A parallel between classical computability theory and effective definability in abstract structures

Alexandra A. Soskova

Faculty of Mathematics and Computer Science
Sofia University

July 2014

A parallel between classical computability theory and effective definability in abstract structures

A close parallel between notions of classical computability theory and of the theory of effective definability in abstract structures:

1. The notion of "c.e. in" corresponds to the notion of Σ_1 definability;
2. The Σ^0_{n+1} sets correspond to the sets definable by means of computable Σ_{n+1} formulae.

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x \rangle \in W_e \wedge D_v \subseteq Y)).$$

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subseteq Y)).$$

Proposition

X is c.e. in Y if and only if $X \leq_e Y \oplus \overline{Y} = Y^+$.

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subseteq Y)).$$

Proposition

X is c.e. in Y if and only if $X \leq_e Y \oplus \overline{Y} = Y^+$.

Given a set A can we find a set M such that $X \leq_e A$ if and only if X is *c.e. in M*?

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subseteq Y)).$$

Proposition

X is c.e. in Y if and only if $X \leq_e Y \oplus \overline{Y} = Y^+$.

Given a set A can we find a set M such that $X \leq_e A$ if and only if X is *c.e. in* M?

There are sets A which are not enumeration equivalent to any set of the form $M \oplus \overline{M}$, so the answer is “No”.

Abstract structures

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- $f^{-1}(X) = \{\langle x_1 \ldots x_a\rangle : (f(x_1), \ldots, f(x_a)) \in X\}$ for any $X \subseteq A^a$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$ computes the positive atomic diagram of an isomorphic copy of $\mathfrak{A}$.

Definition

A set $X \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ (X c.e. in $\mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(X)$ is c.e. in $f^{-1}(\mathfrak{A})$.

Abstract structures

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- $f^{-1}(X) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}$ for any $X \subseteq A^a$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$ computes the positive atomic diagram of an isomorphic copy of $\mathfrak{A}$.

Definition

A set $X \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ (X c.e. in $\mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(X)$ is c.e. in $f^{-1}(\mathfrak{A})$.

By Ash, Knight, Manasse, Slaman and independantly Chisholm we have that X is c.e. in $\mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a computable infinitary Σ_1 formula with parameters.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

$X \leq_e \mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a positive computable infinitary Σ_1 formula with parameters.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

$X \leq_e \mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a positive computable infinitary Σ_1 formula with parameters.
Given a structure $\mathfrak{A} = (A; R_1, \dots R_n)$ let $\mathfrak{A}^+ = (A; R_1, \overline{R_1}, \dots R_n, \overline{R_n})$.

Proposition

For every $X \subseteq A$, X c.e. in $\mathfrak{A}$ if and only if $X \leq_e \mathfrak{A}^+$.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

$X \leq_e \mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a positive computable infinitary Σ_1 formula with parameters.
Given a structure $\mathfrak{A} = (A; R_1, \dots R_n)$ let $\mathfrak{A}^+ = (A; R_1, \overline{R_1}, \dots R_n, \overline{R_n})$.

Proposition

For every $X \subseteq A$, X c.e. in $\mathfrak{A}$ if and only if $X \leq_e \mathfrak{A}^+$.

Question (1.)

Given a structure $\mathfrak{A}$, does there exist a structure $\mathfrak{M}$, such that for all $R \subseteq |\mathfrak{A}|$, $R \leq_e \mathfrak{A}$ if and only if R is relatively intrinsically Σ_1 in $\mathfrak{M}$?

From sets to sequences of sets

Definition

A sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $A \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.

From sets to sequences of sets

Definition

A sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $A \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.

Theorem (Selman)

$X \leq_e A$ if an only if for every B, if A is c.e. in B then X is c.e. in B.

From sets to sequences of sets

Definition

A sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $A \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.

Theorem (Selman)

$X \leq_e A$ if an only if for every B, if A is c.e. in B then X is c.e. in B.

Definition

(i) Given a set X of natural numbers and a sequence $\mathcal{Y}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets B, $\mathcal{Y}$ is c.e. in B implies X is Σ^0_{n+1} in B;

(ii) Given sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers, say that $\mathcal{X}$ is ω-enumeration reducible to $\mathcal{Y}$ ($\mathcal{X} \leq_\omega \mathcal{Y}$) if for all sets B, $\mathcal{Y}$ is c.e. in B implies $\mathcal{X}$ is c.e. in B.

Sequences of sets

Ash presents a characterization of "$\leq_n$" and "$\leq_\omega$" using computable infinitary propositional sentences. Soskov and Kovachev give another characterizations in terms of enumeration computability.

Definition

The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

Sequences of sets

Ash presents a characterization of "$\leq_n$" and "$\leq_\omega$" using computable infinitary propositional sentences. Soskov and Kovachev give another characterizations in terms of enumeration computability.

Definition

The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

Theorem (Soskov)

1. *$X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.*
2. *$\mathcal{X} \leq_\omega \mathcal{Y}$ if and only if for every n, $X_n \leq_e \mathcal{P}_n(\mathcal{Y})$ uniformly in n.*

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \ldots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \ldots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

Definition

For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$.

Soskov and Baleva show that this is equivalent to R is definable by a computable infinitary formula Σ^+_{n+1} with predicates only from the first n structures, such that the predicates for the k-th appear for the first time at level $k+1$ positively.

Sequences of structures

$\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \ldots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

Sequences of structures

$\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$.

Sequences of structures

$\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$.

Soskov and Baleva show that this is equivalent to Y_n is uniformly in n definable by a computable Σ^+_{n+1} formula: a computable infinitary formula with predicates only from the first n structures, such that the predicates for the k-th appear for the first time at level $k+1$ positively.

Questions 2. and 3.

Question (2.)

Given a sequence of structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$, such that the Σ_{n+1} definable in $\mathfrak{M}$ sets coincide with sets $R \leq_n \vec{\mathfrak{A}}$?

Question (3.)

Given a sequence of structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$, such that for every sequence $\mathcal{X}$ of subsets of $A = \bigcup_n A_n$,
$\mathcal{X} \leq_\omega \vec{\mathfrak{A}}$ if and only if $\mathcal{X}$ c.e. in $\mathfrak{M}$?
Here $\mathcal{X}$ c.e. in $\mathfrak{M}$ if for each enumeration f of $\mathfrak{M}$, $f^{-1}(X_n)$ is c.e. in $f^{-1}(\mathfrak{M})^{(n)}$ uniformly in n.

Joint Spectra

Definition

The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.

Joint Spectra

Definition

The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.
The k-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}_k(\mathfrak{A}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{Sp}(\mathfrak{A})\}$.

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be arbitrary countable abstract structures.

Definition

The Joint spectrum of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\begin{aligned}\mathrm{JSp}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \quad & \mathfrak{A}_n) = \\ & \{\mathbf{a} : \mathbf{a} \in \mathrm{Sp}(\mathfrak{A}_0), \mathbf{a}' \in \mathrm{Sp}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{Sp}(\mathfrak{A}_n)\}.\end{aligned}$$

Joint Spectra

Definition

The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.
The k-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}_k(\mathfrak{A}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{Sp}(\mathfrak{A})\}$.

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be arbitrary countable abstract structures.

Definition

The Joint spectrum of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\begin{aligned}\mathrm{JSp}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \quad & \mathfrak{A}_n) = \\ & \{\mathbf{a} : \mathbf{a} \in \mathrm{Sp}(\mathfrak{A}_0), \mathbf{a}' \in \mathrm{Sp}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{Sp}(\mathfrak{A}_n)\}.\end{aligned}$$

Proposition

The joint spectrum of $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ *is the set*
$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists \{f_k\}_{k \leq n})(\forall k \leq n)(f_k^{-1}(\mathfrak{A}_k)$ *is c.e. in* $B^{(k)})\}$.

Co-spectra of structures

Definition

Let $\mathfrak{A}$ be a countable structure and $k \in \mathbb{N}$. The k-th co-spectrum of $\mathfrak{A}$ is the set

$$\mathrm{CoSp}_k(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_e \wedge (\forall \mathbf{b} \in \mathrm{Sp}_k(\mathfrak{A}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

Definition

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of structures.
The k-th co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{CoJSp}_k(\vec{\mathfrak{A}}) = \left\{\mathbf{a} \in \mathcal{D}_e \mid \forall \mathbf{x} \in \mathrm{JSp}_k(\vec{\mathfrak{A}})(\mathbf{a} \leq_e \mathbf{x})\right\},$$

where

$$\mathrm{JSp}_k(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{JSp}(\vec{\mathfrak{A}})\}.$$

Co-spectra of Joint spectra of structures

Proposition

For any set $X \subseteq \mathbb{N}$ *the following equivalence holds*

$$d_e(X) \in \mathrm{CoJSp}_k(\vec{\mathfrak{A}}) \iff X \leq_e \mathcal{P}_k(\vec{f}^{-1}(\vec{\mathfrak{A}})) \text{ for every sequence } \vec{f} = \{f_k\}_{k \leq n} \text{ of enumerations of } \vec{\mathfrak{A}}.$$

Proposition

$d_e(X) \in \mathrm{CoJSp}_k(\vec{\mathfrak{A}})$ *iff there exists a computable sequence of* Σ^+_{k+1} *formulae* $\{\Phi^{\gamma(x)}(W_1, \dots, W_r)\}$ *and parameters* $t_1, \dots, t_r$ *s.t.:*
$x \in X \iff (\vec{\mathfrak{A}}) \models \Phi^{\gamma(x)}(W_1/t_1, \dots, W_r/t_r)$.

Relative Spectra of Structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of countable structures. Denote by $A = \bigcup_k A_k$.

Definition

The relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists f \text{ enumeration of } A)(\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \text{ is c.e. in } B^{(k)})\}$$

where $f^{-1}(\mathfrak{A}_k) = f^{-1}(A_k) \oplus f^{-1}(R_1^k) \oplus \cdots \oplus f^{-1}(R_{m_k}^k)$.

Relative Spectra of Structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of countable structures. Denote by $A = \bigcup_k A_k$.

Definition

The relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists f \text{ enumeration of } A)(\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \text{ is c.e. in } B^{(k)})$$

where $f^{-1}(\mathfrak{A}_k) = f^{-1}(A_k) \oplus f^{-1}(R_1^k) \oplus \cdots \oplus f^{-1}(R_{m_k}^k)$.
The k-th jump spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{RSp}_k(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}})\}.$$

Relative Co-spectra of Structures

Definition

The Relative co-spectrum of $\vec{\mathfrak{A}}$ is the following set of enumeration degrees:

$$\mathrm{CoRSp}(\vec{\mathfrak{A}}) = \{\mathbf{b} \in \mathcal{D}_e \mid (\forall \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}}))(\mathbf{b} \le \mathbf{a})\}.$$

Proposition

For every $X \subseteq \mathbb{N}$, the following are equivalent:

1. $d_{\mathrm{e}}(X) \in \mathrm{CoRSp}_k(\vec{\mathfrak{A}})$.
2. $X \le_{\mathrm{e}} \mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))$, *for every enumeration f of A.*
3. *there exists a computable sequence of Σ^+_{k+1} formulae $\{\Phi^{\gamma(x)}(W_1, \dots, W_r)\}$ and parameters $t_1, \dots, t_r$ from A s.t.:* $x \in X \iff (\vec{\mathfrak{A}}) \models \Phi^{\gamma(x)}(W_1/t_1, \dots, W_r/t_r)$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

Let M be a set which is Σ^0_3, but not Σ^0_2. Fix an enumeration of the elements of M, $M = \{j_0, \dots, j_i, \dots\}$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

Let M be a set which is Σ^0_3, but not Σ^0_2. Fix an enumeration of the elements of M, $M = \{j_0, \dots, j_i, \dots\}$.

Finally let $\mathfrak{A}_1 = (\mathbb{N}, P)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

Let M be a set which is Σ^0_3, but not Σ^0_2. Fix an enumeration of the elements of M, $M = \{j_0, \ldots, j_i, \ldots\}$.

Finally let $\mathfrak{A}_1 = (\mathbb{N}, P)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

- $d_e(M) \notin \mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$.

The connection between the co-spectra of the Joint Spectra and Relative Spectra

For every $\vec{\mathfrak{A}}$ we have $\mathrm{CoJSp}(\vec{\mathfrak{A}}) = \mathrm{CoRSp}(\vec{\mathfrak{A}})$.
However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

Let M be a set which is Σ^0_3, but not Σ^0_2. Fix an enumeration of the elements of M, $M = \{j_0, \ldots, j_i, \ldots\}$.

Finally let $\mathfrak{A}_1 = (\mathbb{N}, P)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

- $d_e(M) \notin \mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$.
- $d_e(M) \in \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$, since if $t_0 = \langle 0, 0\rangle$,

$$\begin{aligned} j \in \quad & M \iff \exists Y_0 \ldots \exists Y_i \exists Z_0 \ldots \exists Z_j (Y_0 = t_0 \;\&\; L(Y_0, Y_1) \;\&\; \ldots \;\&\\ & L(Y_{i-1}, Y_i) \& \; Y_i = Z_0 \;\&\; R(Z_0, Z_1) \;\&\; \ldots \;\&\; R(Z_{j-1}, Z_j) \;\&\; P(Z_j)). \end{aligned}$$

Spectra of sequences of structures

More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

Spectra of sequences of structures

More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

Definition

The Joint spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists \{f_n\}_{n<\omega} \text{ enumerations of } \vec{\mathfrak{A}})\\ (\forall n)(f_n^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f_n^{-1}(\mathfrak{A}_n) = f_n^{-1}(A_n) \oplus f_n^{-1}(R_1^n) \oplus \cdots \oplus f_n^{-1}(R_{m_n}^n)$.

Spectra of sequences of structures

More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

Definition

The Joint spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists \{f_n\}_{n<\omega} \text{ enumerations of } \vec{\mathfrak{A}}) \\ (\forall n)(f_n^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f_n^{-1}(\mathfrak{A}_n) = f_n^{-1}(A_n) \oplus f_n^{-1}(R_1^n) \oplus \cdots \oplus f_n^{-1}(R_{m_n}^n)$.
The n-th jump spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{JSp}_n(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{JSp}(\vec{\mathfrak{A}})\}.$$

Spectra of sequences of structures

More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

Definition

The Joint spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists\{f_n\}_{n<\omega} \text{ enumerations of } \vec{\mathfrak{A}})\\ (\forall n)(f_n^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f_n^{-1}(\mathfrak{A}_n) = f_n^{-1}(A_n) \oplus f_n^{-1}(R_1^n) \oplus \cdots \oplus f_n^{-1}(R_{m_n}^n)$.
The n-th jump spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{JSp}_n(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{JSp}(\vec{\mathfrak{A}})\}.$$

If $\vec{\mathfrak{A}}$ and $\vec{\mathfrak{A}}^*$ are such that for every n $\mathfrak{A}_n \cong \mathfrak{A}_n^*$ then $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{JSp}(\vec{\mathfrak{A}}^*)$.

Spectra of sequences of structures

Let $A = \bigcup_n A_n$.

Definition

The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A)\\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f^{-1}(\mathfrak{A}_n) = f^{-1}(A_n) \oplus f^{-1}(R_1^n) \oplus \cdots \oplus f^{-1}(R_{m_n}^n)$.

Spectra of sequences of structures

Let $A = \bigcup_n A_n$.

Definition

The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A)\\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f^{-1}(\mathfrak{A}_n) = f^{-1}(A_n) \oplus f^{-1}(R_1^n) \oplus \cdots \oplus f^{-1}(R_{m_n}^n)$.
The n-th relative spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{RSp}_n(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}})\}.$$

Spectra of sequences of structures

Let $A = \bigcup_n A_n$.

Definition

The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A)\\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f^{-1}(\mathfrak{A}_n) = f^{-1}(A_n) \oplus f^{-1}(R_1^n) \oplus \cdots \oplus f^{-1}(R_{m_n}^n)$.
The n-th relative spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{RSp}_n(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}})\}.$$

Omega enumeration co-spectra

Definition

The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x}) \right\}.$$

Omega enumeration co-spectra

Definition

The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x}) \right\}.$$

For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition

For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:
$d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ iff $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$, for every enumeration f of A.

Omega enumeration co-spectra

Definition

The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{\mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x})\right\}.$$

For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition

For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:
$d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff* $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$*, for every enumeration f of A.*

Proposition

$d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff there exists a computable sequence* $\{\Phi^{\gamma(n,x)}(W_1, \ldots, W_r)\}$ *of* Σ^+_{n+1} *formulae and elements* $t_1, \ldots, t_r$ *of A s.t.:*
$x \in X_n \iff (\vec{\mathfrak{A}}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \ldots, W_r/t_r)$.

The Question 4.

Question (4.)

Given a sequence of structures $\vec{\mathfrak{A}}$,

1. does there exist a structure $\mathfrak{M}$, such that $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?
2. does there exist a structure $\mathfrak{M}$, such that $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \ldots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \ldots$
$h_n : (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \cdots \cup X_n; X_0, X_1, \ldots X_n, M_n)$.

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

If n is even then:
$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \ldots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \ldots$
$h_n : (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \cdots \cup X_n; X_0, X_1, \ldots X_n, M_n)$.

If n is even then:
$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(\bar{a}, x_0, x_1) \notin G_{h_1}] \iff$

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

If n is even then:

$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(\bar{a}, x_0, x_1) \notin G_{h_1}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

If n is even then:

$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(\bar{a}, x_0, x_1) \notin G_{h_1}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(\bar{a}, x_0, \dots x_n)]$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \ldots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \ldots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \ldots R_m, P_1, P_2, \ldots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.
3. Set $\mathfrak{M}(\vec{\mathfrak{A}})$ to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.
3. Set $\mathfrak{M}(\vec{\mathfrak{A}})$ to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A.

Two steps (Soskov)

Lemma

For every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ *there is an enumeration g of* $\vec{\mathfrak{A}}$*:*

1. $\mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(n)}$ *uniformly in n;*
2. $\bigoplus_n \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_T (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(\omega)}$.

Two steps (Soskov)

Lemma

For every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ *there is an enumeration g of* $\vec{\mathfrak{A}}$*:*

1. $\mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(n)}$ *uniformly in n;*
2. $\bigoplus_n \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_T (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(\omega)}$.

Theorem

Let g be an enumeration of $\vec{\mathfrak{A}}$ *and* $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$. *There is an enumeration f of* $\mathfrak{M}(\vec{\mathfrak{A}})$*:*

1. $\bigoplus_n \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \equiv_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}})))^{(\omega)}$.
2. $\mathcal{Y}$ *is not c.e. in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.

Two steps (Soskov)

Lemma

For every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ there is an enumeration g of $\vec{\mathfrak{A}}$:

1. $\mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(n)}$ *uniformly in n;*
2. $\bigoplus_n \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \leq_T (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))^+)^{(\omega)}$.

Theorem

Let g be an enumeration of $\vec{\mathfrak{A}}$ and $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$. There is an enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$:

1. $\bigoplus_n \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}})) \equiv_e (f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}})))^{(\omega)}$.
2. $\mathcal{Y}$ *is not c.e. in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.

Theorem

A sequence $\mathcal{Y}$ of subsets of A is (r.i.) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if and only if $\mathcal{Y}$ is (r.i) c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Generalized Goncharov and Khoussainov Lemma

Proposition

Let $n \geq 0$ and R be a $\Sigma^0_{n+1}(B)$ set with an infinite computable subset. Then there exists bijections $k_0, \ldots, k_n$ such that the graph of k_n is computable in B, uniformly in an index for R and n and
$k_0 : R \to \mathbb{N}$.
$k_1 : \mathbb{N}^2 \setminus G_{k_0} \to \mathbb{N} \ldots$
$k_n : \mathbb{N}^{n+1} \setminus G_{k_{n-1}} \to \mathbb{N}$.

Lemma (Soskov, M. Soskova)

Let R be $\Sigma^0_2(X)$ and $S \subseteq R$ be infinite and computable. There exists a bijection $k : R \to \mathbb{N}$ such that $\mathbb{N}^2 \setminus G_k$ is $\Sigma^0_1(X)$ and has an infinite computable subset.

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

Example

Let $\mathcal{R} = \{R_n\}_{n<\omega}$ be a seq. of sets. Dfeine $\vec{\mathfrak{A}}$ the seq. of structures:

- $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$;
- $\mathfrak{A}_n = (\mathbb{N}; R_n)$ for $n \geq 1$.

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

Example

Let $\mathcal{R} = \{R_n\}_{n<\omega}$ be a seq. of sets. Dfeine $\vec{\mathfrak{A}}$ the seq. of structures:

- $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$;
- $\mathfrak{A}_n = (\mathbb{N}; R_n)$ for $n \geq 1$.

Since every enumeration g of $\vec{\mathfrak{A}}$ is computable from $g^{-1}(G_s)$, we have that $\mathcal{P}_n(\mathcal{R}) \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))$ uniformly in n.

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

Example

Let $\mathcal{R} = \{R_n\}_{n<\omega}$ be a seq. of sets. Dfeine $\vec{\mathfrak{A}}$ the seq. of structures:

- $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$;
- $\mathfrak{A}_n = (\mathbb{N}; R_n)$ for $n \geq 1$.

Since every enumeration g of $\vec{\mathfrak{A}}$ is computable from $g^{-1}(G_s)$, we have that $\mathcal{P}_n(\mathcal{R}) \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))$ uniformly in n.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \{d_\omega(\mathcal{Y}) \mid \mathcal{Y} \leq_\omega \mathcal{R}\}$.

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the *n*-th jump degree of $\mathfrak{M}$.
The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the *n*-th co-degree of $\mathfrak{M}$.

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the n-th jump degree of $\mathfrak{M}$. The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the n-th co-degree of $\mathfrak{M}$.

Example

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the *n*-th jump degree of $\mathfrak{M}$. The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the *n*-th co-degree of $\mathfrak{M}$.

Example

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

- $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the *n*-th jump degree of $\mathfrak{M}$. The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the *n*-th co-degree of $\mathfrak{M}$.

Example

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

- $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.

Consider the *almost zero* sequence $\mathcal{R}$:

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the *n*-th jump degree of $\mathfrak{M}$. The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the *n*-th co-degree of $\mathfrak{M}$.

Example

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

- $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.

Consider the *almost zero* sequence $\mathcal{R}$:

1. $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every *n*. Hence the *n*-th co-degree of $\mathfrak{M}$ is $\mathbf{0}_e^{(n)}$.

Example: continued

Definition

The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the *n*-th jump degree of $\mathfrak{M}$.
The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the *n*-th co-degree of $\mathfrak{M}$.

Example

Richter, Knight: linear orderings have co-degree $\mathbf{0}_e$ and first co-degree $\mathbf{0}'_e$ but not always a degree or a jump degree.

- $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.

Consider the *almost zero* sequence $\mathcal{R}$:

1. $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every *n*. Hence the *n*-th co-degree of $\mathfrak{M}$ is $\mathbf{0}_e^{(n)}$.
2. $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$. Hence $\mathfrak{M}$ has no *n*-th jump degree for any *n*.

The positive answers of Soskov for the questions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$, $A = \bigcup_n |\mathfrak{A}_n|$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$.

Theorem

For every structure $\mathfrak{A}$, $R \subseteq |\mathfrak{A}|$, $R \leq_e \mathfrak{A}$ if and only if R is relatively intrinsically Σ_1 in $\mathfrak{M}$. Take $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ where $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

Theorem

For every $R \subseteq A$, $R \leq_n \vec{\mathfrak{A}} \iff R$ is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$.

Theorem

For every sequence $\mathcal{R}$ of subsets of A, $\mathcal{R} \leq_\omega \vec{\mathfrak{A}} \iff \mathcal{R} \leq_{c.e.} \mathfrak{M}$.

Theorem

1. *There is a structure $\mathfrak{M}_1$ with $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_1)$.*
2. *There is a structure $\mathfrak{M}_2$ with $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_2)$.*

Degree structures

- The enumeration degree of set X is $d_e(X) = \{Y \mid X \equiv_e Y\}$.

  The structure of the enumeration degrees $\mathcal{D}_e$ is an upper semi-lattice with jump operation.

  The Turing degrees are embedded in to the enumeration degrees by: $\iota(d_T(X)) = d_e(X^+)$.
- This embedding agrees with the jump operation since $(K^X)^+ \equiv_e (X^+)'$.

Degree structures

- The ω-enumeration degree of a sequence $\mathcal{X}$ is $d_\omega(\mathcal{X}) = \{\mathcal{Y} = \{Y_n\}_{n<\omega} \mid \mathcal{X} \equiv_\omega \mathcal{Y}\}$

  The structure of the ω-enumeration degrees $\mathcal{D}_\omega$ is an upper semi-lattice with jump operation.

  The enumeration degrees are embedded in to the ω-enumeration degrees by: $\kappa(d_e(X)) = d_\omega(\{X^{(n)}\}_{n<\omega})$.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets X which are not enumeration equivalent to any set of the form $Y \oplus \bar{Y}$.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets X which are not enumeration equivalent to any set of the form $Y \oplus \bar{Y}$.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets X which are not enumeration equivalent to any set of the form $Y \oplus \bar{Y}$.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{W_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $W_e^{[n]}$ is the n-th column of W_e.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets X which are not enumeration equivalent to any set of the form $Y \oplus \bar{Y}$.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{W_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $W_e^{[n]}$ is the n-th column of W_e.

$$R_n = \begin{cases} \{1\}, & \text{if } 0 \in W_n^{[n]}(\emptyset^{(n)}); \\ \{0\}, & \text{otherwise.} \end{cases} .$$

this property are called *almost zero*.

$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- There are sets X which are not enumeration equivalent to any set of the form $Y \oplus \bar{Y}$.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{W_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $W_e^{[n]}$ is the n-th column of W_e.

$$R_n = \begin{cases} \{1\}, & \text{if } 0 \in W_n^{[n]}(\emptyset^{(n)}); \\ \{0\}, & \text{otherwise.} \end{cases} .$$

this property are called *almost zero*.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$.
$\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- There is a structure $\mathfrak{M}_\mathcal{R}$ such that $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- There is a structure $\mathfrak{M}_\mathcal{R}$ such that $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.
- $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \{ d_T(B) \mid \mathcal{R} \text{ is c.e. in } B \}$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- There is a structure $\mathfrak{M}_\mathcal{R}$ such that $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.
- $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \{ d_T(B) \mid \mathcal{R} \text{ is c.e. in } B \}$.

Then

$$\begin{array}{lr} \mathcal{R} \leq_\omega \mathcal{X} & \iff \\ \{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\} \supseteq \{d_T(B) \mid \mathcal{X} \text{ is c.e. in } B\} & \iff \\ \mathrm{Sp}(\mathfrak{M}_\mathcal{R}) \supseteq \mathrm{Sp}(\mathfrak{M}_\mathcal{X}) & . \end{array}$$

Let $\mu(d_\omega(\mathcal{R})) = \mathrm{Sp}(\mathfrak{M}_\mathcal{R})$.

Spectrum with all non *low*$_n$ degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

Spectrum with all non *low*$_n$ degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

$\mathrm{Sp}(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_0)$, $\mathrm{Sp}_1(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_1), \dots, \mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n) \dots$

Spectrum with all non low_n degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

$\mathrm{Sp}(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_0)$, $\mathrm{Sp}_1(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_1)$, $\ldots$, $\mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n) \ldots$

Apply this to the sequence $\vec{\mathfrak{A}}$, where $\mathfrak{A}_n$ is obtained by Wehner's construction relativized to $\mathbf{0}^{(n)}$.

Spectrum with all non low_n degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ there exists a structure $\mathfrak{M}$ such that $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

$\mathrm{Sp}(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_0)$, $\mathrm{Sp}_1(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_1)$, ..., $\mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n)\dots$

Apply this to the sequence $\vec{\mathfrak{A}}$, where $\mathfrak{A}_n$ is obtained by Wehner's construction relativized to $\mathbf{0}^{(n)}$.

Theorem (Soskov)

There is a structure $\mathfrak{M}$ with $\mathrm{Sp}(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} > \mathbf{0}^{(n)})\}$.

A. A. Soskova and I. N. Soskov
Co-spectra of joint spectra of structures.
Ann. Univ. Sofia, **96** (2004) 35–44.

I. N. Soskov
Degree spectra and co-spectra of structures.
Ann. Univ. Sofia, **96** (2004) 45–68.

A. A. Soskova
Relativized degree spectra.
Journal of Logic and Computation, **17** (2007) 1215–1234.

I. N. Soskov
Effective properties of Marker's Extensions.
Journal of Logic and Computation, **23** (6), (2013) 1335–1367.