The Local Structure of the Enumeration Degrees

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02.07.08
Historical background (Ambos Spies, Fejer: *Degrees of unsolvability*).

Main definition: Enumeration reducibility, Enumeration degrees, etc.

Properties of the local structure.

The priority method using a tree of strategies.

- Solves the *Entscheidungs* problem.
- Introduces relativized computation with oracle Turing machines.
S. C. Kleene, 1936: General recursive functions of natural numbers.
S. C. Kleene, 1943: Recursive predicates and quantifiers.
E. L. Post, 1944: Recursively enumerable sets of positive integers and their decision problems.
The computably enumerable degrees

- Natural problems from other parts of mathematics.
- Post’s theorem: The degrees below $0'$ are exactly the $\Delta^0_2$ Turing degrees.
- Post’s Problem: Are there intermediate c.e. Turing degrees?
- Friedberg and Mučnik 1956-7: The priority method, the hallmark of the field.
Structural properties

Let \( \langle \mathcal{A}, 0, 1, <, \lor \rangle \) be an upper semi-lattice.

**Definition**

If \( a \lor b = c \) and \( a, b < c \) then we shall say that \( a \) cups \( b \) to \( c \).
We shall also say that the pair \((a, b)\) is a splitting of \( c \).
In the special case when \( c = 1 \), we shall simply say that \( a \) cups \( b \).

**Definition**

If \( a \land b = c \) and \( a, b > c \) then we shall say that \( a \) caps \( b \) to \( c \).
We shall also say that \( a \) and \( b \) form a minimal pair above \( c \).
In the special case when \( c = 0 \), we shall simply say that \( a \) caps \( b \) and that \((a, b)\) is a minimal pair.
Further advancements

- Infinite injury priority method.
- Sack’s Density (1963) and Splitting (1964) theorems.
- Shoenfield’s conjecture 1965: The c.e. Turing degrees are a decidable dense homogeneous partial order, reminiscent of the rational numbers.
- Lachlan and Yates 1966: There are minimal pairs of c.e. Turing degrees.
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Complications

- 0'' priority method.
- Cooper and Yates’ Non-cuppable theorem (1973).
Density and splitting cannot be combined, Lachlan’s Non-splitting theorem (1975):
There is a pair of c.e. Turing degrees $a < b$ such that $b$ cannot be split in the c.e. Turing degrees above $a$.
First use of a tree of strategies.
Harrington’s non-splitting theorem (1980): There is a c. e. degree $a < 0'$ such that no pair of c.e. degrees $b, c \geq a$ split $0'$.

Harrington and Shelah (1982): The theory of the c.e. Turing degrees is not decidable.

Harrington and Slaman: The theory of first order arithmetic can be interpreted in the theory of the c.e. Turing degrees.
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Underlying idea

- Developing the methods
- Structural properties
  - Strength of the theory
  - Finding definable classes
- Understand the structure
Alternative approaches to formalizing information content

- Algorithmic randomness.
  - Low for random degrees: zooming into the Turing structure
- Strong reducibilities: restricted computation.
  - Many-one reducibility, Truth table reducibility.
  - Computational complexity: $P = ? NP$. 
Definition (Friedberg, Rogers 1959)

A set $B$ is \textit{enumeration reducible} ($\leq_e$) to a set $A$ if there is a c.e. set $\Phi$ (e-operator) such that:

\[
 n \in B \iff \exists u(\langle n, u \rangle \in \Phi \land D_u \subseteq A),
\]

where $D_u$ denotes the finite set with code $u$ under the standard coding of finite sets.
Enumeration Degrees

“... enumeration reducibility is the fundamental, general concept of relative computability in as much as the nature of the computable universe is intimately bound up with the set of enumeration operators.”

Cooper, 1990

- Enumeration equivalence: $A \equiv_e B \iff A \leq_e B \land B \leq_e A$.
- Enumeration degree: $d_e(A) = \{ B \mid A \equiv_e B \}$.
- Least upper bound: $d_e(A) \lor d_e(B) = d_e(A \oplus B)$.
- Jump operator: $d_e(A)' = d_e(K_A \oplus A)$.
- Upper semi-lattice with jump: $\langle D_e, 0_e, \leq, \cup, \prime \rangle$. 
The local structure of the enumeration degrees $\mathcal{D}_e(\leq 0'_e)$

$\Sigma^0_2$ e-degrees

$\Delta^0_2$ e-degrees

$\Pi^0_1$ e-degrees
Transferring results from the Turing degrees

Natural embedding: \( \iota(d_t(A)) = d_e(A \oplus \overline{A}) \).
Putting words into actions

Theorem (S, Cooper)
There exists a $\Pi^0_1$ enumeration degree $a < 0'_e$ such that there exists no nontrivial splitting of $0'_e$ by a pair of a $\Pi^0_1$ enumeration degree and a $\Sigma^0_2$ enumeration degree both above $a$.

Corollary (Extending Harrington’s Non-splitting Theorem)
There exists a computably enumerable degree $a < 0'$ such that there is no nontrivial splitting of $0'$ by a pair of a c.e. degree and a $\Delta^0_2$ degree both above $a$. 
Putting words into actions

\[ \Delta_2^0 \ni \iota(u) \lor \iota(w) \neq 0_e' \]

\[ \iota(u) \ni \Sigma_2^0 \]

\[ \Pi_1^0 \ni 0_e \]

\[ \iota^{-1}(a) \ni \Delta_2^0 \]

\[ C.E \]

\[ D_e \leftrightarrow D_T : \iota \]
Zooming in: The $\Delta^0_2$ enumeration degrees

- Cooper, Sorbi, Yi (1996): Every nonzero $\Delta^0_2$ enumeration degree is cuppable.
- Cooper, Sorbi, Li, Yang (2006): Every nonzero $\Delta^0_2$ enumeration degree bounds a minimal pair.
- Arslanov, Sorbi (1999): There is a $\Delta^0_2$ splitting of $0'_e$ above each incomplete $\Delta^0_2$ enumeration degree.
- Arslanov, Sorbi, Kalimullin (2001): The $\Delta^0_2$ enumeration degrees are dense.
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Zooming out: The $\Sigma^0_2$ enumeration degrees

- Cooper (1984): The $\Sigma^0_2$ enumeration degrees are dense.
- Cooper, Sorbi, Yi (1996): There is a non-cuppable nonzero $\Sigma^0_2$ enumeration degree.
- Cooper, Sorbi, Li, Yang (2006): There is a nonzero $\Sigma^0_2$ enumeration degree that does not bound a minimal pair.
- An ideal of properly $\Sigma^0_2$ enumeration degrees and $0_e$: 
  \[ \mathcal{I} = \{ a \mid a > 0_e \Rightarrow (\forall x, y \leq a) [0_e < x \land 0_e < y \Rightarrow (\exists d)[d \leq x \land d \leq y \land d \neq 0_e]] \}. \]
Genericity

Definition
A set \( A \) is 1-generic if for every c.e. set \( W \) there exists a finite string \( \lambda \subset \chi_A \) such that:

\[
\lambda \in W \lor (\forall \mu \supseteq \lambda)(\mu \notin W).
\]

Degrees of 1-generic sets are called 1-generic degrees.

Theorem (S)

There exists a 1-generic \( \Sigma^0_2 \) enumeration degree \( a \) that does not bound a minimal pair in the semi-lattice of the enumeration degrees.
Completing the picture

**Theorem (S)**

*There is a $\Sigma^0_2$ enumeration degree $a < 0'_e$ such that $0'_e$ cannot be split in the enumeration degrees above the degree $a$.***

A filter of properly $\Sigma^0_2$ enumeration degrees and $0'_e$:

$$\mathcal{F} = \{ a \mid a < 0'_e \Rightarrow (\forall u, v) \ [a \leq u < 0'_e \land a \leq v < 0'_e \Rightarrow u \lor v \neq 0'_e]\}.$$
Things are not so simple

- Slaman, Woodin (1997): The theory of the $\Sigma^0_2$ enumeration degrees is undecidable.
- Cooper’s conjecture: The structures of the $\Sigma^0_2$ e-degrees and the c.e. Turing degrees are elementary equivalent.
- Ahmad: The diamond can be embedded in the $\Sigma^0_2$ enumeration degrees.
- Ahmad and Lachlan: Non-splittable degrees exist.
- Kent 2005: The theory of the $\Delta^0_2$ enumeration degrees is undecidable.
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Cupping properties of the $\Delta_2^0$ enumeration degrees

**Theorem (S, Wu)**

*Every nonzero $\Delta_2^0$ enumeration degree can be cupped by a partial low $\Delta_2^0$ enumeration degree.*
Reaching the first limit

Theorem (S)

Let \( \{a_i\}_{i<\omega} \) be a \( \Delta^0_2 \)-computably enumerable sequence of enumeration degrees. There exists a nonzero \( \Delta^0_2 \) enumeration degree \( b \) such that for every \( i < \omega \) if \( a_i \) is incomplete then \( a_i \vee b \neq 0'_e \).

Here a class \( \{a_i\}_{i<\omega} \) of \( \Delta^0_2 \) enumeration degrees is \( \Delta^0_2 \)-computably enumerable if there is a computable sequence of \( \Delta^0_2 \) approximations \( \{A_i[s]\}_{i,s<\omega} \) to representatives \( A_i \) of every degree \( a_i \) in the class.
The Difference Hierracy

Definition (Ershov)

1. A set $A$ is $n$-c.e. if there is a computable function $f$ such that for each $x$, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq n$ and $A(x) = \lim_s f(x, s)$.

2. $A$ is $\omega$-c.e. if there are two computable functions $f(x, s), g(x)$ such that for all $x$, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq g(x)$ and $\lim_s f(x, s) = A(x)$.

3. A degree $a$ is $n$-c.e.$(\omega$-c.e.) if it contains a $n$-c.e.$(\omega$-c.e.) set.
The Difference Hierarachy

\[ \Pi^0_1 = 2 \text{-c.e.} \]

\[ \Delta^0_2 \]

\[ \Sigma^0_2 \]

\[ 3 \text{-c.e.} \]

\[ \omega \text{-c.e.} \]
Corollary

There exists a nonzero $\Delta^0_2$ enumeration degree that cannot be cupped by any incomplete $\omega$-c.e. degree.

Theorem (S, Wu)

For every nonzero $\omega$-c.e. enumeration degree $a$ there exists an incomplete $3$-c.e. enumeration degree $b$ that cups $a$. 
Cupping classes of enumeration degrees

(Cooper, Seetapun and Li): There exists a single incomplete $\Delta^0_2$ Turing degree that cups every nonzero c.e. Turing degree.
The second limitation

For any larger subclass, which contains the nonzero 3-c.e enumeration degrees this cannot be done as:

**Theorem (S)**

Let $a$ be an incomplete $\Sigma^0_2$ enumeration degree. There exists a nonzero 3-c.e. enumeration degree $b$ such that $a \lor b \neq 0'_e$. 
The $n$-c.e. enumeration degrees are far from simple

An analog of Lachlan’s non-splitting theorem for every class of $n$-c.e. enumeration degrees, where $n \geq 3$.

Theorem (Arslanov, Cooper, Kalimullin, S)

There exists a pair of a $\Pi^0_1$ enumeration degree $a$ and a 3-c.e. enumeration degree $b < a$ such that $a$ cannot be split by a pair of enumeration degrees above $b$. 
Questions

1. What is the exact theoretical complexity of any of the classes we considered?
2. Can we define a smaller class within a larger class?
3. What is the precise role of genericity in the enumeration degrees?
4. What is the mathematical reason for the similarities and differences between the local structures $\mathcal{D}_T(\leq 0')$ and $\mathcal{D}_e(\leq 0'_e)$?
The priority method

- Friedberg and Mučnik 1956-7. Solution to Post’s problem.
- The main method used in $\mathcal{D}_T(\leq 0')$ and $\mathcal{D}_e(\leq 0'_e)$.
- Construction of representatives of degrees in the local structure.

**Theorem**

There exist incomparable $\Pi^0_1$ enumeration degrees.
Step 1: Formalizing the requirements

We shall construct two $\Pi^0_1$ sets $A$ and $B$ so that ultimately $d_e(A)$ and $d_e(B)$ are incomparable.

The sets $A$ and $B$ should be incomparable: $A \nleq_e B$ and $B \nleq_e A$.

Let $\{\Phi_e\}_{e<\omega}$ be a computable enumeration of all c.e. sets:

1. $P_e: A \neq \Phi^B_e$;
2. $Q_e: B \neq \Phi^A_e$. 
Approximations: The computable content of the constructions

Definition
A $\Pi^0_1$ approximation to a set $A$ is a computable sequence of cofinite sets $\{A[s]\}_{s<\omega}$, such that:

- $A[0] = \mathbb{N}$.
- $n \notin A[s] \Rightarrow (\forall t \geq s)[n \notin A[t]]$.
- $n \in A \iff (\forall t)[n \in A[t]]$.

The construction runs in stages:
- At every stage $s$ we only have finite (computable) information to the given sets: $\Phi_e[s]$.
- We construct $A[s]$ and $B[s]$ based on the finite amount of information given.
Step 2: Designing the basic modules

To every requirement we associate a finite set of instructions:

- Similar requirements have similar basic modules.
- Actions:
  - Modify own parameters;
  - Modify the approximations to the constructed sets;
  - Impose restrictions.
- If executed infinitely many times, guarantee satisfaction of the corresponding requirement.
Basic module for $\mathcal{P}_e$

$\mathcal{P}_e : A \neq \Phi^B_e$.

1. If the witness $x_e$ is not selected, then let $x_e$ be a fresh number, one that has not appeared in the construction so far.

2. If $x_e \notin \Phi^B_e[s]$ then do nothing.

3. If $x_e \in \Phi^B_e[s]$ then there is an axiom $\langle x_e, D \rangle \in \Phi_e[s]$ with $D \subseteq B[s]$. Extract $x_e$ from $A[s]$ and restrain $D$ in $B$. 
Step 3: Identifying the outcomes

- More than one possible method for satisfying a requirement.
- Strategies choose their method with respect to the current situation and the methods chosen by other strategies.
- The choice of a particular method corresponds to an outcome.
- Two outcomes for $P_e$:
  - Wait forever for $x_e \in \Phi^B_e$: Outcome $w$.
  - At some stage $x$ enters $\Phi^B_e$: Outcome $f$. 
Conflicts

\( P_e \)-strategy

1. If the witness \( x_e \) is not selected, then let \( x_e \) be a fresh number, one that has not appeared in the construction so far.

2. If \( x_e \notin \Phi^B_e[s] \) then do nothing.

3. If \( x_e \in \Phi^B_e[s] \) via axiom \( \langle x_e, D \rangle \) then extract \( x_e \) from \( A[s] \) and restrain \( D \) in \( B \).

\( Q_j \)-strategy

1. If the witness \( x_j \) is not selected, then let \( x_j \) be a fresh number, one that has not appeared in the construction so far.

2. If \( x_j \notin \Phi^A_j[s] \) then do nothing.

3. If \( x_j \in \Phi^A_j[s] \) via axiom \( \langle x_j, F \rangle \) then extract \( x_j \) from \( B[s] \) and restrain \( F \) in \( A \).
Resolving the conflicts: Priority ordering

- We order the set of requirements $\mathcal{R}$ linearly:
  \[ P_0 < Q_0 < P_1 < Q_1 < P_2 \ldots \]
- Requirements in earlier positions have higher priority.
- Lower priority requirements respect the restrictions imposed by higher priority requirements.
- They assume that the method chose by higher priority strategies is final.
- If they are wrong - we say that they are injured. An injured requirement is initialized and starts work from the beginning under the changed assumptions.
Step 4: The tree of strategies

- Injury appears when a strategy decides to change its method (outcome).
- We order the set of outcomes \( O = \{ w, f \} \) linearly:
  \[
  f <_L w
  \]
  The set \( O^{\leq \omega} \) has an induced lexicographical order \(<\).

Definition
The tree of strategies is a computable function \( T \) with domain \( D(T) \) a downwards closed subset of \( O^{< \omega} \) and range \( R(T) = \mathcal{R} \), such that:

- For every path \( f \subseteq D(T) \) we have \( R(T \upharpoonright f) = \mathcal{R} \).
- Higher priority requirements are assigned to nodes at higher levels of the tree.
Each node on the tree has its own instance of a strategy associated with it.
The construction

- At stage 0 all nodes are initialized.
- At each stage $s > 0$ we construct a finite path $\delta[s]$ of length $s$ through the domain of $T$ starting at the root of the tree.
- Nodes $\alpha \subseteq \delta[s]$ are activated at stage $s$.
  - They run their instance of the basic module.
  - Select an outcome $o$.
  - Initialize lower priority requirements which have not predicted the outcome correctly.
- The next node visited at stage $s$ will be $\alpha^o$. 
Visualizing the construction

Stage 0

All nodes are initialized.

(A = \mathbb{N} \\
\Phi_0[0] = \emptyset)
Visualizing the construction

Stage 1

\[ A = \mathbb{N} \]
\[ \Phi_0[1] = \{ \langle 2, \{3, 5\} \rangle \} \]

Visit node \( \emptyset \). Select a witness for node \( \emptyset \): \( x_\emptyset = 6 \).
Check if \( x_\emptyset \in \Phi_0^B[1] \). The answer is no, outcome is \( w \). End stage 1.
Visualizing the construction

Stage 2

\[ A = \mathbb{N} \]
\[ \Phi_0[2] = \{ \langle 2, \{3, 5\} \rangle, \langle 11, \{1, 12\} \rangle \} \]

Visit node \( \emptyset \). Check if \( x_\emptyset \in \Phi_0^B[2] \). The answer is no, outcome is \( w \).
Visualizing the construction

Stage 2

Visit node $w$. Select a witness $x_w$ for the node $w$.
Check if $x_w \in \Phi^A_0[2]$. The answer is no, outcome is $w$. End stage 2.
Visualizing the construction

Stage 19

Visit node $\emptyset$. Visit node $\emptyset$. Check if $x_\emptyset \in \Phi_0^B[19]$. The answer is no, outcome is $w$. 

\[ A = \mathbb{N} \]
\[ \Phi_0[19] = \{ \langle 2, \{3, 5\}\rangle, \langle 11, \{1, 17\}\rangle, \langle 13, \{2, 21, 88\}\rangle \ldots \} \]

\[ B = \mathbb{N} \]
Visualizing the construction

Stage 19

\[
A = \mathbb{N} \\
\Phi_0[19] = \{\langle 2, \{3, 5\} \rangle, \langle 11, \{1, 17\} \rangle, \langle 13, \{2, 21, 88\} \rangle \ldots \}
\]

Visit node \(w\). Check if \(x_w \in \Phi_0^A[19]\). The answer is Yes. Extract \(x_w\) from \(B\), restrain \(\{2, 21, 88\}\) in \(A\). The outcome is \(f\). Initialize all nodes to the right of \(f\).
Visualizing the construction
Stage 19

$A = \mathbb{N}$
$B = \mathbb{N} \setminus \{13\}$
$
\Phi_0[19] = \{\langle 2, \{3, 5\} \rangle, \langle 11, \{1, 17\} \rangle, \langle 13, \{2, 21, 88\} \rangle \ldots \} 
\Phi_1[19] = \{\langle 1, \{2, 4\} \rangle, \langle 10, \{5, 16\} \rangle, \ldots \}
$

Visit node $wf$. Select a witness $x_{wf}$. Check if $x_{wf} \in \Phi_1^B[19]$. The answer is no, outcome is $w$. 
The key point: The true path

Lemma (True path lemma)

There exists an infinite path $h$ in the tree of strategies, called the true path, with the following properties:

1. $(\forall n)(\exists \infty s)[ h \uparrow n \subseteq \delta[s] ];$
2. $(\forall n)(\exists s_i(n))(\forall s > s_i(n))[ h \uparrow n \text{ is not initialized at stage } s ].$

In our case: the leftmost path of nodes visited infinitely often is the true path.
Every node along the true path satisfies its requirement.
Thank you!