Generalization of the notion of jump sequence of sets for sequences of structures

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A parallel between classical computability theory and effective definability in abstract structures

A close parallel between notions of classical computability theory and of the theory of effective definability in abstract structures:

1. The notion of “c.e. in” corresponds to the notion of Σ_1 definability;
2. The “Σ^0_{n+1} in” sets correspond to the sets definable by means of computable infinitary Σ_{n+1} formulae.

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x \rangle \in W_e \wedge D_v \subseteq Y)).$$

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Proposition

X is c.e. in Y if and only if $X \leq_e Y \oplus \overline{Y} = Y^+$.
X is computable in Y if and only if $X \oplus \overline{X} \leq_e Y \oplus \overline{Y}$.

Abstract structures

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- $f^{-1}(X) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}$ for any $X \subseteq A^a$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$ computes the positive atomic diagram of an isomorphic copy of $\mathfrak{A}$.

We always consider $\mathfrak{A} = (A; R_1, \bar{R}_1 \ldots, R_k, \bar{R}_k)$.

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Definition

For every $X \subseteq A$ and f, g enumerations of A let

$$E_X^{f,g} = \{\langle x, y \rangle \mid f(x) = g(y) \in X\}.$$

Relatively intrinsically c.e. in $\mathfrak{A}$ sets

Definition

A set $R \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ if and only if $f^{-1}(R)$ is c.e. in $f^{-1}(\mathfrak{A})$ for every enumeration f of $\mathfrak{A}$.

Theorem (Ash, Knight, Manasse, Slaman, Chisholm)

A set $R \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ by means of a computable infinitary Σ_1^c formula with parameters.

Sequences of sets of natural numbers

Definition

A sequence of sets of natural numbers $\mathcal{Y} = \{Y_n\}_{n<\omega}$ is *c.e. in* a set $Z \subseteq \mathbb{N}$ if for every n, Y_n is c.e. in $Z^{(n)}$ uniformly in n.

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Theorem (Selman)

Let $X, Y \subseteq \mathbb{N}$.
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Definition

Given a set X of natural numbers and a sequence $\mathcal{Y}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets $Z \subseteq \mathbb{N}$,
$\mathcal{Y}$ is c.e. in Z implies X is Σ^0_{n+1} in Z.

The relation $\leq_n$

Ash presents a characterization of "$\leq_n$" using computable infinitary propositional sentences. Soskov and Kovachev give another characterization in terms of enumeration reducibility.

Definition

Let $\mathcal{X} = \{X_n\}_{n<\omega}$. The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})'_e \oplus X_{n+1}$.

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Theorem (Soskov)

$X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.

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An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.

Denote by $f^{-1}(\vec{\mathfrak{A}})$ the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

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Definition

For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$ i.e. $f^{-1}(R) \leq_e \mathcal{P}_n(f^{-1}(\vec{\mathfrak{A}}))$.

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Theorem (Soskov)

For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure $\mathfrak{M}$, such that for each n, the relatively intrinsically Σ_{n+1} sets in $\mathfrak{M}$ sets coincide with sets $R \leq_n \vec{\mathfrak{A}}$.

The structure $\mathfrak{M}$ is the Marker's extension of the sequence of structures $\vec{\mathfrak{A}}$.

Equivalent structures

Definition

We call two structures $\mathfrak{A}$ and $\mathfrak{B}$ equivalent: $\mathfrak{A} \equiv \mathfrak{B}$ if they have the same relatively intrinsically c.e. subsets of the common part of the domains of $\mathfrak{A}$ and $\mathfrak{B}$.

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Definition

Given a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i<\omega}$ the *n*-th polynomial of $\vec{\mathfrak{A}}$ is a structure $\mathfrak{P}_n(\vec{\mathfrak{A}})$ defined inductively:

- $\mathfrak{P}_0(\vec{\mathfrak{A}}) = \mathfrak{A}_0$,
- $\mathfrak{P}_{n+1}(\vec{\mathfrak{A}}) = \mathfrak{P}_n(\vec{\mathfrak{A}})' \oplus \mathfrak{A}_{n+1}$.

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Theorem

For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure $\mathfrak{M}$ such that for every n we have $\mathfrak{P}_n(\vec{\mathfrak{A}}) \equiv \mathfrak{M}^{(n)}$.

The Moschovakis extension

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$.

- Let $\bar{0} \notin A$. Set $A_0 = A \cup \{\bar{0}\}$.
- Let $\langle ., . \rangle$ be a pairing function: each element of A_0 is not a pair.
- Let $A^\star$ be the least set containing A_0 and closed under $\langle ., . \rangle$.
- $0^* = \bar{0}$, $(n+1)^* = \langle \bar{0}, n^* \rangle$.
  The set of all n^* we denote by N^*.
- The decoding functions: $L(\langle s, t \rangle) = s$ & $R(\langle s, t \rangle) = t$,
  $L(\bar{0}) = R(\bar{0}) = 0^\star$ $(\forall t \in A)[L(t) = R(t) = 1^\star]$.

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  $L(\bar{0}) = R(\bar{0}) = 0^\star$ $(\forall t \in A)[L(t) = R(t) = 1^\star]$.

Definition

The Moschovakis extension of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^\star = (A^\star; A_0, R_1^*, \dots, R_k^*, G_{\langle .,. \rangle}, G_L, G_R).$$

$$R_i^*(t) \iff (\exists a_1 \in A) \dots (\exists a_{r_i} \in A)(t = \langle a_1, \dots, a_{r_i} \rangle \ \& \ R_i(a_1, \dots, a_{r_i})).$$

The set $K^{\mathfrak{A}}$

A new predicate $K^{\mathfrak{A}}$ (analogue of Kleene's set).
For $e, x \in \mathbb{N}$ and finite part τ, let

$$\tau \Vdash F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})}$$

$$\tau \Vdash \neg F_e(x) \leftrightarrow (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x))$$

$$K^{\mathfrak{A}} = \{\langle \delta^*, e^*, x^* \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

$$\mathfrak{A}' = (\mathfrak{A}^\star, K^{\mathfrak{A}}), \;\; \mathfrak{A}^{(n+1)} = (\mathfrak{A}^{(n)})'.$$

Proposition

For every $R \subseteq A$ we have

- *R is relatively intrinsically c.e. on $\mathfrak{A}'$ iff R is relatively intrinsically Σ_2 on $\mathfrak{A}$.*
- *R is relatively intrinsically c.e. on $\mathfrak{A}^{(n)}$ iff R is relatively intrinsically Σ_{n+1} on $\mathfrak{A}$.*

The jump structure $\mathfrak{A}'$

$$\mathfrak{A}' = (\mathfrak{A}^\star, K^{\mathfrak{A}}).$$

Proposition

For every enumeration f of $\mathfrak{A}$ there exists an enumeration g of $\mathfrak{A}'$, such that

1. $g^{-1}(\mathfrak{A}') \leq_{\mathrm{T}} (f^{-1}(\mathfrak{A}))'_T$;
2. $E_A^{f,g}$ *is c.e. in* $(f^{-1}(\mathfrak{A}))'_T$.

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2. $E_A^{f,g}$ *is c.e. in* $g^{-1}(\mathfrak{A}')$.

The *n*th polynomial of a sequence of structures

Definition

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ and $\mathfrak{B} = (B; P_1, \dots, P_m)$ are structures and $A \cap B = \emptyset$. *The join of* $\mathfrak{A}$ *and* $\mathfrak{B}$ we call the structure $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; A, B, R_1, \dots, R_k, P_1, \dots, P_m)$.

Definition

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i \in \omega}$ be a sequence of structures with disjoint domains $A_i \cap A_j = \emptyset$ for $i \neq j$. *The nth polynomial of* $\vec{\mathfrak{A}}$ we call the structure $\mathfrak{P}_n(\vec{\mathfrak{A}})$, defined inductively:

1. $\mathfrak{P}_0(\vec{\mathfrak{A}}) = \mathfrak{A}_0$
2. $\mathfrak{P}_{n+1}(\vec{\mathfrak{A}}) = (\mathfrak{P}_n(\vec{\mathfrak{A}}))' \oplus \mathfrak{A}_{n+1}$.

Our goal is to prove that if $\mathfrak{M}(\vec{\mathfrak{A}})$ is the Marker's extension of the sequence $\vec{\mathfrak{A}}$ then

$$(\forall n \in \mathbb{N})(\mathfrak{M}(\vec{\mathfrak{A}})^{(n)} \equiv \mathfrak{P}_n(\vec{\mathfrak{A}})).$$

The definability in $\mathfrak{P}_n(\vec{\mathfrak{A}})$

If f is an enumeration of $\vec{\mathfrak{A}}$ denote by $f^{-1}(\vec{\mathfrak{A}})$ the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.
Denote by $A_0^n = \bigcup_{i=0}^n A_i$.

Proposition

For every enumeration f of $\vec{\mathfrak{A}}$ and each $n \in \mathbb{N}$ there exists an enumeration g of $\mathfrak{P}_n(\vec{\mathfrak{A}})$ such that:

1. $g^{-1}(\mathfrak{P}_n(\vec{\mathfrak{A}})) \leq_{\mathrm{T}} \mathcal{P}_n(f^{-1}(\vec{\mathfrak{A}}))$,
2. $E_{A_0^n}^{f,g}$ *is c.e. in* $\mathcal{P}_n(f^{-1}(\vec{\mathfrak{A}}))$.

Proposition

For every enumeration g of $\mathfrak{P}_n(\vec{\mathfrak{A}})$ there exists an enumeration f of the set A_0^n such that:

1. $\mathcal{P}_n(f^{-1}(\vec{\mathfrak{A}})) \leq_{\mathrm{T}} g^{-1}(\mathfrak{P}_n(\vec{\mathfrak{A}}))$,
2. $E_{A_0^n}^{g,f}$ *is c.e. in* $g^{-1}(\mathfrak{P}_n(\vec{\mathfrak{A}}))$.

The connection between $\leq_{\mathrm{n}}$ and $\mathfrak{P}_n(\vec{\mathfrak{A}})$

Theorem

Let $n \in \mathbb{N}$ and $R \subseteq \bigcup_{i=0}^{n} A_i$. The following equivalence is true:
R is relatively intrinsically c.e. in $\mathfrak{P}_n(\vec{\mathfrak{A}}) \iff R \leq_{\mathrm{n}} \vec{\mathfrak{A}}$.

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

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The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$.

Fix bijections: $h_0 : R \to X_0$

$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$

$h_n : (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

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If n is even then:

$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(\bar{a}, x_0, \dots x_n)]$.

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Theorem (Soskov)

For each $n \in \mathbb{N}$ and every $R \subseteq A$
$R \leq_{\mathrm{n}} \vec{\mathfrak{A}}$ iff R is relatively intrinsically Σ_{n+1} in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\ldots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \cdots \cup \mathfrak{M}_n(R_{m_n}^n)$.
3. Set $\mathfrak{M}(\vec{\mathfrak{A}})$ to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A.

Theorem (Soskov)

For each $n \in \mathbb{N}$ and every $R \subseteq A$
$R \leq_{\mathrm{n}} \vec{\mathfrak{A}}$ iff R is relatively intrinsically Σ_{n+1} in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Proposition (Vatev)

R is relatively intrinsically Σ_{n+1} in $\mathfrak{M}(\vec{\mathfrak{A}})$ iff R is relatively intrinsically c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})^{(n)}$.

Theorem

$\mathfrak{P}_n(\vec{\mathfrak{A}}) \equiv \mathfrak{M}(\vec{\mathfrak{A}})^{(n)}$ *for every $n \in \mathbb{N}$.*

Strong reducibility of structures

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures and $A \subseteq B$. The structure $\mathfrak{A}$ is *strong reducible* in the structure $\mathfrak{B}$: $\mathfrak{A} \leq \mathfrak{B}$ if the following conditions hold:

1. for each enumeration g of $\mathfrak{B}$ there is an enumeration f of $\mathfrak{A}$, such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{B})$ and
2. the set $E_A^{g,f}$ is c.e. in $g^{-1}(\mathfrak{B})$.

Proposition

If $\mathfrak{A} \leq \mathfrak{B}$ then for all $R \subseteq A$ if R is definable by means of an infinitary Σ_1^c formula in $\mathfrak{A}$ then R is definable by Σ_1^c formula in $\mathfrak{B}$

Strong reducibility of structures

Theorem (Terziivanov)

For every sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i\in\omega}$, *where* $\mathfrak{A}_i = (A_i; R_{1,i}, \dots, R_{m_i,i})$ *with disjoint domains and each* $n \in \mathbb{N}$,

$$\mathfrak{P}_n(\vec{\mathfrak{A}}) \leq \mathfrak{M}(\vec{\mathfrak{A}})^{(n)}.$$

The question here when the opposite is true?

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