Definability and interpretability in the Σ^0_2 enumeration degrees

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joint work with H. Ganchev

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Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $A \equiv_e B$ iff $A \leq_e B$ and $B \leq_e A$.
- $d_e(A) = [A]_{\equiv_e}$ and $D_e = \{d_e(A) \mid A \subseteq \mathbb{N}\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

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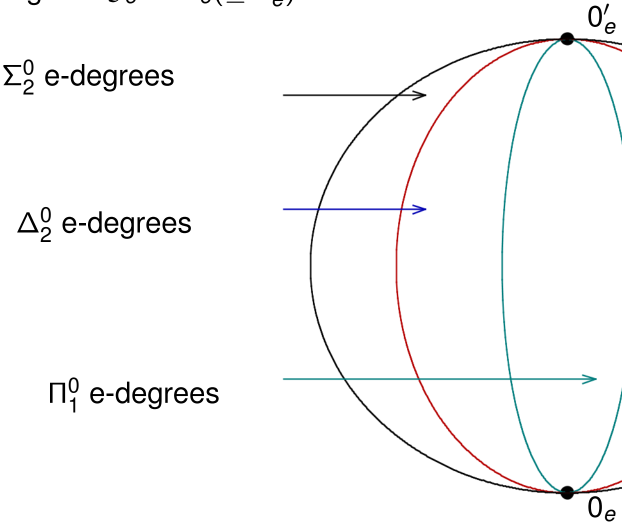
Preliminaries: The local structure

The jump operation gives rise to the local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$.



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The local structure $\mathcal{G}_e$ can be partitioned into classes with respect to the jump hierarchy:

Definition

A degree $\mathbf{a} \in \mathcal{G}_e$ is low if $\mathbf{a}' = \mathbf{0}'_e$.

Or in terms of its relation ship to the Turing degrees.

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

The sub structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

- Every low e-degree is Δ^0_2.
- Every total e-degree in $\mathcal{G}_e$ is Δ^0_2.
- There are properly Σ^0_2 degrees.

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Characterizing the theory of these structures

Theorem (Slaman and Woodin)

The theory of $\mathcal{D}_e$ is computably isomorphic to the theory of second order arithmetic.
The theory of $\mathcal{G}_e$ is undecidable.

Theorem (Kent)

The theory of the Δ^0_2 enumeration degrees is computably isomorphic to the theory of first order arithmetic.

Question

Is the theory of $\mathcal{G}_e$ computably isomorphic to first order arithmetic?

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The general plan: Coding standard models of arithmetic

Given a sentence in the langauge of true arithmetic φ we want to be able to computably translate it into a sentence φ_e in the langauge of the $\mathcal{G}_e$ so that:

$$\langle \mathbb{N}, +, * \rangle \vDash \varphi \text{ iff } \mathcal{G}_e \vDash \varphi_e$$

I Represent $\langle \mathbb{N}, +, * \rangle$ as a partial order (PO).

II Embed this partial order in $\mathcal{G}_e$ and code it with a finite number of parameters.

III Find a first order condition on the parameters, which ensures that they code a SMA.

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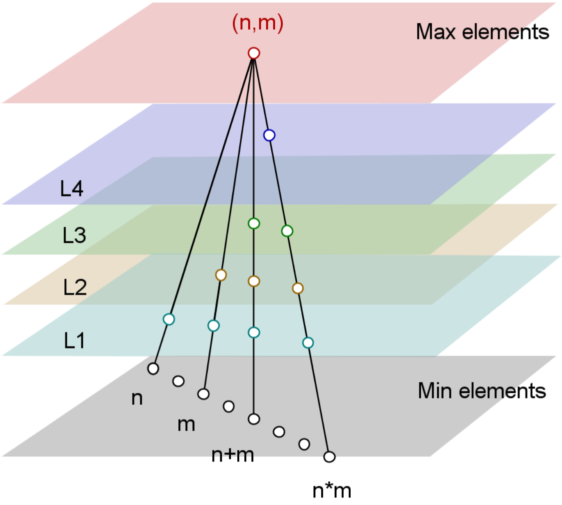
A special type of partial order

We can represent an SMA $\langle \mathbb{N}, +, * \rangle$ as follows:



First tool: Coding antichains

$$\varphi_{SW}(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}) \iff \boldsymbol{x} \leq \boldsymbol{a} \text{ is a minimal solution to}$$

$$\boldsymbol{x} \neq (\boldsymbol{x} \vee \boldsymbol{p}) \wedge (\boldsymbol{x} \vee \boldsymbol{q}).$$

Theorem (Slaman, Woodin)

Let $\{X_i \mid i \in \mathbb{N}\}$ be a system of incomparable sets uniformly enumeration reducible to a low set A with degree $\mathbf{a}$. There are Σ^0_2 e-degrees $\mathbf{p}$ and $\mathbf{q}$, such that for arbitrary Σ^0_2 degree $\mathbf{x}$

$$\mathcal{G}_e \models \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}, \mathbf{q}) \iff \exists i [X_i \in \mathbf{x}].$$

Goal: Embed the PO so that each level is *well presented*.

Second tool: $\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A and B be a pair sets of natural numbers. The pair (A, B) is a $\mathcal{K}$-pair (e-ideal) if there exists a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.

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$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

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An order theoretic characterization of $\mathcal{K}$-pairs

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

A pair of degrees $(\mathbf{a}, \mathbf{b})$ will be called a $\mathcal{K}$-pair if and only if there are representatives $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that (A, B) is a $\mathcal{K}$-pair.

Properties of $\mathcal{K}$-pairs

Theorem (Kallimulin)

1. *If* $(\mathbf{a}, \mathbf{b})$ *are a nontrivial* Σ^0_2 $\mathcal{K}$*-pair then* $\mathbf{a}$ *and* $\mathbf{b}$ *are low and do not bound any total degree.*
2. *Every nontrivial* $\mathcal{K}$*-pair is a minimal pair.*
3. *Every nonzero* Δ^0_2 *enumeration degree bounds a* $\mathcal{K}$*-pair.*
4. *The set of degrees* $\mathbf{b}$ *which form a* $\mathcal{K}$*-pair with a fixed degree* $\mathbf{a}$ *is an ideal.*

$\mathcal{K}$-systems

Definition

We shall say that a system of nonzero degrees $\{\mathbf{a}_i \mid i \in I\}$ ($|I| \geq 2$) is a $\mathcal{K}$-system, if $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$ for each $i, j \in I$, such that $i \neq j$.

- Every $\mathcal{K}$-system is an antichain.
- If $\{\mathbf{a}_i \mid i \in I\}$ is a $\mathcal{K}$-system and $i_1 \neq i_2 \in I$ then $\{\mathbf{a}_{i_1} \vee \mathbf{a}_{i_2}\} \cup \{\mathbf{a}_i \mid i \in I, i \neq i_1, i_2\}$ is a $\mathcal{K}$-system .

Theorem

Let A be a Δ^0_2 non-c.e. set. There is a sequence $\{A_i\}_{i<\omega}$ uniformly enumeration reducible to A such that $\{d_e(A_i)\}_{i<\omega}$ is a $\mathcal{K}$-system.

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Coding an SMA below any half of a $\mathcal{K}$-pair

Construction:

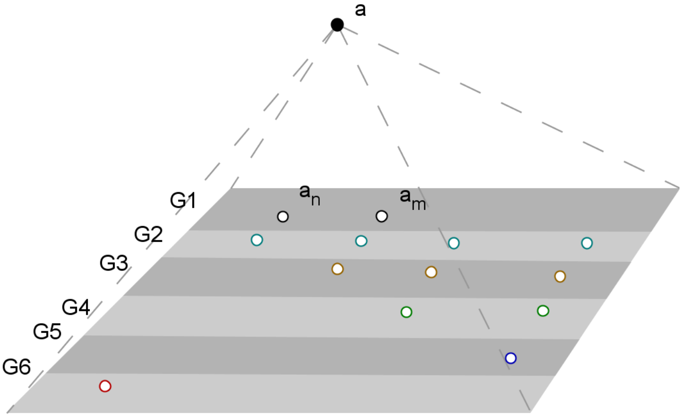
Let $\mathbf{a} = d_e(A)$ be half of a nontrivial $\mathcal{K}$-pair. (Hence a low nonzero Δ^0_2 enumeration degree.)

Let $\{A_i\}_{i<\omega}$ be the uniformly e-reducible to A sequence whose degrees $\{\mathbf{a}_i\}_{i<\omega}$ form a $\mathcal{K}$-system. This is a *well presented system*.

We computably divide the system $\{\mathbf{a}_i\}_{i<\omega}$ into six infinite groups.

Coding an SMA below any half of a $\mathcal{K}$-pair

To every pair of elements from G1 we assign 4 unique elements of G2, 3 of G3, 2 of G4 and 1 of each G5 and G6.



K-system

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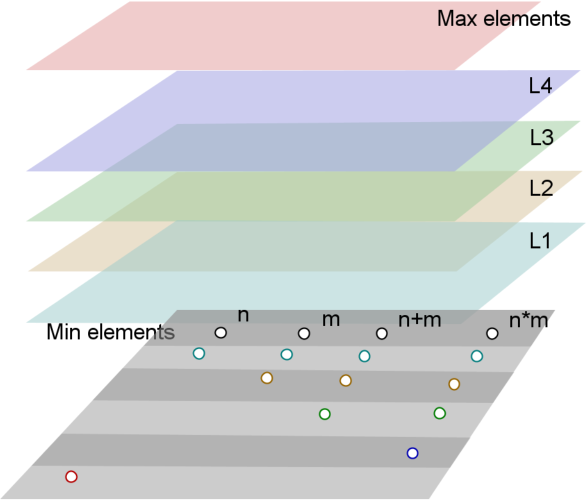

The elements of G1 will represent the natural numbers. There are parameters $\mathbf{p}_0$ and $\mathbf{q}_0$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_0, \mathbf{q}_0)$ defines them.

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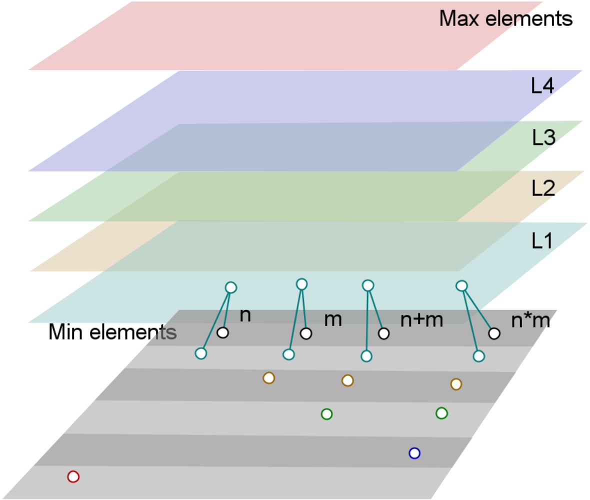

L1 is constructed from lub's of elements from G1 and G2. There are parameters $\mathbf{p}_1$ and $\mathbf{q}_1$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_1, \mathbf{q}_1)$ defines them.

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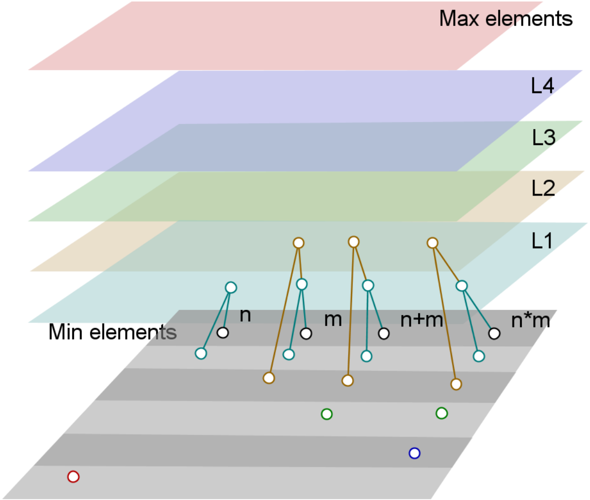
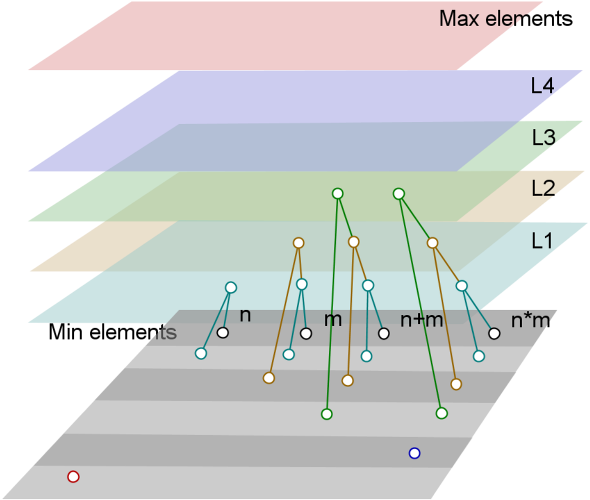


L2 is constructed from lub's of elements from L1 and G3. There are parameters $\mathbf{p}_2$ and $\mathbf{q}_2$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_2, \mathbf{q}_2)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair



L3 is constructed from lub's of elements from L2 and G4. There are parameters $\mathbf{p}_3$ and $\mathbf{q}_3$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_3, \mathbf{q}_3)$ defines them.

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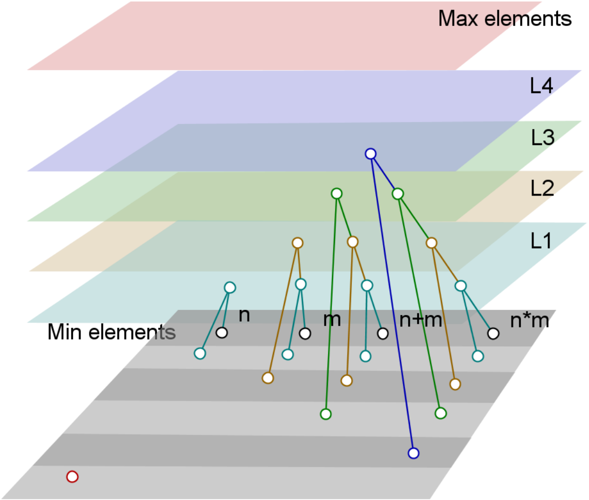
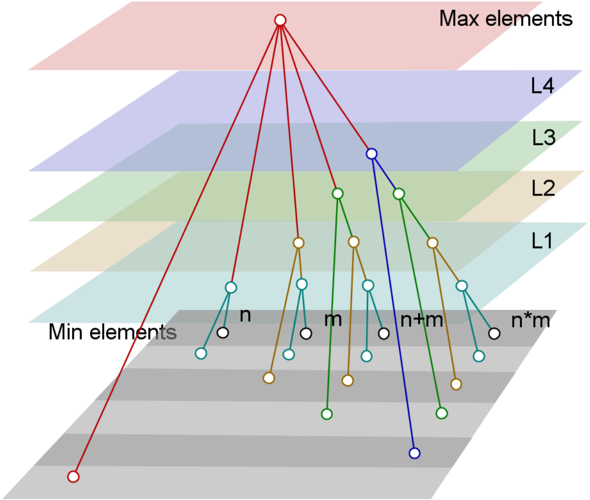


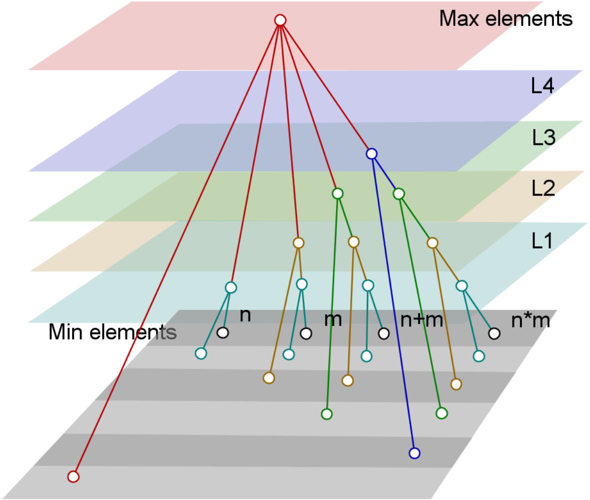
L4 is constructed from lub's of elements from L3 and G5. There are parameters $\mathbf{p}_4$ and $\mathbf{q}_4$ such that $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_4, \mathbf{q}_4)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair



Finally the maximal elements are constructed from lub's of elements from L1, L2, L3, L4 and G6. $\varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_5, \mathbf{q}_5)$ defines them.

Coding an SMA below any half of a $\mathcal{K}$-pair



So the parameters $\mathbf{a}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$ $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$, $\mathbf{q_5}$ code a partial order, which represents a standard model of arithmetic $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$.

The other direction

Given parameters $\mathbf{a}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{p_2}$ $\mathbf{p_3}$, $\mathbf{p_4}$, $\mathbf{p_5}$, $\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, $\mathbf{q_4}$, $\mathbf{q_5}$, let $PO = \{\mathbf{x} \mid \varphi_{SW}(\mathbf{x}, \mathbf{a}, \mathbf{p}_i, \mathbf{q}_i)$ for some $i = 0, 1, 2, 3, 4, 5\}$.
We can define a first order condition $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ so that the partial order $(PO, \leq)$ satisfies:

(M1) Every element belongs to one of six levels in the PO.

(M2) For every pair of minimal elements there exists a unique maximal element above them at distance 1 from the first and 2 from the second.

(M3) For every maximal element m there are 4 unique minimal elements below it, such that the first one is at distance 1 from m, the second is at distance 2, the third at distance 3 and the fourth at distance 4 from m.

The other direction

If $(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ satisfy $M1, M2, M3$ then we have definable relations which represent two binary operations:

- R_+ The relation $R_+(x, y, z) =_{def} min(x) \& \min(y) \& \min(z) \& \exists m(\max(m) \& x <_1 m \;\&\; y <_2 m \;\&\; z <_3 m)$ defines an operation $+$;
- R_* The relation $R_*(x, y, z) =_{def} min(x) \& \min(y) \& \min(z) \& \exists m(\max(m) \& x <_1 m \;\&\; y <_2 m \;\&\; z <_4 m)$ defines an operation $*$;

Then $(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ codes a structure defined as $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}) = \langle \{x \in PO \mid \min(x)\}, +, * \rangle$.

We add requirements to ST_0 which ensure that $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is a model of arithmetic which contains a standard part.

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Isolating parameters which code SMA'a

Suppose that we can ask additionally that $ST_0(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ ensures:

- **a** is half of a nontrivial $\mathcal{K}$-pair;
- The minimal elements in *PO* form a $\mathcal{K}$-system.

Let **b** be such that **a** and **b** are a $\mathcal{K}$-pair.

If we ask additionally that the model coded bolew **a** is embedded in all models coded below **b**, then $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ will be embedded into a SMA and hence will be itself a SMA.

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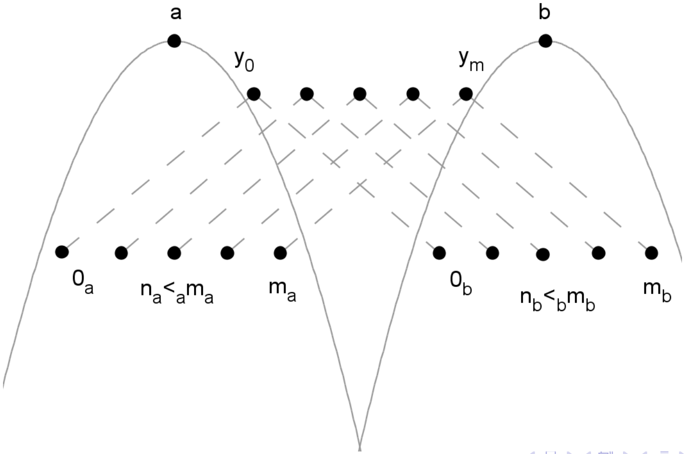
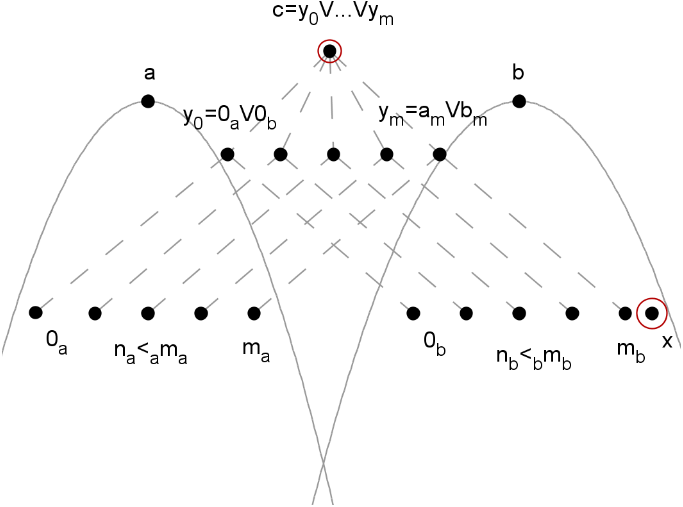
Comparison maps

For every model $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ we ask that $\mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ holds: $\forall m_a \in \mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ there is an $m_b \in \mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ and an antichain $(y_0, y_1, \ldots y_m)$ coded by parameters $\mathbf{c}$, $\mathbf{p}''$ and $\mathbf{q}''$ such that:



Comparison maps

If $\mathfrak{A}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}})$ is an SMA then for every $\mathfrak{A}(\mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ the condition $\mathcal{M}(\mathbf{a}, \overline{\mathbf{p}}, \overline{\mathbf{q}}, \mathbf{b}, \overline{\mathbf{p}}', \overline{\mathbf{q}}')$ is true.



SMA condition

If the property "$\mathbf{x}$ and $\mathbf{y}$ form a $\mathcal{K}$-pair" is first order definable in the Σ^0_2 e-degrees by the formula $\mathcal{LK}(\mathbf{x},\mathbf{y})$ then:

Theorem

There are first order conditions ST_0 and $\mathcal{M}$ such that if $\mathbf{a}$, $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ satisfy:

$$ST_0(\mathbf{a},\overline{\mathbf{p}},\overline{\mathbf{q}})$$

and

$$\exists \mathbf{b}(\mathcal{LK}(\mathbf{a},\mathbf{b}) \ \& \ \forall \overline{\mathbf{p}'}, \forall \overline{\mathbf{q}'}[ST_0(\mathbf{b},\overline{\mathbf{p}}',\overline{\mathbf{q}}') \Longrightarrow \mathcal{M}(\mathbf{a},\overline{\mathbf{p}},\overline{\mathbf{q}},\mathbf{b},\overline{\mathbf{p}}',\overline{\mathbf{q}}')]$$

then $\mathfrak{A}(\mathbf{a},\overline{\mathbf{p}},\overline{\mathbf{q}})$ is a standard model of arithmetic.

An order theoretic characterization of $\mathcal{K}$-pairs

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_{\mathbf{e}})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to check that:

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Definability of $\mathcal{K}$-pairs

Theorem (Kalimullin)

If (A, B) is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C)) \neq d_e(C)$$

- If **a** and **b** are Δ^0_2 then C is also Δ^0_2 and $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures "**a** and **b** are a true $\mathcal{K}$-pair".
- If **a** and **b** are properly Σ^0_2 then C is at best Δ^0_3. So it is possible that there is a fake $\mathcal{K}$-pair **a** and **b** such that

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg\mathcal{K}(\mathbf{a}, \mathbf{b})$$

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Cupping properties

Definition

A Σ_2^0 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ_2^0 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

Proposition (The $\mathcal{K}$-cupping property)

Let $\mathbf{a}$ *and* $\mathbf{b}$ *are* Σ_2^0 *degrees such that* $\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$.
If $\mathbf{c}$ *is a* Σ_2^0 *degree, such that* $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'_e$ *then* $\mathbf{a} \leq \mathbf{c}$.

Proof:

$$\mathbf{c} = (\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \vee \mathbf{c}) \wedge \mathbf{0}'_e = \mathbf{a} \vee \mathbf{c}$$

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Cupping properties

Theorem (S,Wu)

Every nonzero Δ^0_2 enumeration degree $\mathbf{a}$ is low-cuppable, i.e. there is a low $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable nonzero Σ^0_2 enumeration degrees.

Question

Are all cuppable degrees also low-cuppable?

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Cupping $\mathbf{0}'_e$-splittings

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Proof:
Uses a construction very similar to the construction of a non-splitting enumeration degree.

A non-splitting theorem

Theorem (S)

There is a degree $\mathbf{a} < \mathbf{0}'_e$ *such that no pair of incomplete* Σ^0_2 *degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *above* $\mathbf{a}$ *splits* $\mathbf{0}'_e$.

We build a Σ^0_2 set A and an auxiliary Π^0_1 set E so that:

$$\mathcal{N}_\Phi : \Phi(A) \neq E$$

$$\mathcal{P}_{\Theta,U,V} : \Theta(U \oplus V) = E \Rightarrow \exists \Gamma, \Lambda(\Gamma(U \oplus A) = \overline{K} \vee \Lambda(V \oplus A) = \overline{K})$$

Corollary

There exists an incomplete Σ^0_2 *e-degree* $\mathbf{a}$, *such that for every pair of* Σ^0_2 *enumeration degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *with* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *either* $\mathbf{u} \vee \mathbf{a} = \mathbf{0}'_e$ *or* $\mathbf{v} \vee \mathbf{a} = \mathbf{0}'_e$

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Cupping $\mathbf{0}'_e$-splittings

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Proof:
Fix U, V such that $U \oplus V \equiv_e \overline{K}$.

We construct an auxiliary Π^0_1 set E and find an e-operator Θ such that $\Theta(U \oplus V) = E$.

First we try to construct a 1-generic Δ^0_2 set A such that $A \oplus U \equiv_e \overline{K}$.

If this plan fails we have acquired sufficient information to construct a 1-generic Δ^0_2 set B such that $B \oplus V \equiv_e \overline{K}$.

Cupping $\mathbf{0}'_e$-splittings

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Proof:
Fix U, V such that $U \oplus V \equiv_e \overline{K}$.

We construct an auxiliary Π^0_1 set E and find an e-operator Θ such that $\Theta(U \oplus V) = E$.

First we try to construct a 1-generic Δ^0_2 set A such that $A \oplus U \equiv_e \overline{K}$.

If this plan fails we have acquired sufficient information to construct a 1-generic Δ^0_2 set B such that $B \oplus V \equiv_e \overline{K}$.

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Defining a true $\mathcal{K}$-pair

Corollary

If $\mathbf{a}, \mathbf{b}$ *are nonzero* Σ^0_2 *degrees such that* $\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$ *and* $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$ *then* $(\mathbf{a}, \mathbf{b})$ *is a true* $\mathcal{K}$*-pair.*

Proof:

By the previous theorem **a** is low-cuppable or **b** is low-cuppable.

b is low-cuppable $\Rightarrow$ **a** is low $\Rightarrow$ **a** is Δ^0_2 $\Rightarrow$
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In either case both **a** and **b** are Δ^0_2 and hence $\mathcal{K}(\mathbf{a}, \mathbf{b})$ ensures that they form a true $\mathcal{K}$-pair.

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A locally definable set of low degrees

Kallimulin has proved that there is a true nontrivial $\mathcal{K}$-pair $(\mathbf{a}, \mathbf{b})$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'$, so:

Theorem

The formula

$$\mathcal{L}(\mathbf{a}) \Leftrightarrow \mathbf{a} > \mathbf{0}_e \ \& \ (\exists \mathbf{b} > \mathbf{0}_e)(\mathcal{K}(\mathbf{a}, \mathbf{b}) \& (\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e'))$$

defines in $\mathcal{G}_e$ a nonempty set of true halves of nontrivial $\mathcal{K}$-pairs.

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Defining $\mathcal{K}$-pairs

Denote by $\mathcal{L}$ the definable set of all degrees **a**, such that

$$\mathcal{G}_e \models \mathcal{L}(\mathbf{a}).$$

Definition

x is downwards properly Σ^0_2 every $\mathbf{y} \in (\mathbf{0}_e, \mathbf{x}]$ is properly Σ^0_2.

Example

If **x** is not low cuppable then it is downwards properly Σ^0_2.

If $(\mathbf{a}, \mathbf{b})$ is a fake $\mathcal{K}$-pair then i.e.:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg\mathcal{K}(\mathbf{a}, \mathbf{b})$$

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For every nonzero Δ^0_2 degree $\mathbf{b}$ there is a nontrivial $\mathcal{K}$-pair, $(\mathbf{c}, \mathbf{d})$, such that

$$\mathbf{b} \vee \mathbf{c} = \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e.$$

Hence if $(\mathbf{a}, \mathbf{b})$ is a true $\mathcal{K}$-pair of Σ^0_2 e-degrees (hence low and Δ^0_2) we apply this theorem to get a $\mathcal{K}$-pair $(\mathbf{c}, \mathbf{d})$ such that:

- $\mathbf{b} \vee \mathbf{c} = \mathbf{0}'_e$ and hence $\mathbf{a} \leq \mathbf{c}$ by:

$$\mathbf{c} = (\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = (\mathbf{a} \vee \mathbf{c}) \wedge \mathbf{0}'_e = (\mathbf{a} \vee \mathbf{c}).$$

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- If $(\mathbf{a}, \mathbf{b})$ is a fake $\mathcal{K}$-pair then $\mathbf{a}$ and $\mathbf{b}$ are incomparable with all members of $\mathcal{L}$.
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Let $\mathcal{LK}(\mathbf{a}, \mathbf{b}) \Leftrightarrow \mathcal{K}(\mathbf{a}, \mathbf{b})\ \&\ \mathbf{a} > \mathbf{0}_e\ \&\ \mathbf{b} > \mathbf{0}_e\ \&\ \exists \mathbf{c}(\mathbf{c} \geq \mathbf{a}\ \&\ \mathcal{L}(\mathbf{c}))$

Corollary

A pair of Σ^0_2 enumeration degrees $\mathbf{a}$, $\mathbf{b}$ forms a nontrivial $\mathcal{K}$-pair if and only if:

$$\mathcal{G}_e \models \mathcal{LK}(\mathbf{a}, \mathbf{b}).$$

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Two additional application

Defining the downwards properly Σ^0_2 degrees

If **a** is a nonzero Δ^0_2 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** bounds a nonzero Δ^0_2 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ^0_2 degree, then it bounds no $\mathcal{K}$-pair.

Theorem

A degree **a** *is downwards properly* Σ^0_2 *if and only if:*

$$\mathcal{G}_e \models \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{a} \;\&\; \mathbf{c} \leq \mathbf{a}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

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Two additional application

Defining the upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Example

1. If $\mathbf{a}$ is a non-splitting degree then it is upwards properly Σ^0_2.
2. (Cooper, Copestake) There is a properly Σ^0_2 degree that is incomaparable with all nonzero incomplete Δ^0_2 degrees.
3. (Bereznyuk, Coles, Sorbi) For every enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there exists an upwards properly Σ^0_2 degree $\mathbf{c} \geq \mathbf{a}$.

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Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

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Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

Two additional application

The least upper bound of every $\mathcal{K}$-pair is a Δ_2^0 degree.

So a degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if no element above it other than $\mathbf{0}_e'$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Theorem

A degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if:

$$\mathcal{G}_e \models \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{a} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e').$$

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Theorem

A degree **a** *is upwards properly* Σ_2^0 *if and only if:*

$$\mathcal{G}_e \models \forall \mathbf{c}, \mathbf{d} (\mathcal{LK}(\mathbf{c}, \mathbf{d}) \;\&\; \mathbf{a} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e').$$

The end

Thank you!