Introduction to Harrington's Nonsplitting Theorem
Part One

Mariya I. Soskova

University of Leeds
Department of Pure Mathematics

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The Priority Method

- 1958 Post's Problem and Friedberg and Muchnik's solution
- 1963 Sacks splitting theorem

Theorem

For any c.e. degree b and noncomputable c.e. degree $c \leq 0'$ there exist incomparable c.e. degrees a_0 and a_1 such that $b = a_0 \vee a_1$ and $c \not\leq a_0$ and $c \not\leq a_1$.

- 1963 Sacks Jump Theorem and 1964 Density theorem

Theorem

The c.e. degrees are dense.

- 1975 Lachlan and the priority tree method

Lachlan's Nonsplitting theorem

Theorem

There exist c.e. degrees $a < b$ such that b can not be split over a.

$$\begin{tikzcd}
 & b \arrow[dl, dotted] \arrow[dr, dotted] & \\
d \arrow[dr] & & c \arrow[dl] \\
 & a &
\end{tikzcd}$$

The priority tree method

- D is a set of requirements.
  $D = \{R_1(e)\}_{e<\omega} \cup \dots \cup \{R_k(e)\}_{e<\omega}$
- Aim: Build a set A, satisfying all requirements $R_i(e)$.
- Strategies and outcomes: $R \Rightarrow S_1 \dots S_k \Rightarrow O_1 \dots O_k$
- The tree of strategies: a computable tree T with $Dom(T) \subset O^{<\omega}$ and $Range(T) = D$, such that:
  - Every infinite path $f \subset T$, has $\mathrm{Range}(f) = D$.
  - If $\alpha \in \mathrm{Dom}(T)$ and $T(\alpha) = R$, then $\alpha\hat{\ } o \in \mathrm{Dom}(T)$ for all $o \in O_R$.

The construction

- Finite path through the tree $\delta_s \in Dom(T)$ injuring all strategies to the right.
  $\delta_s^0 = \varnothing, \dots, \delta_s^n, \dots, \delta_s^s$.
- Approximation to the set A
- Approximation to the true path:
  An infinite path $f \subset Dom(T)$, such that
  - $\forall n \exists s_n \forall s > s_n (\delta_s \not<_L f \upharpoonright n)$
  - $\forall n \overset{\infty}{\exists} s (f \upharpoonright n \subseteq \delta_s)$

Harrington's Nonsplitting theorem

Theorem

There exists a c.e. degree a such that $0'$ can not be split over a.

$$
\begin{tikzcd}
 & 0' \arrow[dl, dotted] \arrow[dr, dotted] & \\
d \arrow[dr] & & c \arrow[dl] \\
 & a &
\end{tikzcd}
$$

The Requirements

We will construct the c.e. sets A and E

- $N_\Psi : E \neq \Psi^A$ - hence A is not complete
- $P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U,A} \vee K = \Lambda^{V,A}]$
    1. Assume $A <_T U, V <_T K$ and $U \oplus V \equiv_T K$
    2. Then $E <_T U \oplus V$, hence $E = \Theta^{U,V}$
    3. But $K \equiv_T U \oplus A \equiv_T U$ or $K \equiv_T V \oplus A \equiv_T V$

Definitions

- Use function: Given a computation $\Phi^A(x) = \varepsilon$, then $\phi(x) = \mu n[\Phi^{(A\upharpoonright n)}(x) = \varepsilon]$.
- Length of agreement: Given sets C and D, $l(C, D) = \max(n)[\chi_C \upharpoonright n = \chi_D \upharpoonright n]$, where χ_C and χ_D are the characteristic functions of C and D.

The naive N strategy

- Select a witness x for N_Ψ
- Wait for $\Psi^A(x) = 0$
- Enumerate x in E and restrain each $y \in A \upharpoonright \psi(x)$.

The naive P strategy

- Wait for an expansionary stage at which $l = l(E, \Theta^{U,V})$ is greater than at any previous stage.
- Construct a Turing operator Γ, so that $\Gamma^{U,A} = K$
- For each $z < l$: axiom $\Gamma^{U\upharpoonright(u(z)+1), A\upharpoonright(\gamma(z)+1)}(z) = K(z)$
- If $K(z)$ changes, enumerate $\gamma(z)$ in A and define new axiom

Combining the two strategies

- A-restraint by N_Ψ conflicts the need to rectify Γ
- Choose threshold d and try to achieve $\gamma(n) > \psi(x)$ for all $n \geqslant d$
- Enumerate x in E. Return of $l(E, \Theta^{U,V})$ forces U or V to change.
- U-change: lift the gamma markers and preserve the restraint
- V-change - start over with new witness, implement the backup strategy which insures $\Lambda^{V,A} = K$

The detailed $(P_\ominus, \Gamma)$ strategy

- Working interval (L, R)
- Operator Γ:
  $n \Rightarrow u_s(n), \gamma_s(n) \Rightarrow \Gamma^{U_s \upharpoonright u_s(n), A_s \upharpoonright \gamma_s(n)}(n) = K(n)$
- Conditions:
  1. Correctness of Γ: a new axiom for n only if $\Gamma^{U,A}(n) \uparrow$
  2. Γ must be total: $lim_s u_s(n) < \infty$ and $lim_s(\gamma_s(n)) < \infty$.
- Individual γ marker set A_G and $\gamma(n) \in A_G$.

The detailed (P_Θ, Γ) strategy

$$\begin{tikzcd} & (P_\Theta,\Gamma) \arrow[dl, no head] \arrow[dr, no head] & \\ e & & l \end{tikzcd}$$

1. Wait for an expansionary stage. $(o = l)$
2. Choose $n < l(\Theta^{U,V}, E)$ in turn $(n = 0, 1, \dots)$ and perform following actions:

- Check if the markers are defined, define them if not.
- If $\Gamma^{(U,A)}(n)\uparrow$, define $\gamma(n)$ new and an axiom $\Gamma^{(U\restriction u(n)+1, A\restriction\gamma(n)+1)}(n) = K(n)$.
- If $\Gamma^{(U,A)}(n) \neq K(n)$, then enumerate $\gamma(n)$ in A, define the new axiom $\Gamma^{(U\restriction u(n)+1, A\restriction\gamma(n)+1)}(n) = K(n)$.

The detailed N_Ψ strategy

Initialization

(N_Ψ,Γ)

g-(P_Θ,Λ) h - (N_Ψ,FM) f w

1. Choose a new threshold bigger than any defined until now d such that $L < d < R$.
2. Choose a new witness $x > d$, $x \notin E$.
3. Wait for $x < l(E, \Theta^{U,V})$. $(o = w)$

The detailed N_Ψ strategy

Honestification

- Enumerating x in E produces a change in $(U, V) \upharpoonright \theta(x)$.
- We need a change in $U \upharpoonright u(d)$. Hence first insure that $\theta(x) < u(x)$.
- Problem: $\theta(x)$ grows unbounded
- Solution: then $\Theta^{U,V}(x) \uparrow$ - P_Θ is satisfied, N_Ψ can be satisfied by a simple strategy.

The detailed N_Ψ strategy

Honestification

(N_Ψ, Γ)

g-(P_Θ, Λ) h - (N_Ψ, FM) f w

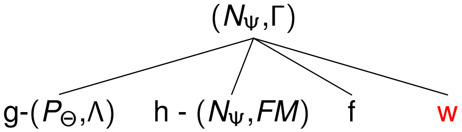
- Check if $\theta(x)$ has grown since the last stage (o = h)
- Check if $u(d) < \theta(x)$, if so:

1. Enumerate $\gamma(d)$ in A. Redefine $u(d) = \theta(x) + 1$.
2. Cancel all markers $u(n)$ for $n > d$ and $n \notin K$.

- Wait for $\Psi^A(x) = 0$ with $\psi(x) < R$ (o = w).

The detailed N_Ψ strategy

Honestification



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- Wait for $\Psi^A(x) = 0$ with $\psi(x) < R$ (o = w).

The detailed N_Ψ strategy

Attack

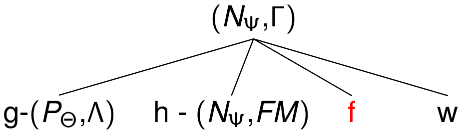
(N_Ψ, Γ)

g-(P_Θ, Λ) h - (N_Ψ, FM) f w

- Enumerate x in E and restrain A on $\psi(x)$. (o = g)
- Wait for the next expansionary stage
- Successful attack - $U \upharpoonright \theta(x)$ changed. (o = f).
- Unsuccessful attack. Enumerate $\gamma(d)$ in A. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage (o = g).

The detailed N_Ψ strategy

Attack



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The detailed N_Ψ strategy

Attack

(N_Ψ, Γ)

g-(P_Θ, Λ) h - (N_Ψ, FM) f w

- Enumerate x in E and restrain A on $\psi(x)$. (o = g)
- Wait for the next expansionary stage
- Successful attack - $U \restriction \theta(x)$ changed. (o = f).
- Unsuccessful attack. Enumerate $\gamma(d)$ in A. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage (o = g).

The outcomes

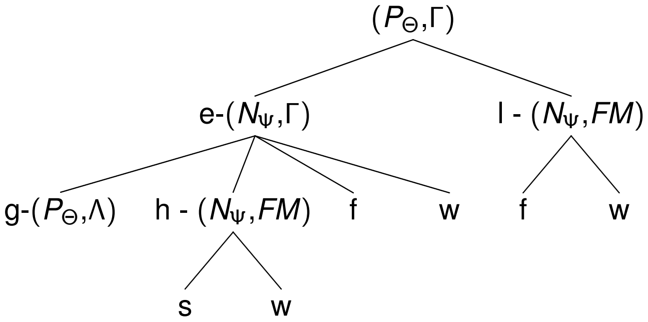
P_Θ

- l - $lim_{sup} l(\Theta^{U,V}, E) < \infty$. Then P_Θ is trivially satisfied. Satisfaction of N_Ψ with simpler strategy working within boundaries (L, ∞).
- e - infinitely many expansionary stages. P_Θ remains intact.

The outcomes

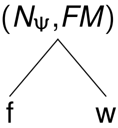
N_Ψ

- w - Infinite wait for $\Psi^A(x) = 0$. N_Ψ is satisfied. P_Θ remains intact. Successive strategies work within boundaries $(L = \gamma(d), R = \infty)$
- f - N_Ψ is satisfied, P_Θ- remains intact. Successive strategies work within boundaries $(L = \gamma(d), R = \infty)$
- h - Infinitely many occurrences of Honestification, precluding an occurrence of Attack. P_Θ is satisfied. Simple strategy for N_Ψ working within boundaries $(x, \gamma(d))$.
- g - Infinitely many unsuccessful attacks. A backup strategy for P_Θ is activated. A copy of N_Ψ works below the backup strategy in boundaries $(L = d, R = x)$.


(P_Θ, Γ)
e-(N_Ψ, Γ)
l - (N_Ψ, FM)
g-(P_Θ, Λ)
h - (N_Ψ, FM)
f
w
f
w
s
w

The backup strategies

(N_Ψ, FM)



- Choose a new witness x, such that $x \notin E$ and $L < x < R$.
- Wait for $\Psi^A(x) = 0$ with $\psi(x) < R$. $(o = w)$
- Enumerate x in E and restrain $A \upharpoonright \psi(x)$. $(o = f)$

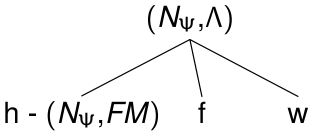
The backup strategies

$(P_\ominus, \Lambda)$

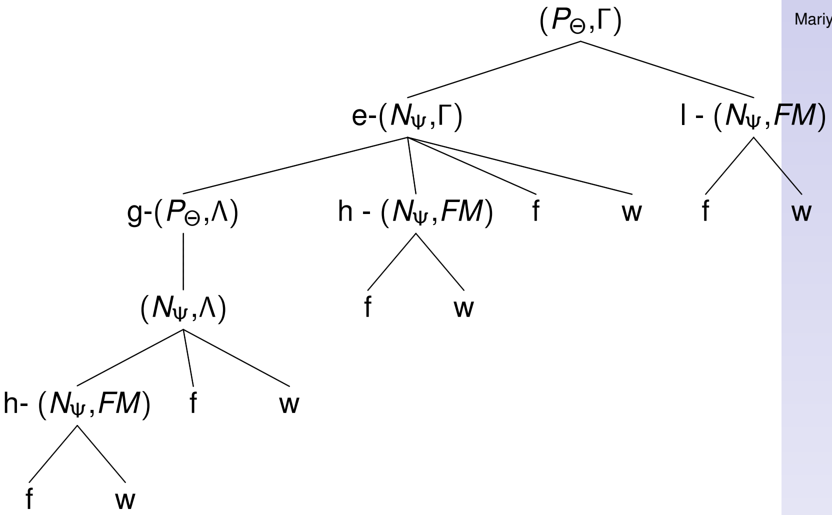
$$\begin{tikzcd}
(P_\ominus,\Lambda) \arrow[d, no head] \\
s
\end{tikzcd}$$

- Construct Λ, so that $\Lambda^{V,A} = K$.
- Markers $v(n)$ and $\lambda(n)$
- Axioms similar to the Γ- markers

N_Ψ working below (P_Θ, Λ)



- Active and Nonactive stages
- Initialization, Honestification, Attack
- The witness $\widehat{x}$ chosen before x.
- Attacks only on nonactive stages, synchronized with attacks by the original copy of N_Ψ.
- Every attack is successful.


(P_Θ, Γ)
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