Embedding Partial Orderings in Degree Structures

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Embeddability in the Turing degrees

Definition (Kleene, Post 1954)
A sequence of sets \( \{A_i\}_{i<\omega} \) is called computably independent if for every \( i \):

\[
A_i \not\leq_T \bigoplus_{j \neq i} A_j.
\]

- Mostowski 1938: There exists a computable partial ordering \( \mathcal{R} = \langle \mathbb{N}, \leq \rangle \) in which every countable partial ordering can be embedded.
- Sacks 1963: The existence of a computably independent sequence of sets gives an embedding of any computable partial ordering.
Embeddability in the Turing degrees

Localizing independent sequences of sets:

Corollary

*Every countable partial ordering can be embedded*

1. Kleene and Post 1954: in the Turing degrees, even in the $\Delta^0_2$ Turing degrees.
3. Robinson 1971: densely in the c.e. Turing degrees, i.e. in any nonempty interval of c.e. Turing degrees.
The enumeration degrees

- Case 1971: Any countable partial ordering can be embedded in the e-degrees below the degree of any generic function.
- Copestake 1988: below any 1-generic enumeration degree.
- Lageman 1972: below any nonzero $\Delta^0_2$ e-degree.
- Bianchini 2000: densely in the $\Sigma^0_2$ enumeration degrees.

**Theorem**

*Let $b < a$ be enumeration degrees such that $a$ contains a member with a good approximation. Then every countable partial ordering can be embedded in the interval $[b, a]$.***

**Method:** e-independent sequences of sets.
The general picture
The $\omega$ e-degrees: Basic definitions

Let $S$ be the set of all sequences of sets of natural numbers.

**Definition**

Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets of natural numbers and $V$ be an e-operator. The result of applying the enumeration operator $V$ to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- **Enumeration reducibility**: for every $n$ we have that $A_n \leq_e B_n$ via, say, $\Gamma_n$.
- **Uniformity**: the sequence $\{\Gamma_n\}_{n<\omega}$ is uniform.
Basic definitions

With every member $\mathcal{A} \in S$ we connect a *jump sequence* $P(\mathcal{A})$.

**Definition**

The *jump sequence* of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:

- $P_0(\mathcal{A}) = \mathcal{A}_0$.
- $P_{n+1}(\mathcal{A}) = \mathcal{A}_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$. 
The $\omega$-enumeration degrees

Let $A, B \in S$.

Definition

- $\omega$-enumeration reducibility: $A \leq_\omega B$, if $A \leq_e P(B)$.
- $A \equiv_\omega B$ iff $A \leq_\omega B$ and $B \leq_\omega A$.
- $d_\omega(A) = \{B \mid A \equiv_\omega B\}$
- $\mathcal{D}_\omega$ is an upper semi-lattice with jump operation and least element $0_\omega = d_\omega(((\emptyset, \emptyset, \ldots)))$. 
The e-degrees as a substructure

\[ \langle \mathcal{D}_e, \leq_e, \lor, ' \rangle \] can be embedded in \[ \langle \mathcal{D}_\omega, \leq_\omega, \lor, ' \rangle \] via the embedding \( \kappa \) defined as follows:

\[ \kappa(d_e(A)) = d_\omega((A, \emptyset, \emptyset, \ldots)) = d_\omega((A, A', A'', \ldots)). \]

Theorem (Soskov, Ganchev)

- The structure \( \mathcal{D}_1 = \kappa(\mathcal{D}_e) \) is first order definable in \( \mathcal{D}_\omega \).
- The structures \( \mathcal{D}_e \) and \( \mathcal{D}_\omega \) with jump operation have isomorphic automorphism groups.
The embeddability question

Consider the structure $\mathcal{L}_\omega$ consisting of all degrees reducible to $0'_\omega = d_\omega(((\emptyset', \emptyset'', \emptyset''', \ldots))$ also called the $\Sigma^0_2 \omega$-enumeration degrees.

Theorem (Soskov)

The structure $\mathcal{L}_\omega$ is dense.

Theorem

Let $b <_\omega a \leq_\omega 0'_\omega$. Every countable partial ordering can be embedded in the interval $[b, a]$.

Proof techniques: Independent sequences of sequences sets, embeddability results in the enumeration degrees, good approximations for sequences, recursion theorem.
The c.e. degrees modulo iterated jump

Definition (Jockusch, Lerman, Soare and Solovay)
Let \( a \) and \( b \) be c.e. Turing degrees. \( a \sim \infty b \) iff there exists a natural number \( n \) such that \( a^n = b^n \).

- Induced degree structure \( \mathcal{R}/\sim_\infty \) with \( [a]_{\sim_\infty} \leq [b]_{\sim_\infty} \) if and only if there exists a natural number \( n \) such that \( a^n \leq_T b^n \).
- Least element \( L = \bigcup_{n<\omega} L_n \).
- Greatest element \( H = \bigcup_{n<\omega} H_n \).
- \( \mathcal{R}/\sim_\infty \) is a dense structure.
- Lempp: There is a splitting of the highest \( \infty \)-degree and a minimal pair of \( \infty \)-degrees.
Starting with other classes of degree

- $\mathcal{L}_T/\sim_\infty$: the $\Delta^0_2$ Turing degrees modulo iterated jump. Shoenfield, Sacks: The range of the jump operator restricted to the c.e. Turing degrees coincides with the range of the jump operator restricted to the $\Delta^0_2$ Turing degrees. It is namely the set of all Turing degrees c.e. in and above $0'$. Hence:

\[ \mathcal{L}_T/\sim_\infty \cong \mathcal{R}/\sim_\infty. \]

- $\mathcal{L}_e/\sim_\infty$: the $\Sigma^0_2$ e-degrees modulo iterated jump. McEvoy: The range of the enumeration jump operator restricted to the $\Sigma^0_2$-enumeration degrees coincides with the range of the enumeration jump operator restricted to the $\Pi^0_1$ enumeration degrees. Hence:

\[ \mathcal{R}/\sim_\infty \cong (\Pi^0_1 \text{ e-degrees})/\sim_\infty \cong \mathcal{L}_e/\sim_\infty. \]
The $\omega$-enumeration degrees modulo iterated jump

Consider $\mathcal{L}_\omega / \sim_\infty$. $\mathcal{R} / \sim_\infty$ embeds in $\mathcal{L}_\omega / \sim_\infty$.

\[ \mathcal{R} \subseteq \mathcal{L}_T \hookrightarrow \iota(\mathcal{L}_T) = \text{Tot} \subseteq \mathcal{L}_e \hookrightarrow \kappa(\mathcal{L}_e) = \mathcal{D}_1 \subseteq \mathcal{L}_\omega \]

Lemma

Let $a$ and $b$ be two $\Sigma^0_2$ $\omega$-enumeration degrees.

1. If $a \leq_\omega b$ then $[a]_{\sim_\infty} \leq [b]_{\sim_\infty}$.
   
   Proof idea: Monotonicity of the jump.

2. If $[a]_{\sim_\infty} \leq [b]_{\sim_\infty}$ then there is a representative $c \in [a]_{\sim_\infty}$ such that $c \leq_\omega b$.
   
   Proof idea: Existence of least jump inverts.
The almost degrees

Definition
Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets of natural numbers. We shall say that the sequence $\mathcal{B} = \{B_n\}_{n<\omega}$ is almost-$\mathcal{A}$ if for every $n$ we have that $P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$. If $\mathcal{A}$ is almost-$\mathcal{B}$ then we shall say that $d_\omega(\mathcal{A})$ is almost-$d_\omega(\mathcal{B})$.

Properties:
- If $a <_\omega b$ and $a <_\infty b$ then there exists an almost-$a$ degree $z$ such that $a <_\omega z \leq_\omega b$.
- $\omega$-reducibility and $\infty$-reducibility coincide when restricted to the almost $a$-degrees.
The final result

Theorem

1. $\mathcal{L}_\omega / \sim_\infty$ properly extends $\mathcal{R} / \sim_\infty$.

2. Every countable partial ordering can be embedded densely in $\mathcal{L}_\omega / \sim_\infty$. 
The final result

Theorem

1. $\mathcal{L}_\omega / \sim_\infty$ properly extends $\mathcal{R} / \sim_\infty$.
2. Every countable partial ordering can be embedded densely in $\mathcal{L}_\omega / \sim_\infty$. 
Thank you!