Definability, automorphisms and enumeration degrees

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In honor of Ivan Soskov's 60'th birthday

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Enumeration reducibility

Reducibility	Oracle set B	Reduced set
$A \leq_T B$	Complete information	Complete information
A c.e. in B	Complete information	Positive information
$A \leq_e B$	Positive information	Positive information

Definition (Friedberg, Rogers (59))

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \ \&\ D \subseteq B)\}.$$

The structures of the Turing degrees $\mathcal{D}_T$ and the enumeration degrees $\mathcal{D}_e$ are upper semi-lattices with least element and jump operation.

The automorphism problem

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Theorem (Slaman, Woodin)

The rigidity of $\mathcal{D}_T$ is equivalent to its biinterpretability with second order arithmetic.

Biinterpretability

Theorem (Simpson, Slaman and Woodin)

The first order theories of $\mathcal{D}_T$ and $\mathcal{D}_e$ are each computably isomorphic to the theory of Second order arithmetic.

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$\mathcal{D}_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{\mathbf{p}}, \mathbf{x})$ defined by "$\vec{\mathbf{p}}$ codes a standard model of arithmetic with a unary predicate for the set Y and Y is of the same degree as $\mathbf{x}$" is definable in $\mathcal{D}_T$.

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$Aut(\mathcal{D}_T)$ is countable and every member has an arithmetically definable presentation.
Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A \oplus \overline{A}$ *is c.e. in* $B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

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The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

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$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

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Question (Rogers (67))

Is the set of total enumeration degrees first order definable in $\mathcal{D}_e$?

The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

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- The total enumeration degrees are an automorphism base for $\mathcal{D}_e$.
- If $\mathcal{TOT}$ is definable then a nontrivial automorphism of $\mathcal{D}_e$ implies a nontrivial automorphism of $\mathcal{D}_T$.

Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function s_A, such that $s_A(x,y) \in \{x,y\}$ and if $\{x,y\} \cap A \neq \emptyset$ then $s_A(x,y) \in A$.

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Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-computable set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

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- If X is not computable then there is a semi-computable set A with $d_e(X \oplus \overline{X}) = d_e(A) \vee d_e(\overline{A})$.

Kalimullin pairs

Definition (Kalimullin)

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Theorem (Kalimullin)

A pair of sets A, B is a $\mathcal{K}$-pair if and only if their enumeration degrees **a** *and* **b** *satisfy:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

Definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ *is the largest degree which can be represented as the least upper bound of a triple* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*, such that* $\mathcal{K}(\mathbf{a}, \mathbf{b})$*,* $\mathcal{K}(\mathbf{b}, \mathbf{c})$ *and* $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

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Corollary (Kalimullin)

1. *The enumeration jump is first order definable in $\mathcal{D}_e$.*

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$\mathbf{0}_e'$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

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1. *The enumeration jump is first order definable in $\mathcal{D}_e$.*
2. *The set of total enumeration degrees above $\mathbf{0}_e'$ is first order definable in $\mathcal{D}_e$.*

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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2. *The low enumeration degrees are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.*

Maximal $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if for every $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ with $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

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Corollary

In $\mathcal{D}_e(\leq \mathbf{0}'_e)$ a nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

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In particular the total enumeration degrees are definable with parameters in $\mathcal{D}_e$.

Defining total enumeration degrees in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set C, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

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The goal: $A = \{m \mid \exists q \in C(q \text{ is labeled by } m)\}$ and
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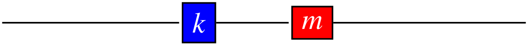
While $(m, k) \notin W$: $\mathbb{Q}$:



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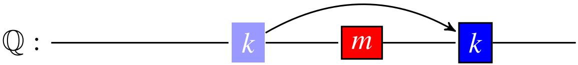
If $(m, k) \in W$:



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Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

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The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- *If $\mathcal{D}_T$ is rigid then $\mathcal{D}_e$ is rigid.*
- *The automorphism analysis for the enumeration degrees follows.*
- *The total degrees below $\mathbf{0}_e^{(5)}$ are an automorphism base of $\mathcal{D}_e$.*

The relation *c.e. in*

Definition

A Turing degree $\mathbf{a}$ is *c.e. in* a Turing degree $\mathbf{x}$ if some $A \in \mathbf{a}$ is c.e. in some $X \in \mathbf{x}$.

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Theorem (Cai, Ganchev, Lempp, Miller, S)

The set $\{\langle \iota(\mathbf{a}), \iota(\mathbf{x})\rangle \mid \mathbf{a} \text{ is c.e. in } \mathbf{x}\}$ *is first order definable in* $\mathcal{D}_e$.

1. Ganchev, S had observed that if $\mathcal{TOT}$ is definable by maximal $\mathcal{K}$-pairs then the image of the relation 'c.e. in' is definable for non-c.e. degrees.
2. A result by Cai and Shore allowed us to complete this definition.

Local structures of Turing degrees

Definition

$\mathcal{R}$ is the substructure of the computably enumerable degrees.

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$\mathcal{D}_T(\leq \mathbf{0}')$ is the substructure of all degrees that are bounded by $\mathbf{0}'$, the Δ^0_2 Turing degrees.

The local coding theorem

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Definition

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq \mathbf{0}')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function f such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

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Example: If $\bigoplus_{i<\omega} A_i$ is low then $\mathcal{A} = \{d_T(A_i) \mid i < \omega\}$ is uniformly low.

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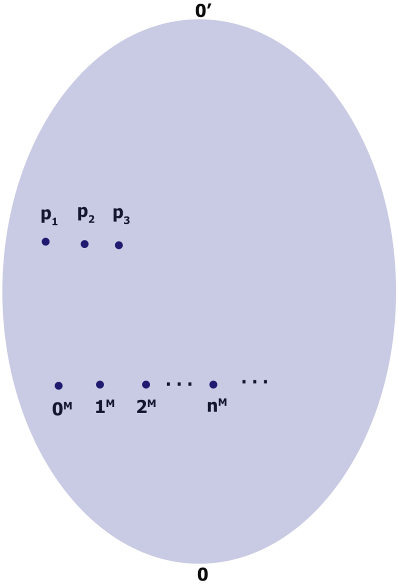
Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from finitely many parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

Applications of the coding theorem



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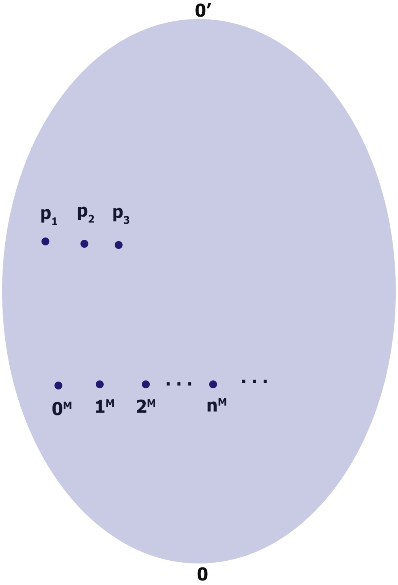

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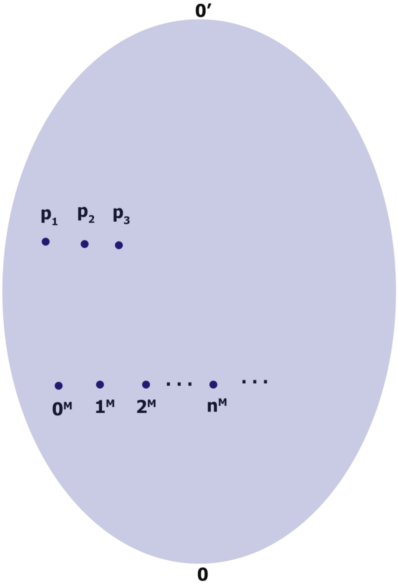

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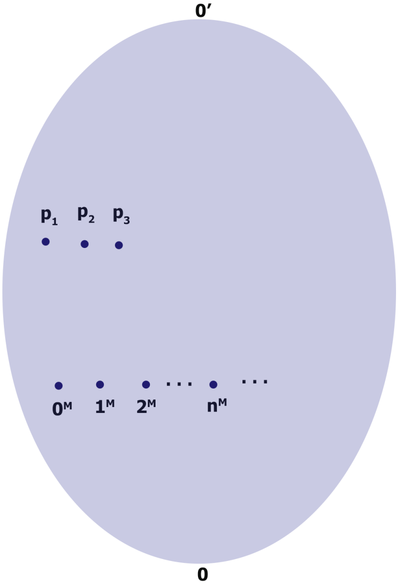

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3. $\mathbb{N} \models \varphi$ iff $\mathcal{D}_T(\leq \mathbf{0}') \models \varphi_T(\vec{\mathbf{p}})$

Applications of the coding theorem

If $\mathcal{Z} \subseteq \mathcal{D}_T(\leq \mathbf{0}')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.

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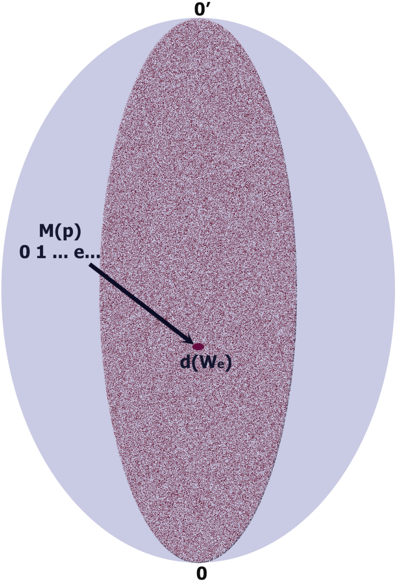
Theorem (Slaman and Woodin)

There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

An indexing of the c.e. degrees



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Extend this result to an indexing φ of the Δ^0_2 Turing degrees.

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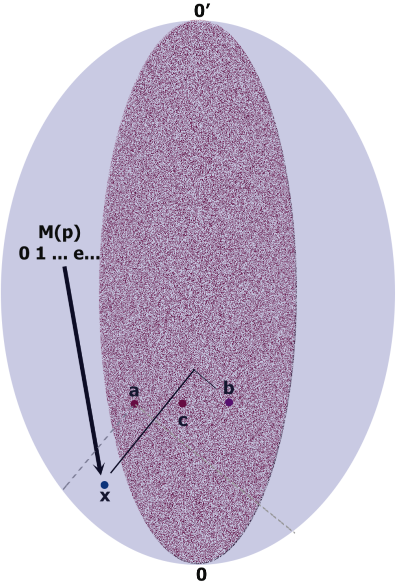

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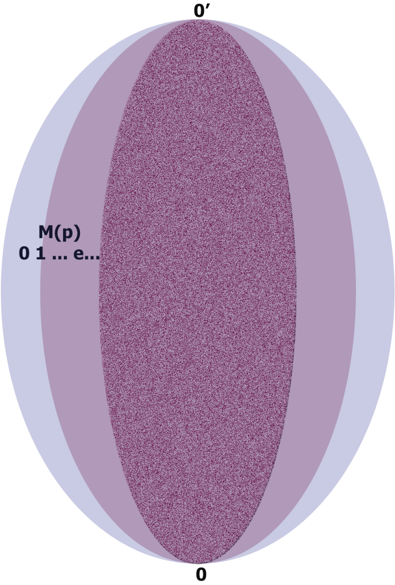

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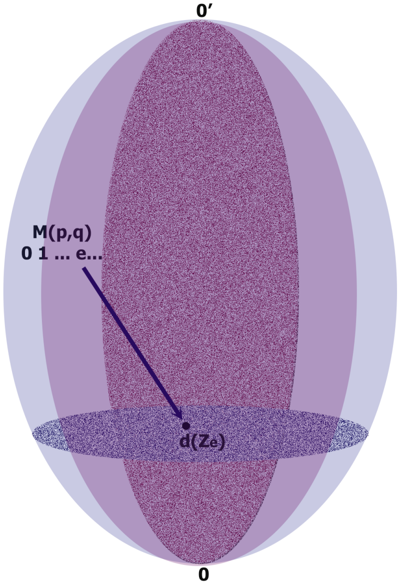

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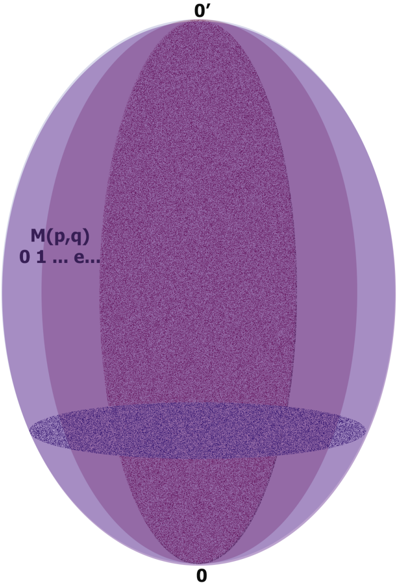

Biinterpretability with parameters

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   If $\mathbf{x}, \mathbf{y} \leq \mathbf{0}'$, $\mathbf{x}' = \mathbf{0}'$ and $\mathbf{y} \not\leq \mathbf{x}$ then there are $\mathbf{g}_i \leq \mathbf{0}'$, c.e. degrees $\mathbf{a}_i$ and Δ^0_2 degrees $\mathbf{c}_i, \mathbf{b}_i \in \mathcal{Z}$ for $i = 1, 2$ such that:

   1. $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$.
   2. $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$.
   3. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

Applications

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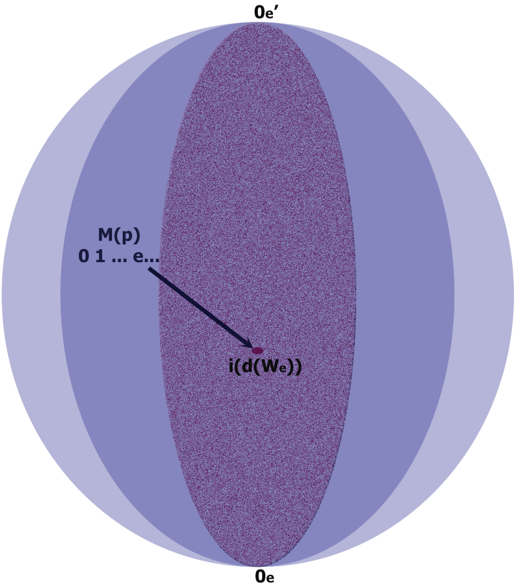
5. *$\mathcal{D}_T(\leq \mathbf{0}')$ is rigid if and only if $\mathcal{D}_T(\leq \mathbf{0}')$ is biinterpretable with first order arithmetic.*

Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, Woodin)

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees.



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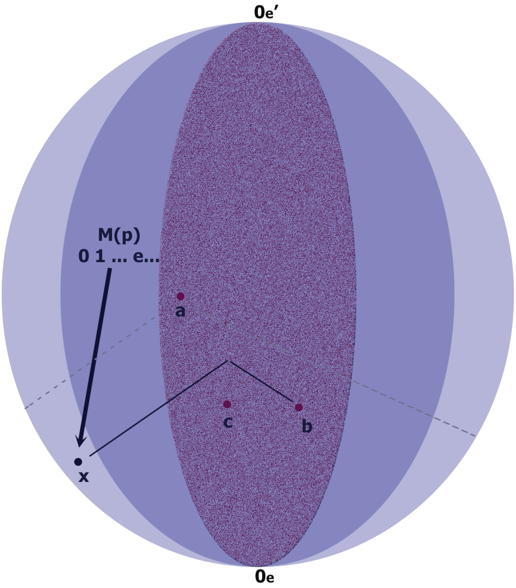
Idea: In the wider context of $\mathcal{D}_e$ we can reach more elements: non-total elements.

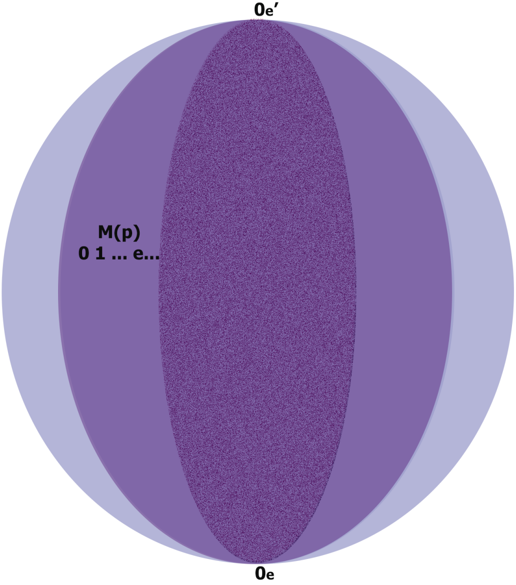


Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, S)

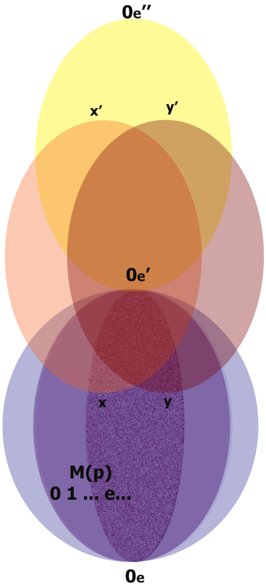
If $\vec{p}$ defines a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total Δ^0_2 enumeration degrees.

Proof flavour:
The image of the c.e. degrees
$\rightarrow$ The low 3-c.e. e-degrees
$\rightarrow$ The low Δ^0_2 e-degrees
$\rightarrow$ The total Δ^0_2 e-degrees



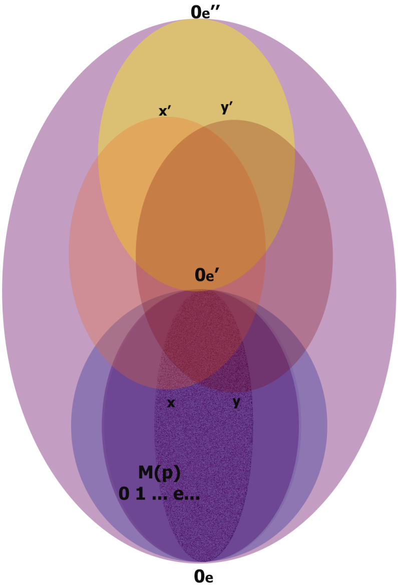
Moving outside the local structure

1. Extend to an indexing of all total degrees that are "c.e. in " and above some total Δ^0_2 enumeration degree.
   - The jump is definable.
   - The image of the relation "c.e. in " is definable.
2. Relativizing the previous theorem extend to an indexing of $\bigcup_{\mathbf{x} \leq_T \mathbf{0}'} \iota([\mathbf{x}, \mathbf{x}'])$.

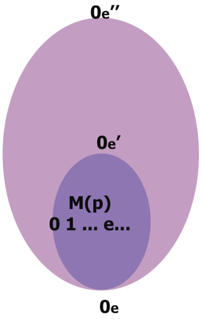


Moving outside the local structure

3. Extend to an indexing of all total degrees below $\mathbf{0}_e''$.



And now we iterate



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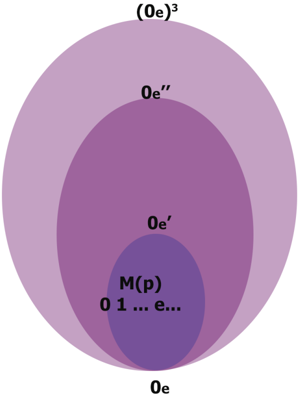

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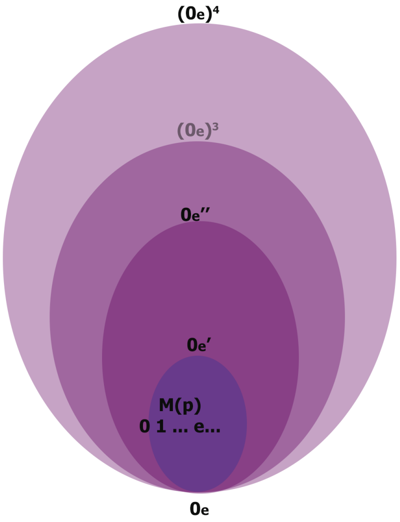

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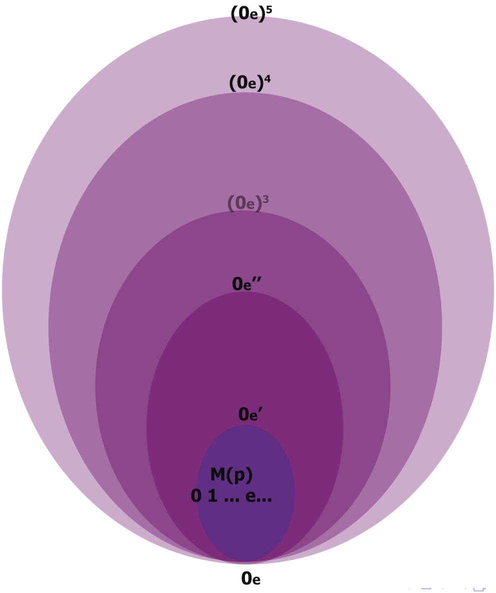

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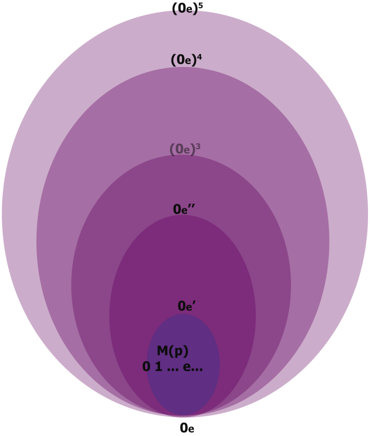

Theorem (Slaman, S)

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} degrees.

Consequences

Theorem (Slaman, S)

1. *There is a finite automorphism base for the enumeration degrees consisting of total Δ^0_2 enumeration degrees.*

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1. *There is a finite automorphism base for the enumeration degrees consisting of total Δ^0_2 enumeration degrees.*
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1. *There is a finite automorphism base for the enumeration degrees consisting of total Δ^0_2 enumeration degrees.*
2. *The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.*
3. *If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.*

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Question

Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?

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The best puzzles are the ones that will never be completely solved.

-Ivan Soskov