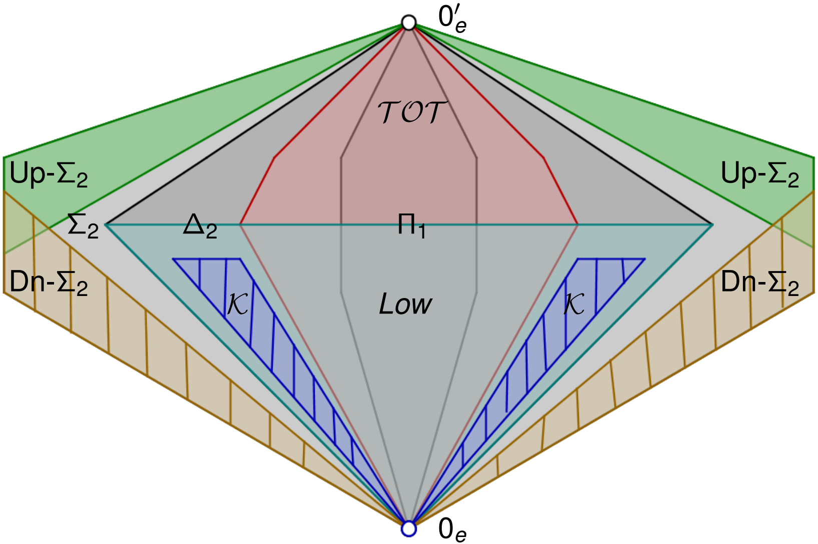
If **a** bounds a nonzero Δ_2^0 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ_2^0 degree, then it bounds no $\mathcal{K}$-pair.

Corollary

The class of downwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{DP}\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{x} \ \& \ \mathbf{c} \leq \mathbf{x}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$



The second example of a definable class of degrees in the local structure: Downwards properly Σ^0_2 degrees.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ^0_2 degrees

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

So a degree $\mathbf{a}$ is upwards properly Σ^0_2 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Corollary

The class of upwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{UP}\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$

The upwards properly Σ_2^0 degrees

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}_e'$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}_e'$.

So a degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if no element above it other than $\mathbf{0}_e'$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Corollary

The class of upwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{UP}\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e').$$

The upwards properly Σ_2^0 degrees

Theorem (Arslanov, Cooper, Kalimullin)

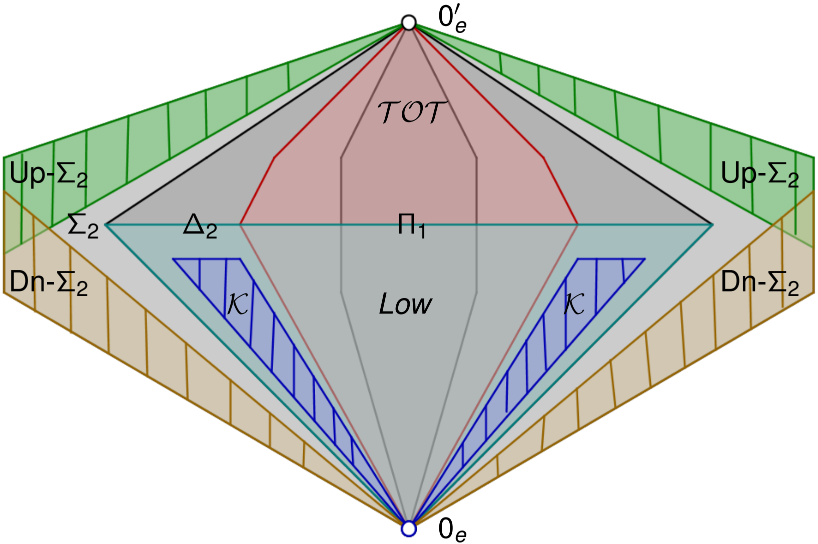
For every Δ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

So a degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Corollary

The class of upwards properly Σ_2^0 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{U}P\Sigma_2^0(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \,\&\, \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$



The third example of a definable class of degrees in the local structure: Upwards properly Σ^0_2 degrees.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.

Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.

By the ideal property A forms a $\mathcal{K}$-pair with D.

Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.

Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.

By the ideal property A forms a $\mathcal{K}$-pair with D.

Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Semi-recursive sets revisited

Proposition (Kalimullin)

If A and B form a nontrivial Δ^0_2 $\mathcal{K}$-pair then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.
Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.
By the ideal property A forms a $\mathcal{K}$-pair with D.
Hence $D \leq_e \overline{A}$.

Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if for every $\mathcal{K}$-pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Local definability of the total degrees

Theorem (G, S)

For every nontrivial Δ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Denote by $\mathcal{MK}(\mathbf{x}, \mathbf{y})$ the first order formula that defines in $\mathcal{G}_e$ the set of degrees of maximal $\mathcal{K}$-pairs.

Corollary

The class of total degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TOT}(\mathbf{x}) \rightleftharpoons \mathbf{x} = \mathbf{0}_e \ \vee \ \exists \mathbf{c} \exists \mathbf{d} [\mathcal{MK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} = \mathbf{c} \vee \mathbf{d}.]$$

Local definability of the total degrees

Theorem (G, S)

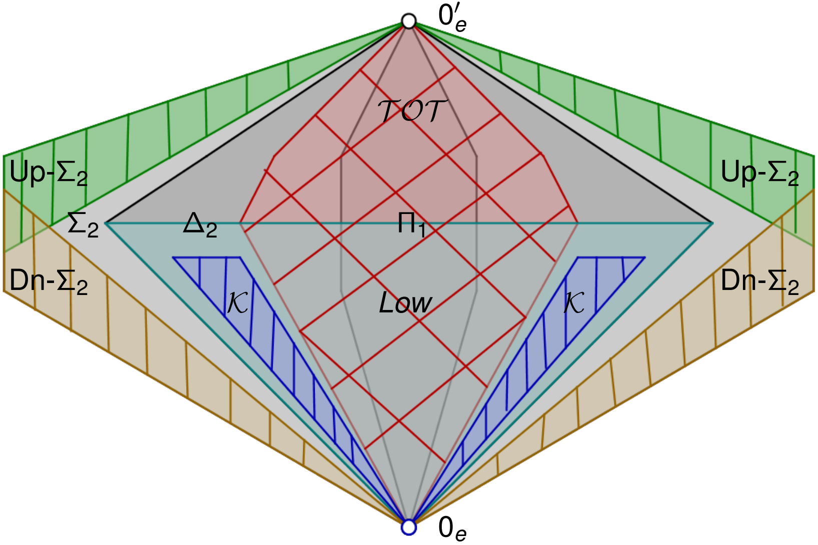
For every nontrivial Δ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Denote by $\mathcal{MK}(\mathbf{x}, \mathbf{y})$ the first order formula that defines in $\mathcal{G}_e$ the set of degrees of maximal $\mathcal{K}$-pairs.

Corollary

The class of total degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TOT}(\mathbf{x}) \rightleftharpoons \mathbf{x} = \mathbf{0}_e \ \vee \ \exists \mathbf{c} \exists \mathbf{d} [\mathcal{MK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} = \mathbf{c} \vee \mathbf{d}.]$$



The fourth example of a definable class of degrees in the local structure: The total degrees.

One final consequence

Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ^0_2 enumeration degree.

Corollary

The class of low total e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TL}(\mathbf{x}) \rightleftharpoons \mathcal{TOT}(\mathbf{x}) \ \&\ \forall \mathbf{c} \leq \mathbf{x}[\neg \mathcal{D}P\Sigma^0_2(\mathbf{c})]$$

One final consequence

Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ^0_2 enumeration degree.

Corollary

The class of low total e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TL}(\mathbf{x}) \rightleftharpoons \mathcal{TOT}(\mathbf{x}) \;\&\; \forall \mathbf{c} \leq \mathbf{x}[\neg \mathcal{D}P\Sigma^0_2(\mathbf{c})]$$

One final consequence

Theorem (Soskov)

For every enumeration degree $\mathbf{x}$ *there is a total enumeration degree* $\mathbf{y}$*, such that* $\mathbf{x} < \mathbf{y}$ *and* $\mathbf{x}' = \mathbf{y}'$.

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it.

Theorem (G, S)

The class of low e-degrees is first order definable in $\mathcal{G}_e$ *by the formula:*

$$\mathcal{LOW}(\mathbf{x}) \rightleftharpoons \exists \mathbf{y}[\mathbf{x} \leq \mathbf{y} \ \& \ \mathcal{TL}(\mathbf{y})]$$

One final consequence

Theorem (Soskov)

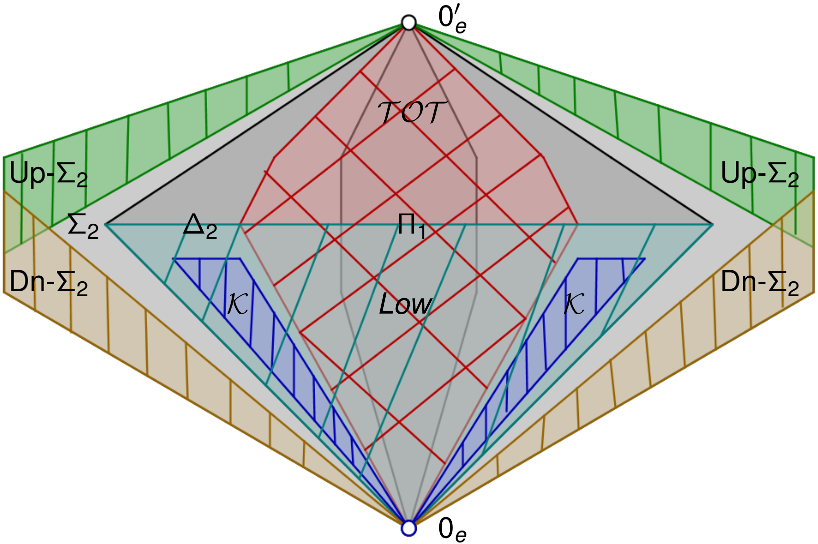
For every enumeration degree $\mathbf{x}$ *there is a total enumeration degree* $\mathbf{y}$*, such that* $\mathbf{x} < \mathbf{y}$ *and* $\mathbf{x}' = \mathbf{y}'$*.*

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it.

Theorem (G, S)

The class of low e-degrees is first order definable in $\mathcal{G}_e$ *by the formula:*

$$\mathcal{LOW}(\mathbf{x}) \rightleftharpoons \exists \mathbf{y}[\mathbf{x} \leq \mathbf{y} \ \& \ \mathcal{TL}(\mathbf{y})]$$



$0'_e$
$\mathcal{TOT}$
Up-Σ_2
Up-Σ_2
Σ_2
Δ_2
Π_1
Dn-Σ_2
Dn-Σ_2
$\mathcal{K}$
Low
$\mathcal{K}$
0_e

PART II

THE ω-ENUMERATION DEGREES

The ω e-degrees: Basic definitions

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}$ and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Γ_n.
- Uniformity: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.

The ω e-degrees: Basic definitions

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}$ and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Γ_n.
- Uniformity: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.

The ω e-degrees: Basic definitions

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}$ and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Γ_n.
- Uniformity: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.

Basic definitions: The jump sequence

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P_n'(\mathcal{A})$, where $P_n'(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.

Basic definitions: The jump sequence

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P_n'(\mathcal{A})$, where $P_n'(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.

Basic definitions: The jump sequence

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.

The ω-enumeration degrees

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

Definition

- ω-enumeration reducibility: $\mathcal{A} \leq_\omega \mathcal{B}$, if $\mathcal{A} \leq_e P(\mathcal{B})$.
- ω-enumeration equivalence: $\mathcal{A} \equiv_\omega \mathcal{B}$ if $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$.
- ω-enumeration degrees: $d_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$.
- The structure of the ω-enumeration degrees: $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$, where $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{B})$ if $\mathcal{A} \leq_\omega \mathcal{B}$.
- The least ω-enumeration degree: $\mathbf{0}_\omega = d_\omega(\{\emptyset, \emptyset, \emptyset, \dots\})$ or equivalently $d_\omega(\{\emptyset, \emptyset', \emptyset'', \dots\})$.

The ω-enumeration degrees

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

Definition

- ω-enumeration reducibility: $\mathcal{A} \leq_\omega \mathcal{B}$, if $\mathcal{A} \leq_e P(\mathcal{B})$.
- ω-enumeration equivalence: $\mathcal{A} \equiv_\omega \mathcal{B}$ if $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$.
- ω-enumeration degrees: $d_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$.
- The structure of the ω-enumeration degrees: $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$, where $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{B})$ if $\mathcal{A} \leq_\omega \mathcal{B}$.
- The least ω-enumeration degree: $\mathbf{0}_\omega = d_\omega(\{\emptyset, \emptyset, \emptyset, \dots\})$ or equivalently $d_\omega(\{\emptyset, \emptyset', \emptyset'', \dots\})$.

$\mathcal{D}_\omega$ as an upper semi-lattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.
- Iterating the jump: for every k, $d_\omega(\mathcal{A})^k = d_\omega(\mathcal{A}^k)$, where $\mathcal{A}^k = \{P_{n+k}(\mathcal{A})\}_{n<\omega}$.
- The jump inversion operation: for every k, let $I^k(d_\omega(\mathcal{A}^k)) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{k}, P_k(\mathcal{A}), P_{k+1}(\mathcal{A}), \dots\})$.

  Then $I^k(\mathbf{a}^k)$ is the least degree $\mathbf{b}$, such that $\mathbf{b}^k = \mathbf{a}^k$.

$\mathcal{D}_\omega$ as an upper semi-lattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.
- Iterating the jump: for every k, $d_\omega(\mathcal{A})^k = d_\omega(\mathcal{A}^k)$, where $\mathcal{A}^k = \{P_{n+k}(\mathcal{A})\}_{n<\omega}$.
- The jump inversion operation: for every k, let $I^k(d_\omega(\mathcal{A}^k)) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{k}, P_k(\mathcal{A}), P_{k+1}(\mathcal{A}), \dots\})$.

  Then $I^k(\mathbf{a}^k)$ is the least degree $\mathbf{b}$, such that $\mathbf{b}^k = \mathbf{a}^k$.

$\mathcal{D}_\omega$ as an upper semi-lattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.
- Iterating the jump: for every k, $d_\omega(\mathcal{A})^k = d_\omega(\mathcal{A}^k)$, where $\mathcal{A}^k = \{P_{n+k}(\mathcal{A})\}_{n<\omega}$.
- The jump inversion operation: for every k, let $I^k(d_\omega(\mathcal{A}^k)) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{k}, P_k(\mathcal{A}), P_{k+1}(\mathcal{A}), \dots\})$.

  Then $I^k(\mathbf{a}^k)$ is the least degree $\mathbf{b}$, such that $\mathbf{b}^k = \mathbf{a}^k$.

$\mathcal{D}_\omega$ as an upper semi-lattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.
- Iterating the jump: for every k, $d_\omega(\mathcal{A})^k = d_\omega(\mathcal{A}^k)$, where $\mathcal{A}^k = \{P_{n+k}(\mathcal{A})\}_{n<\omega}$.
- The jump inversion operation: for every k, let $I^k(d_\omega(\mathcal{A}^k)) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{k}, P_k(\mathcal{A}), P_{k+1}(\mathcal{A}), \dots\})$.

  Then $I^k(\mathbf{a}^k)$ is the least degree $\mathbf{b}$, such that $\mathbf{b}^k = \mathbf{a}^k$.

$\mathcal{D}_\omega$ as an upper semi-lattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.
- Iterating the jump: for every k, $d_\omega(\mathcal{A})^k = d_\omega(\mathcal{A}^k)$, where $\mathcal{A}^k = \{P_{n+k}(\mathcal{A})\}_{n<\omega}$.
- The jump inversion operation: for every k, let $I^k(d_\omega(\mathcal{A}^k)) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{k}, P_k(\mathcal{A}), P_{k+1}(\mathcal{A}), \dots\})$.

  Then $I^k(\mathbf{a}^k)$ is the least degree $\mathbf{b}$, such that $\mathbf{b}^k = \mathbf{a}^k$.

The e-degrees as a substructure of $\mathcal{D}_\omega$

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega(\{A, \emptyset, \emptyset, \dots\}) = d_\omega(\{A, A', A'', \dots\}).$$

Theorem (Soskov, Ganchev)

- *The structure $\mathcal{D}_1 = \kappa(\mathcal{D}_e)$ is first order definable in $\mathcal{D}_\omega$.*
- *The structures $\mathcal{D}_e$ and $\mathcal{D}_\omega$ with jump operation have isomorphic automorphism groups.*

The e-degrees as a substructure of $\mathcal{D}_\omega$

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega(\{A, \emptyset, \emptyset, \dots\}) = d_\omega(\{A, A', A'', \dots\}).$$

Theorem (Soskov, Ganchev)

- *The structure $\mathcal{D}_1 = \kappa(\mathcal{D}_e)$ is first order definable in $\mathcal{D}_\omega$.*
- *The structures $\mathcal{D}_e$ and $\mathcal{D}_\omega$ with jump operation have isomorphic automorphism groups.*

The e-degrees as a substructure of $\mathcal{D}_\omega$

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega(\{A, \emptyset, \emptyset, \dots\}) = d_\omega(\{A, A', A'', \dots\}).$$

Theorem (Soskov, Ganchev)

- *The structure $\mathcal{D}_1 = \kappa(\mathcal{D}_e)$ is first order definable in $\mathcal{D}_\omega$.*
- *The structures $\mathcal{D}_e$ and $\mathcal{D}_\omega$ with jump operation have isomorphic automorphism groups.*

The Local structure $\mathcal{G}_\omega$

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $\mathbf{0}'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees.
The degrees in this local structure can as well be partitioned in terms of the high-low jump hierarchy.

Definition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a}$ is low_n if $\mathbf{a}^n = \mathbf{0}^n_\omega$. The class of all low_n degrees is denoted by L_n.
2. $\mathbf{a}$ is $high_n$ if $\mathbf{a}^n = \mathbf{0}^{n+1}_\omega$. The class of all $high_n$ degrees is denoted by H_n.

There is a copy of the Σ^0_2 enumeration degrees $\mathcal{G}_1 = D_1 \cap \mathcal{G}_\omega$.

The Local structure $\mathcal{G}_\omega$

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $\mathbf{0}'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees. The degrees in this local structure can as well be partitioned in terms of the high-low jump hierarchy.

Definition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a}$ is *low$_n$* if $\mathbf{a}^n = \mathbf{0}^n_\omega$. The class of all *low$_n$* degrees is denoted by L_n.
2. $\mathbf{a}$ is *high$_n$* if $\mathbf{a}^n = \mathbf{0}^{n+1}_\omega$. The class of all *high$_n$* degrees is denoted by H_n.

There is a copy of the Σ^0_2 enumeration degrees $\mathcal{G}_1 = D_1 \cap \mathcal{G}_\omega$.

The Local structure $\mathcal{G}_\omega$

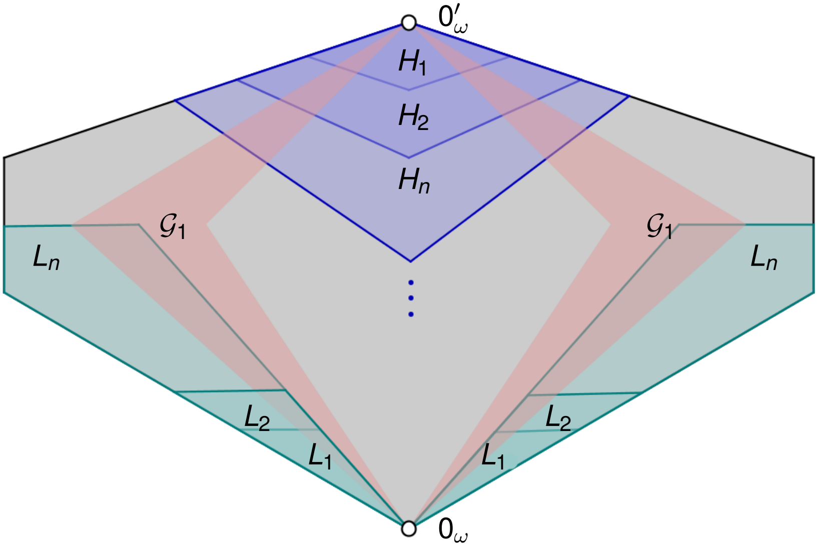
Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $\mathbf{0}'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees. The degrees in this local structure can as well be partitioned in terms of the high-low jump hierarchy.

Definition

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a}$ is *low*$_n$ if $\mathbf{a}^n = \mathbf{0}^n_\omega$. The class of all *low*$_n$ degrees is denoted by L_n.
2. $\mathbf{a}$ is *high*$_n$ if $\mathbf{a}^n = \mathbf{0}^{n+1}_\omega$. The class of all *high*$_n$ degrees is denoted by H_n.

There is a copy of the Σ^0_2 enumeration degrees $\mathcal{G}_1 = D_1 \cap \mathcal{G}_\omega$.


$0'_\omega$
H_1
H_2
H_n
$\mathcal{G}_1$
$\mathcal{G}_1$
L_n
L_n
L_2
L_2
L_1
L_1
0_ω

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega((\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots))$.

Theorem

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{o}_n \leq_\omega \mathbf{a}$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: 1: If $\mathbf{o}_n \leq_\omega \mathbf{a}$ then $\mathbf{0}_\omega^{n+1} = \mathbf{o}_n^n \leq_\omega \mathbf{a}^n$. Hence $\mathbf{a} \in H_n$.
If $\mathbf{a} \in H_n$ then $\mathbf{a}^n = \mathbf{0}_\omega^{n+1}$. But $\mathbf{o}_n$ is the least degree whose n-th jump is $\mathbf{0}_\omega^{n+1}$, so $\mathbf{o}_n \leq_\omega \mathbf{a}$.

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega((\underbrace{\emptyset,\dots,\emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots))$.

Theorem

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{o}_n \leq_\omega \mathbf{a}$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: 1: If $\mathbf{o}_n \leq_\omega \mathbf{a}$ then $\mathbf{0}_\omega^{n+1} = \mathbf{o}_n^n \leq_\omega \mathbf{a}^n$. Hence $\mathbf{a} \in H_n$.
If $\mathbf{a} \in H_n$ then $\mathbf{a}^n = \mathbf{0}_\omega^{n+1}$. But $\mathbf{o}_n$ is the least degree whose n-th jump is $\mathbf{0}_\omega^{n+1}$, so $\mathbf{o}_n \leq_\omega \mathbf{a}$.

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega((\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots))$.

Theorem

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{o}_n \leq_\omega \mathbf{a}$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: 1: If $\mathbf{o}_n \leq_\omega \mathbf{a}$ then $\mathbf{0}_\omega^{n+1} = \mathbf{o}_n^n \leq_\omega \mathbf{a}^n$. Hence $\mathbf{a} \in H_n$.
If $\mathbf{a} \in H_n$ then $\mathbf{a}^n = \mathbf{0}_\omega^{n+1}$. But $\mathbf{o}_n$ is the least degree whose n-th jump is $\mathbf{0}_\omega^{n+1}$, so $\mathbf{o}_n \leq_\omega \mathbf{a}$.

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega((\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots))$.

Theorem

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{o}_n \leq_\omega \mathbf{a}$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: 1: If $\mathbf{o}_n \leq_\omega \mathbf{a}$ then $\mathbf{0}_\omega^{n+1} = \mathbf{o}_n^n \leq_\omega \mathbf{a}^n$. Hence $\mathbf{a} \in H_n$.
If $\mathbf{a} \in H_n$ then $\mathbf{a}^n = \mathbf{0}_\omega^{n+1}$. But $\mathbf{o}_n$ is the least degree whose n-th jump is $\mathbf{0}_\omega^{n+1}$, so $\mathbf{o}_n \leq_\omega \mathbf{a}$.

The $\mathbf{o}_n$ degrees

Definition

For every n let $\mathbf{o}_n = I^n(\mathbf{0}_\omega^{n+1}) = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots\})$.

Theorem

Let $\mathbf{a} \in \mathcal{G}_\omega$.

1. $\mathbf{a} \in H_n$ *if and only if* $\mathbf{o}_n \leq_\omega \mathbf{a}$
2. $\mathbf{a} \in L_n$ *if and only if* $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$

Proof: For every $\mathcal{A} \equiv_e P(\mathcal{A}) \in \mathbf{a} \in \mathcal{G}_\omega$:

$$\{A_0, A_1 \dots, A_n, A_{n+1}, \dots\} \wedge \{\underbrace{\emptyset, \dots, \emptyset}_{n}, \emptyset^{n+1}, \emptyset^{n+2}, \dots\}$$

$$= \{\underbrace{\emptyset, \dots, \emptyset}_{n}, A_n, A_{n+1}, \dots\} = I^n(\mathcal{A}^n)$$

Hence $\mathbf{a} \wedge \mathbf{o}_n = \mathbf{0}_\omega$ if and only if $I^n(\mathbf{a}^n) = \mathbf{0}_\omega$, if and only if $\mathbf{a}^n = \mathbf{0}_\omega^n$.

A new “lowness” property

Definition

A sequence $\mathcal{A}$ is called almost zero (a.z.) if for every n, $A_n \leq_e \emptyset^n$. A degree is a.z. if it contains an a.z. sequence.

- The a.z. degrees form an ideal.
- If $\mathbf{a} \in \mathcal{G}_\omega$ then $\mathbf{a}$ is a.z. if and only if $\mathbf{a} <_\omega \mathbf{o}_n$ for all n.
- Nonzero a.z. degrees exist.

A new “lowness” property

Definition

A sequence $\mathcal{A}$ is called almost zero (a.z.) if for every n, $A_n \leq_e \emptyset^n$. A degree is a.z. if it contains an a.z. sequence.

- The a.z. degrees form an ideal.
- If $\mathbf{a} \in \mathcal{G}_\omega$ then $\mathbf{a}$ is a.z. if and only if $\mathbf{a} <_\omega \mathbf{o}_n$ for all n.
- Nonzero a.z. degrees exist.

A new "lowness" property

Definition

A sequence $\mathcal{A}$ is called almost zero (a.z.) if for every n, $A_n \leq_e \emptyset^n$. A degree is a.z. if it contains an a.z. sequence.

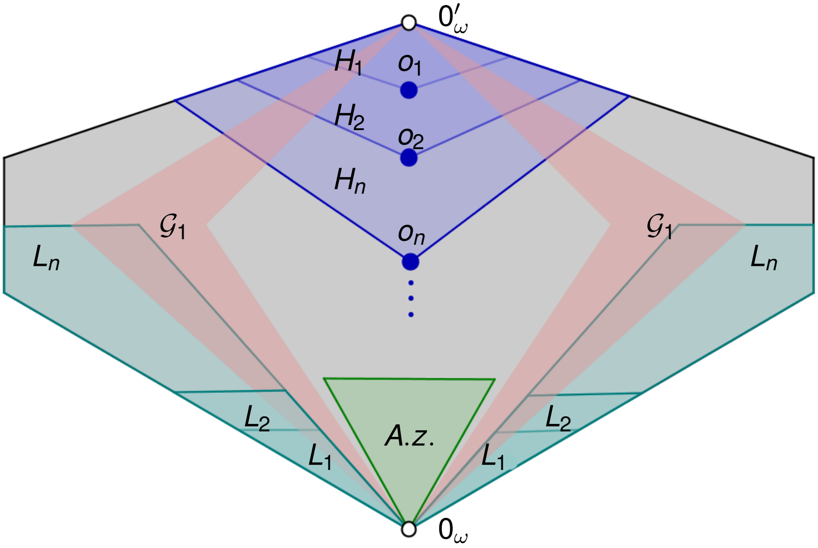
- The a.z. degrees form an ideal.
- If $\mathbf{a} \in \mathcal{G}_\omega$ then $\mathbf{a}$ is a.z. if and only if $\mathbf{a} <_\omega \mathbf{o}_n$ for all n.
- Nonzero a.z. degrees exist.

A new "lowness" property

Definition

A sequence $\mathcal{A}$ is called almost zero (a.z.) if for every n, $A_n \leq_e \emptyset^n$. A degree is a.z. if it contains an a.z. sequence.

- The a.z. degrees form an ideal.
- If $\mathbf{a} \in \mathcal{G}_\omega$ then $\mathbf{a}$ is a.z. if and only if $\mathbf{a} <_\omega \mathbf{o}_n$ for all n.
- Nonzero a.z. degrees exist.


$0'_\omega$
H_1
o_1
H_2
o_2
H_n
o_n
$\mathcal{G}_1$
$\mathcal{G}_1$
L_n
L_n
L_2
L_2
L_1
L_1
$A.z.$
0_ω

$\mathcal{K}$-pairs in $\mathcal{G}_\omega$

Definition

A pair of degrees $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ is called a $\mathcal{K}$-pair if

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{G}_\omega)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Theorem (G,S)

If $d_\omega(\mathcal{A})$ and $d_\omega(\mathcal{B})$ form a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_\omega$ then both $\mathcal{A}$ and $\mathcal{B}$ are a.z. or for some n there exists a $\mathcal{K}$-pair in $\mathcal{D}_e$ A, B over $\emptyset^{(n)}$ such that $\emptyset^n < A, B, A' = B' = \emptyset^{n+1}$ and:

$$\mathcal{A} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\} \text{ and}$$

$$\mathcal{B} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots, \emptyset, \dots\}.$$

$\mathcal{K}$-pairs in $\mathcal{G}_\omega$

Definition

A pair of degrees $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ is called a $\mathcal{K}$-pair if

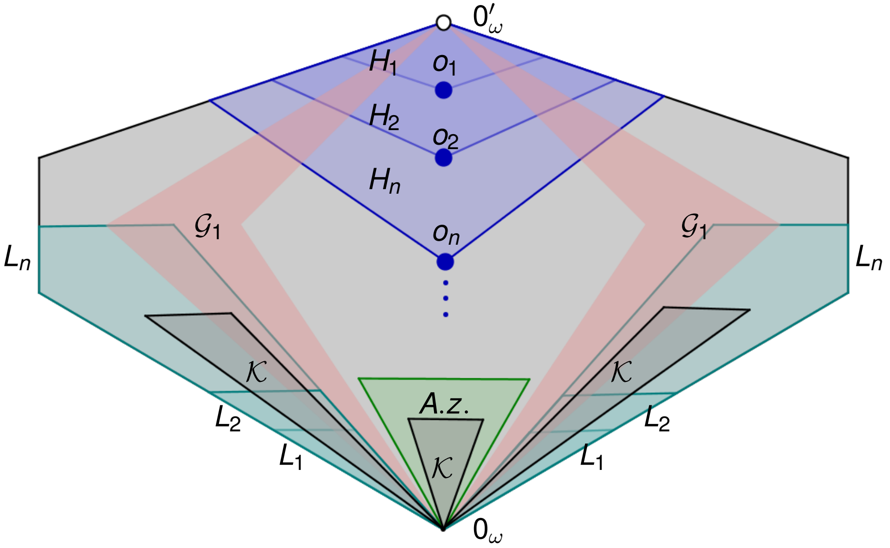
$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{G}_\omega)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Theorem (G,S)

If $d_\omega(\mathcal{A})$ and $d_\omega(\mathcal{B})$ form a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_\omega$ then both $\mathcal{A}$ and $\mathcal{B}$ are a.z. or for some n there exists a $\mathcal{K}$-pair in $\mathcal{D}_e$ A, B over $\emptyset^{(n)}$ such that $\emptyset^n < A, B$, $A' = B' = \emptyset^{n+1}$ and:

$$\mathcal{A} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\} \text{ and}$$

$$\mathcal{B} \equiv_\omega \{\underbrace{\emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots, \emptyset, \dots\}.$$


$0'_\omega$
H_1
o_1
H_2
o_2
H_n
$\mathcal{G}_1$
$\mathcal{G}_1$
o_n
L_n
L_n
$\mathcal{K}$
$\mathcal{K}$
$A.z.$
L_2
L_2
L_1
L_1
$\mathcal{K}$
0_ω

Distinguishing between different types of $\mathcal{K}$-pairs

Theorem (G,S)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ form a nontrivial minimal pair. Then for every natural number n

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof:

Suppose that $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_n$ and $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_{n+1}$. Then

$$\mathbf{a} = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\})$$

and $\emptyset^n < A < \emptyset^{n+1}$ and $A' = \emptyset^{n+1}$. Relativising the low cupping theorem for the Δ^0_2 enumeration degrees, there is an X such that $X' = \emptyset^{n+1}$ and $X \vee A \equiv_e \emptyset^{n+1}$.

$$d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \dots\}) \vee d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, X, \emptyset, \dots\}) = \mathbf{o}_n.$$

Distinguishing between different types of $\mathcal{K}$-pairs

Theorem (G,S)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ form a nontrivial minimal pair. Then for every natural number n

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n [\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof:

Suppose that $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_n$ and $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_{n+1}$. Then

$$\mathbf{a} = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\})$$

and $\emptyset^n < A < \emptyset^{n+1}$ and $A' = \emptyset^{n+1}$. Relativising the low cupping theorem for the Δ^0_2 enumeration degrees, there is an X such that $X' = \emptyset^{n+1}$ and $X \vee A \equiv_e \emptyset^{n+1}$.

$$d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \dots\}) \vee d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, X, \emptyset, \dots\}) = \mathbf{o}_n.$$

Distinguishing between different types of $\mathcal{K}$-pairs

Theorem (G,S)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ *form a nontrivial minimal pair. Then for every natural number* n

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof:

Suppose that $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_n$ and $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_{n+1}$. Then

$$\mathbf{a} = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\})$$

and $\emptyset^n < A < \emptyset^{n+1}$ and $A' = \emptyset^{n+1}$. Relativising the low cupping theorem for the Δ^0_2 enumeration degrees, there is an X such that $X' = \emptyset^{n+1}$ and $X \vee A \equiv_e \emptyset^{n+1}$.

$$d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \dots\}) \vee d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, X, \emptyset, \dots\}) = \mathbf{o}_n.$$

Distinguishing between different types of $\mathcal{K}$-pairs

Theorem (G,S)

Let $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ *form a nontrivial minimal pair. Then for every natural number n*

$$\forall \mathbf{x} \not\leq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \not\leq_\omega \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof:

Suppose that $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_n$ and $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_{n+1}$. Then

$$\mathbf{a} = d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots\})$$

and $\emptyset^n < A < \emptyset^{n+1}$ and $A' = \emptyset^{n+1}$. Relativising the low cupping theorem for the Δ^0_2 enumeration degrees, there is an X such that $X' = \emptyset^{n+1}$ and $X \vee A \equiv_e \emptyset^{n+1}$.

$$d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \dots\}) \vee d_\omega(\{\underbrace{\emptyset, \dots, \emptyset}_{n}, X, \emptyset, \dots\}) = \mathbf{o}_n.$$

Definability of $\mathbf{o}_n$

Theorem (G,S)

For every n $\mathbf{o}_n$ is first order definable in $\mathcal{G}_\omega$.

Proof Sketch: Fix $n \geq 0$. Then $\mathbf{o}_{n+1}$ is the greatest degree which is the least upper bound of a nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ in $\mathcal{G}_\omega$, such that

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n].$$

Corollary

For all n the classes H_n and L_n are first order definable in $\mathcal{G}_\omega$.

Definability of $\mathbf{o}_n$

Theorem (G,S)

For every n $\mathbf{o}_n$ is first order definable in $\mathcal{G}_\omega$.

Proof Sketch: Fix $n \geq 0$. Then $\mathbf{o}_{n+1}$ is the greatest degree which is the least upper bound of a nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ in $\mathcal{G}_\omega$, such that

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n].$$

Corollary

For all n the classes H_n and L_n are first order definable in $\mathcal{G}_\omega$.

Definability of $\mathbf{o}_n$

Theorem (G,S)

For every n $\mathbf{o}_n$ is first order definable in $\mathcal{G}_\omega$.

Proof Sketch: Fix $n \geq 0$. Then $\mathbf{o}_{n+1}$ is the greatest degree which is the least upper bound of a nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ in $\mathcal{G}_\omega$, such that

$$\forall \mathbf{x} \lneq_\omega \mathbf{o}_n [\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n].$$

Corollary

For all n the classes H_n and L_n are first order definable in $\mathcal{G}_\omega$.

Definability of $\mathcal{G}_1$

For every sequence $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ we have that:

$$d_\omega(\mathcal{A}) \vee \mathbf{o}_1 = d_\omega(\{A_0, \emptyset'', \emptyset''', \dots\}).$$

If $\mathcal{A}^* \in \mathcal{G}_1$ and $\mathcal{A}^* = \{A_0, \emptyset, \emptyset, \dots\}$ then $d_\omega(\mathcal{A}^*) \vee \mathbf{o}_1 = d_\omega(\mathcal{A}) \vee \mathbf{o}_1$ and $d_\omega(\mathcal{A}^*) \leq_\omega d_\omega(\mathcal{A})$.

Theorem (G,S)

$\mathcal{G}_1$ is first order definable in $\mathcal{G}_\omega$ by:

$$\mathbf{a} \in \mathcal{G}_1 \iff \forall \mathbf{y}(\mathbf{a} \vee \mathbf{o}_1 = \mathbf{y} \vee \mathbf{o}_1 \Rightarrow \mathbf{a} \leq_\omega \mathbf{y}).$$

Definability of $\mathcal{G}_1$

For every sequence $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ we have that:

$$d_\omega(\mathcal{A}) \vee \mathbf{o}_1 = d_\omega(\{A_0, \emptyset'', \emptyset''', \dots\}).$$

If $\mathcal{A}^* \in \mathcal{G}_1$ and $\mathcal{A}^* = \{A_0, \emptyset, \emptyset, \dots\}$ then $d_\omega(\mathcal{A}^*) \vee \mathbf{o}_1 = d_\omega(\mathcal{A}) \vee \mathbf{o}_1$ and $d_\omega(\mathcal{A}^*) \leq_\omega d_\omega(\mathcal{A})$.

Theorem (G,S)

$\mathcal{G}_1$ is first order definable in $\mathcal{G}_\omega$ by:

$$\mathbf{a} \in \mathcal{G}_1 \iff \forall \mathbf{y}(\mathbf{a} \vee \mathbf{o}_1 = \mathbf{y} \vee \mathbf{o}_1 \Rightarrow \mathbf{a} \leq_\omega \mathbf{y}).$$

The end

Thank you!