Embedding Partial Orderings in Degree Structures

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A very basic result

Theorem (Mostowski 1938)

There exists a computable partial ordering $\mathcal{R} = \langle \mathbb{N}, \leq \rangle$ in which every countable partial ordering can be embedded.

Proof.

Let $\mathcal{R} = \langle \mathbb{Q}^2, \leq \rangle$, where $\langle a, b \rangle \leq \langle c, d \rangle$ if and only if $a \leq c$ and $b \leq d$. $\square$

Conclusion: An embedding of this computable partial ordering gives automatically an embedding of every countable partial ordering.

Independent sequences of sets

Definition (Kleene, Post 1954)

A sequence of sets $\{A_i\}_{i<\omega}$ is called computably independent if for every i:

$$A_i \not\leq_T \bigoplus_{j \neq i} A_j.$$

Theorem (Kleene, Post 1954)

There is a computably independent sequence of sets. This sequence can be constructed uniformly below $0'$.

Theorem (Muchnik 1958)

There is a computably independent sequence of c.e. sets.

Putting the two together

Theorem (Sacks 1963)

The existence of a computably independent sequence of sets gives an embedding of any computable partial ordering in the Turing degrees.

Proof.

Let $\mathcal{R} = \langle \mathbb{N}, \preceq \rangle$ be a computable partial ordering and $\{A_i\}_{i<\omega}$ be a computably independent sequence of sets. The embedding is:

$$\kappa(i) = d_T(\bigoplus_{j \preceq i} A_i).$$

□

The final step..

Corollary

Every countable partial ordering can be embedded

1. *Kleene and Post: in the Turing degrees, even in the Δ^0_2 Turing degrees.*
2. *Muchnik: in the c.e. Turing degrees.*
3. *Robinson 1971: densely in the c.e. Turing degrees, i.e. in any nonempty interval of c.e. Turing degrees.*

The enumeration degrees

- The e-degrees as a proper extension of the Turing degrees, inherit this complexity.
- Case 1971: Any countable partial ordering can be embedded in the e-degrees below the degree of any generic function.
- Copestake 1988: below any 1-generic enumeration degree.
- Cooper and McEvoy 1985: below any nonzero Δ^0_2 e-degree.
- Bianchini 2000: densely in the Σ^0_2 enumeration degrees.

Method: *e*-independent sequences of sets.

The first observation

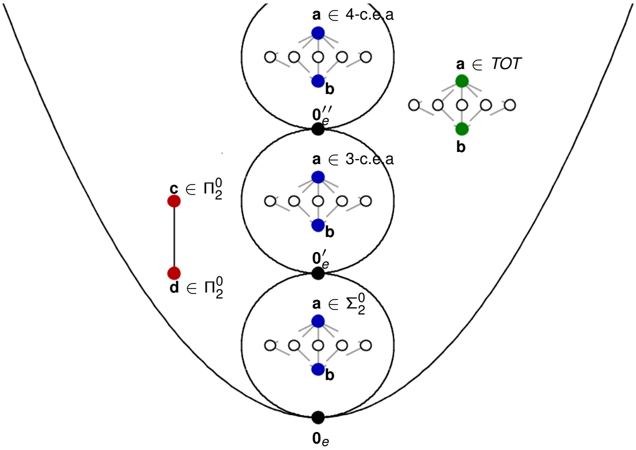
Theorem

Let $\mathbf{b} < \mathbf{a}$ *be enumeration degrees such that* $\mathbf{a}$ *contains a member with a good approximation. Then every countable partial ordering can be embedded in the interval* $[\mathbf{b}, \mathbf{a}]$*.*

Idea: Construct an e-independent sequence of sets above $\mathbf{b}$ and uniformly below $\mathbf{a}$.

Techniques: Good approximations combined with a construction inspired by Cooper's density construction.

The general picture



The ω e-degrees: Basic definitions

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets natural numbers and V be an e-operator. The result of applying the enumeration operator V to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every n we have that $A_n \leq_e B_n$ via, say, Γ_n.
- Uniformity: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.

Basic definitions

With every member $\mathcal{A} \in \mathcal{S}$ we connect a *jump sequence* $P(\mathcal{A})$.

Definition

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.

The ω-enumeration degrees

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$.

Definition

- ω-enumeration reducibility: $\mathcal{A} \leq_\omega \mathcal{B}$, if $\mathcal{A} \leq_e P(\mathcal{B})$.
- ω-enumeration equivalence: $\mathcal{A} \equiv_\omega \mathcal{B}$ if $\mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$.
- ω-enumeration degrees: $d_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$.
- The structure of the ω-enumeration degrees: $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$, where $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{B})$ if $\mathcal{A} \leq_\omega \mathcal{B}$.
- The least ω-enumeration degree: $\mathbf{0}_\omega = d_\omega((\emptyset, \emptyset, \emptyset, \dots))$ or equivalently $d_\omega((\emptyset, \emptyset', \emptyset'', \dots))$.

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$\mathcal{D}_\omega$ as an upper semilattice with jump operation

- The join and least upper bound: $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n<\omega}$. $d_\omega(\mathcal{A} \oplus \mathcal{B}) = d_\omega(\mathcal{A}) \vee d_\omega(\mathcal{B})$.
- The jump operation: $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n<\omega}$.

The e-degrees as a substructure

$\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$ can be embedded in $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$ via the embedding κ defined as follows:

$$\kappa(d_e(A)) = d_\omega((A, \emptyset, \emptyset, \dots)) = d_\omega((A, A', A'', \dots)).$$

Theorem (Soskov, Ganchev)

- *The structure $\mathcal{D}_1 = \kappa(\mathcal{D}_e)$ is first order definable in $\mathcal{D}_\omega$.*
- *The structures $\mathcal{D}_e$ and $\mathcal{D}_\omega$ with jump operation have isomorphic automorphism groups.*

The embeddability question

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $0'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \dots))$ also called the Σ^0_2 ω-enumeration degrees.

Theorem (Soskov)

The structure $\mathcal{G}_\omega$ is dense.

Theorem

Let $\mathbf{b} <_\omega \mathbf{a} \leq_\omega \mathbf{0}'_\omega$. Every countable partial ordering can be embedded in the interval $[\mathbf{b}, \mathbf{a}]$.

The independent sequence method

Definition

Let $\{\mathcal{A}_i\}_{i<\omega}$ be a sequence of sequences of sets

- For every computable set C set $\bigoplus_{i\in C}\mathcal{A}_i = (\bigoplus_{i\in C} A_{0,i}, \bigoplus_{i\in C} A_{1,i}, \bigoplus_{i\in C} A_{2,i}, \dots)$.
- The sequence is ω-independent if for every i we have $\mathcal{A}_i \not\leq_\omega \bigoplus_{j\neq i}\mathcal{A}_j$

Goal: Construct an ω-independent sequence of sequences of sets above **b** and uniformly below **a**.

Good approximations to sequences

Definition (Soskov)

Let $\{A_n^{\{s\}}\}_{n,s<\omega}$ be a uniformly computable matrix of finite sets.
We say that $\{A_n^{\{s\}}\}_{s<\omega}$ is a *good approximation* to the sequence $\mathcal{A} = \{A_n\}_{n<\omega}$ if:

$G0$: $(\forall s, k)[A_k^{\{s\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m^{\{s\}} \subseteq A_m]]$;

$G1$: $(\forall n, k)(\exists s)(\forall m \leq k)[\ A_m \upharpoonright n \subseteq A_m^{\{s\}} \subseteq A_m\ ]$ and

$G2$: $(\forall n, k)(\exists s)(\forall t > s)[A_k^{\{t\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m \upharpoonright n \subseteq A_m^{\{t\}}]]$.

Or more intuitively:

- We have a good approximation to every member of the sequence.
- If $m \leq k$ then every k-good stage is m-good.

Proof idea

Theorem (Soskov)

Every Σ^0_2 ω-enumeration degree contains a member $\mathcal{A}$ such that $\mathcal{A} \equiv_e P(\mathcal{A})$ and $\mathcal{A}$ has a good approximation.

So fix $\mathcal{A} = (A_0, A_1, \dots)$ in **a** with the properties listed in the theorem and $\mathcal{B} = (B_0, B_1, \dots)$ in **b**.
Now $\mathcal{B} <_\omega \mathcal{A}$ can follow by two ways:

- Non-enumeration reducibility: There is an n such that $B_n <_e A_n$.
- Non-uniformity: For every n we have $B_n \equiv_e A_n$ but not uniformly in n.

Easy case: From the e-degrees

Let n be such that $B_n <_e A_n$.

- By first theorem there is an independent sequence of sets $\{C_i\}_{i<\omega}$ above B_n and uniformly below A_n.
- Define $\{\mathcal{C}_i\}_{i<\omega}$ by $\mathcal{C}_i = (B_0, B_1, \ldots, B_{n-1}, C_i, B_{n+1}, \ldots)$.
- $\{\mathcal{C}_i\}_{i<\omega}$ is an ω-independent.
- For every i we have $\mathcal{B} <_\omega \mathcal{C}_i <_\omega \mathcal{A}$.

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- For every i we have $\mathcal{B} <_\omega \mathcal{C}_i <_\omega \mathcal{A}$.

Difficult case

For every n $A_n \equiv_e B_n$.

Idea: Direct construction building on ideas from first result.

Difficulties: Approximate sets of the form $P(V(\mathcal{A}))$, where V is the constructed e-operator.

Techniques: Good approximations for sequences of sets. Length of agreement function for sequences of sets. Fixed point theorem (Recursion theorem).

The c.e. degrees modulo iterated jump

Definition (Jockusch, Lerman, Soare and Solovay)

Let **a** and **b** be c.e. Turing degrees. $\mathbf{a} \sim_\infty \mathbf{b}$ iff there exists a natural number n such that $\mathbf{a}^n = \mathbf{b}^n$.

- Induced degree structure $\mathcal{R}/\sim_\infty$ with $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ if and only if there exists a natural number n such that $\mathbf{a}^n \leq_T \mathbf{b}^n$.
- Least element $L = \bigcup_{n<\omega} L_n$.
- Greatest element $H = \bigcup_{n<\omega} H_n$.
- $\mathcal{R}/\sim_\infty$ is a dense structure.
- Lempp : There is a splitting of the highest ∞-degree and a minimal pair of ∞-degrees.

Starting with other classes of degree

- $\mathcal{G}_T / \sim_\infty$: the Δ^0_2 Turing degrees modulo iterated jump. Shoenfield, Sacks: The range of the jump operator restricted to the c.e. Turing degrees coincides with the range of the jump operator restricted to the Δ^0_2 Turing degrees. It is namely the set of all Turing degrees c.e. in and above $\mathbf{0}'$. Hence:

$$\mathcal{G}_T / \sim_\infty \simeq \mathcal{R} / \sim_\infty .$$

- $\mathcal{G}_e / \sim_\infty$: the Σ^0_2 e-degrees modulo iterated jump. McEvoy: The range of the enumeration jump operator restricted to the Σ^0_2-enumeration degrees coincides with the range of the enumeration jump operator restricted to the Π^0_1 enumeration degrees. Hence:

$$\mathcal{R} / \sim_\infty \simeq (\Pi^0_1 \text{ e-degrees}) / \sim_\infty \simeq \mathcal{G}_e / \sim_\infty .$$

The ω-enumeration degrees modulo iterated jump

Consider $\mathcal{G}_\omega / \sim_\infty$.

- $\mathcal{R} / \sim_\infty$ embeds in $\mathcal{G}_\omega / \sim_\infty$.

$$\mathcal{R} \subseteq \mathcal{G}_T \hookrightarrow \iota(\mathcal{G}_T) = \mathit{Tot} \subseteq \mathcal{G}_e \hookrightarrow \kappa(\mathcal{G}_e) = \mathcal{D}_1 \subseteq \mathcal{G}_\omega$$

- A basic property:

Lemma

Let $\mathbf{a}$ and $\mathbf{b}$ be two Σ^0_2 ω-enumeration degrees.

1. *If $\mathbf{a} \leq_\omega \mathbf{b}$ then $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$.*
2. *If $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ then there is a representative $\mathbf{c} \in [\mathbf{a}]_{\sim_\infty}$ such that $\mathbf{c} \leq_\omega \mathbf{b}$.*

The almost degrees

Definition

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets of natural numbers. We shall say that the sequence $\mathcal{B} = \{B_n\}_{n<\omega}$ is *almost-$\mathcal{A}$* if for every n we have that $P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$.
If $\mathcal{A}$ is almost-$\mathcal{B}$ then we shall say that $d_\omega(\mathcal{A})$ is almost-$d_\omega(\mathcal{B})$.

Lemma

Let $\mathbf{a} \leq 0'_\omega$ *be an* ω*-enumeration degree.*

1. *If* $\mathbf{b}$ *is almost-*$\mathbf{a}$ *and* $\mathcal{A} \in \mathbf{a}$ *then every* $\mathcal{B} \in \mathbf{b}$ *is almost-*$\mathcal{A}$*.*
2. *The class of almost-*$\mathbf{a}$ *degrees is closed under least upper bound.*
3. *If* $\mathbf{a} \leq_\omega \mathbf{c} \leq_\omega \mathbf{b}$ *and* $\mathbf{b}$ *is almost-*$\mathbf{a}$ *then* $\mathbf{c}$ *is almost-*$\mathbf{a}$*.*

The almost degrees

Lemma

4. *If* **a** $\in \mathcal{D}_1$ *then* **a** *is the least almost-***a** Σ^0_2 ω*-enumeration degree.*
5. *If* **b** *and* **c** *are almost-***a** Σ^0_2 ω*-enumeration degrees then* $[\mathbf{b}]_{\sim_\infty} \leq [\mathbf{c}]_{\sim_\infty}$ *if and only if* $\mathbf{b} \leq_\omega \mathbf{c}$*.*
6. *If* $\mathbf{a} <_\omega \mathbf{b}$ *and* $\mathbf{a} <_\infty \mathbf{b}$ *then there exists an almost-***a** *degree* **z** *such that* $\mathbf{a} <_\omega \mathbf{z} \leq_\omega \mathbf{b}$ *.*

Corollary

$\mathcal{G}_\omega / \sim_\infty$ *properly extends* $\mathcal{R} / \sim_\infty$*.*

Embedding partial orders in $\mathcal{G}_\omega / \sim_\infty$

Corollary

Every countable partial ordering can be embedded densely in $\mathcal{G}_\omega / \sim_\infty$.

Proof.

- Let $[\mathbf{a}]_{\sim_\infty} < [\mathbf{b}]_{\sim_\infty}$.
- We may assume that $\mathbf{a} <_\omega \mathbf{b}$.
- Let $\mathbf{z}$ be an almost-$\mathbf{a}$ degree such that $\mathbf{a} <_\omega \mathbf{z} \leq_\omega \mathbf{b}$.
- Then $[\mathbf{a}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$.
- And $[\mathbf{a}, \mathbf{z}]$ consists entirely of almost-$\mathbf{a}$ degrees, hence is isomorphic to $[[\mathbf{a}]_{\sim_\infty}, [\mathbf{z}]_{\sim_\infty}]$.
- By second result we can embed any countable partial ordering in $[\mathbf{a}, \mathbf{z}]$.

□

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$\square$

Thank you!