Degree Spectra and Conservative Extensions of Abstract Structures

Alexandra A. Soskova [1]
joint work with
Ivan N. Soskov and Stefan V. Vatev

Faculty of Mathematics and Computer Science
Sofia University

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Outline

- Degree spectra of structures
- Definability on structures
- Conservative (k, n) Extensions
- Jumps of Structures
- Jump inversion theorem for structures

Degree Spectra

Let $\mathfrak{A} = (A; P_1, \dots, P_k)$ be a denumerable structure. Enumeration of $\mathfrak{A}$ is every one to one mapping of $\mathbb{N}$ onto A.

Given an enumeration f of $\mathfrak{A}$ and a subset of X of A^a, let

$$f^{-1}(X) = \{\langle x_1, \dots, x_a \rangle : (f(x_1), \dots, f(x_a)) \in X\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(P_1) \oplus \cdots \oplus f^{-1}(P_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Definition.(Richter) *The Degree Spectrum of $\mathfrak{A}$* is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

Degree Spectra

Definition.(Knight) The n-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

Proposition.*(Knight) For every automorphically nontrivial structure $\mathfrak{A}$, $DS_n(\mathfrak{A})$ is an upwards closed set of degrees.*

Theorem.*(A. Soskova, I. Soskov) Every jump spectrum is a spectrum of a structure, i.e. for every countable structure $\mathfrak{A}$ there is a structure $\mathfrak{B}$ such that $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$.*

Theorem. *(A. Soskova, I. Soskov) Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{C})$. There exists a structure $\mathfrak{B}$ such that $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$ and $DS(\mathfrak{B}) \subseteq DS(\mathfrak{C})$.*

Formally Σ^c_n-definable sets

Let L be the language of $\mathfrak{A}$. The computable Σ^c_n formulas in L are defined inductively:

- *A computable Σ^c_0 (Π^c_0) formula is a finitary quantifier-free formula in L.*
- *A computable Σ^c_{n+1} formula $\Phi(\overline{x})$ is a disjunction of c.e. set of formulas of the form*

$$(\exists \overline{Y})\Psi(\overline{X}, \overline{Y})$$

  Ψ is a finite conjunction of Σ^c_n and Π^c_n formulas
- *Π^c_{n+1} formulas are the negations of the Σ^c_{n+1} formulas.*

Example

Consider $\mathcal{O} = (\mathbb{N}; =)$ *and* $\mathcal{S} = (\mathbb{N}; G_{Succ}; =)$, *where* G_{Succ} *is the graph of the successor function.*

$$DS(\mathcal{O}) = DS(\mathcal{S})$$

The $\Sigma^c_1(\mathcal{O})$ *are all finite and co-finite sets of natural numbers.*
But all c.e. set are formally Σ^c_1 *definable on* $\mathcal{S}$.
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Enumerations

Definition. The pair $\alpha = (f_\alpha, R_\alpha)$ is *an enumeration* of the set $X \subseteq A$, if R_α is a set of natural numbers, f_α is a partial one-to-one mapping of $\mathbb{N}$ onto X and $\mathrm{dom}(f_\alpha) = f_\alpha^{-1}(X)$ is c.e. in R_α. We denote this by $X \leq \alpha$.

Definition. The pair $\alpha = (f_\alpha, R_\alpha)$ is an *enumeration* of $\mathfrak{A}$ if α is an enumeration of A and $f_\alpha^{-1}(\mathfrak{A})$ is computable in R_α. We denote this by $\mathfrak{A} \leq \alpha$.

For an enumeration $\alpha = (f_\alpha, R_\alpha)$ *of* $\mathfrak{A}$ *we denote by* $\alpha^{(n)} = (f_\alpha, R_\alpha^{(n)})$.

Reformulation

The Degree Spectrum of $\mathfrak{A}$ *is the set*

$$DS(\mathfrak{A}) = \{d_T(R_\alpha) \mid \mathfrak{A} \leq \alpha\}.$$

Theorem. *(Ash, Knigh, Manasse, Slaman, Chisholm)*
For every set $X \subseteq A$,

$$X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow (\forall \alpha)[\mathfrak{A} \leq \alpha \rightarrow X \leq \alpha^{(n)}].$$

Conservative (k, n) Extensions

Let $\alpha = (f_\alpha, R_\alpha)$ *and* $\beta = (f_\beta, R_\beta)$ *be enumerations of the structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *respectively.*
We write $\alpha \leq \beta$ *if*

(i) $R_\alpha \leq_T R_\beta$ *and*

(ii) *the set* $E(f_\alpha, f_\beta) = \{(x, y) \mid x \in Dom(f_\alpha)\ \&\ y \in Dom(f_\beta)\ \&\ f_\alpha(x) = f_\beta(y)\}$ *is c.e. in* R_β.

Conservative (k, n) Extensions

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly with different signatures and $A \subseteq B$.

(i) $\mathfrak{A} \leq_n^k \mathfrak{B}$ iff for every enumeration β of $\mathfrak{B}$ there exists an enumeration α of $\mathfrak{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$.

(ii) $\mathfrak{A} \geq_n^k \mathfrak{B}$ iff for every enumeration α of $\mathfrak{A}$ there exists an enumeration β of $\mathfrak{B}$ such that $\beta^{(n)} \leq \alpha^{(k)}$.

(iii) $\mathfrak{A} \equiv_n^k \mathfrak{B}$ if $\mathfrak{A} \leq_n^k \mathfrak{B}$ and $\mathfrak{A} \geq_n^k \mathfrak{B}$. We shall say that $\mathfrak{B}$ is a (k, n)-conservative extension of $\mathfrak{A}$.

Note that the relation $\equiv_n^k$ is not symmetric.

Proposition. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$.*

(i) *If $\mathfrak{A} \leq_n^k \mathfrak{B}$ then $DS_n(\mathfrak{B}) \subseteq DS_k(\mathfrak{A})$;*

(ii) *If $\mathfrak{A} \geq_n^k \mathfrak{B}$ then $DS_k(\mathfrak{A}) \subseteq DS_n(\mathfrak{B})$;*

(iii) *If $\mathfrak{A} \equiv_n^k \mathfrak{B}$ then $DS_k(\mathfrak{A}) = DS_n(\mathfrak{B})$;*

Corollary.

(i) $k = 1, n = 0$:
If $\mathfrak{A} \equiv_0^1 \mathfrak{B}$ then $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$.

(ii) $k = 0, n = 1$:
If $\mathfrak{A} \equiv_1^0 \mathfrak{B}$ then $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$.

Theorem. *Let for* $\mathfrak{A}$ *and* $\mathfrak{B}$: $A \subseteq B$. *For all* $k, n \in \mathbb{N}$,

(i) *if* $\mathfrak{A} \leq_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathfrak{A}) \rightarrow X \in \Sigma^c_{n+1}(\mathfrak{B})]$;

(ii) *if* $\mathfrak{A} \geq_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{B}) \rightarrow X \in \Sigma^c_{k+1}(\mathfrak{A})]$;

(iii) *if* $\mathfrak{A} \equiv_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{n+1}(\mathfrak{B})]$.

The opposite direction is not always true:

Example.
Consider $\mathcal{O}_A = (A; =)$ *and take* $\mathfrak{A} = \mathfrak{B} = \mathcal{O}_A$.
For every natural number n,
$X \subseteq A$ *is* $\Sigma^c_n(\mathcal{O}_A)$ *iff* X *is a finite or co-finite subset of A.*
Therefore $\Sigma^c_1(\mathcal{O}_A) = \Sigma^c_n(\mathcal{O}_A)$ *and*

$$(\forall n)(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{O}_A) \to X \in \Sigma^c_1(\mathcal{O}_A)].$$

But $(\forall n)[\mathcal{O}_A \leq^n_0 \mathcal{O}_A]$ *is evidently not true.*

Moschovakis' extension

Let $\mathfrak{A} = (A; P_1, \ldots, P_k)$ *and* $\bar{0} \notin A$.
Set $A_0 = A \cup \{\bar{0}\}$.
Let $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A *is a pair and* $A^\star$ *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$.
Let $0^\star = \bar{0}$ *and* $(n+1)^\star = \langle \bar{0}, n^\star \rangle$, $\mathbb{N}^\star = \{n^\star \mid n \in \mathbb{N}\}$.
The decoding functions: $L(\langle s, t \rangle) = s$ & $R(\langle s, t \rangle) = t$
$L(\bar{0}) = R(\bar{0}) = 0^\star \quad (\forall t \in A)[L(t) = R(t) = 1^\star]$.

Moschovakis' extension

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^\star = (A^\star, P_1, \ldots, P_k, A_0, G_{\langle .,. \rangle}, G_L, G_R).$$

Proposition. $\mathfrak{A} \equiv_n^n \mathfrak{A}^\star$ *for every* $n \in \mathbb{N}$.

Proposition. *For every two structures* $\mathfrak{A}$, $\mathfrak{B}$ *with* $A \subseteq B$ *and natural numbers* n, k
$\mathfrak{A} \equiv_n^k \mathfrak{B}$ *iff* $\mathfrak{A}^\star \equiv_n^k \mathfrak{B}^\star$.

Conservative (k, n) Extensions and Definability

Theorem. *(S. Vatev)*
Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$ and $k, n \in \mathbb{N}$. If $(\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B})]$ then $\mathfrak{A} \leq^k_n \mathfrak{B}$.

Corollary. *For any two countable structures $\mathfrak{A}$, $\mathfrak{B}$ with $A \subseteq B$ and $n, k \in \mathbb{N}$,*

$$\mathfrak{A} \leq^k_n \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B}^\star)].$$

The set $K^{\mathfrak{A}}$

A new predicate $K_{\mathfrak{A}}$ (analogue of Kleene's set).
For $e, x \in \mathbb{N}$ and finite part τ, let

$$\tau \Vdash F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})}$$

$$\tau \Vdash \neg F_e(x) \leftrightarrow (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x))$$

$$K^{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

$$\mathfrak{A}' = (\mathfrak{A}^\star, K^{\mathfrak{A}}).$$

Theorem. $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}')$.

Proposition. $\mathfrak{A} \equiv^1_0 \mathfrak{A}'$.

The Forcing Relation

For every $e, x, n \in \mathbb{N}$ and for every finite part $\tau : \mathbb{N} \to A$, we define the forcing relations $\Vdash_n$:

$$
\begin{array}{lcl}
\tau \Vdash_0 F_e(x) & \leftrightarrow & x \in W_e^{\tau^{-1}(\mathfrak{A})} \\
 & & , \\
\tau \Vdash_{n+1} F_e(x) & \leftrightarrow & (\exists v)[\langle x, v\rangle \in W_e \ \&\ (\forall u \in D_v)[\\
 & & (u = \langle e_u, x_u, 1\rangle \ \&\ \tau \Vdash_n F_{e_u}(x_u)) \vee \\
 & & (u = \langle e_u, x_u, 0\rangle \ \&\ \tau \Vdash_n \neg F_{e_u}(x_u))]], \\
\tau \Vdash_n \neg F_e(x) & \leftrightarrow & (\forall \rho \supseteq \tau)(\rho \nVdash_n F_e(x)).
\end{array}
$$

The set $K_n^{\mathfrak{A}}$

Definition.

$$K_n^{\mathfrak{A}} = \{\langle \delta, e, x\rangle \mid (\exists \tau \supseteq \delta)[\tau \Vdash_n F_e(x)]\}.$$

Proposition. *(S. Vatev)*

(i) $K_n^{\mathfrak{A}} \in \Sigma^c_{n+1}(\mathfrak{A}^\star)$ *and* $A^\star \setminus K_n^{\mathfrak{A}} \in \Sigma^c_{n+2}(\mathfrak{A}^\star)$.

(ii) $K_n^{\mathfrak{A}} \notin \Sigma^c_n(\mathfrak{A}^\star)$.

Jumps of Structures

Definition. For every natural number n, we define the n-th jump of the structure $\mathfrak{A}$ in the following way:

$$\mathfrak{A}^{(0)} = \mathfrak{A} \text{ and } \mathfrak{A}^{(n+1)} = (\mathfrak{A}^\star, K_n^{\mathfrak{A}}).$$

Jumps of Structures

Proposition. *For every* $\mathfrak{A}$ *and natural number* n,

(i) $\mathfrak{A} \equiv_0^n \mathfrak{A}^{(n)}$;

(ii) $\mathfrak{A}^{(n)} \leq_0^0 \mathfrak{A}^{(n+1)}$ *and* $\mathfrak{A}^{(n)} \not\equiv_0^0 \mathfrak{A}^{(n+1)}$.

Since $\mathfrak{A} \equiv_n^k \mathfrak{B}$ implies $DS_k(\mathfrak{A}) = DS_n(\mathfrak{B})$, we get the following.

Corollary. *For every* $\mathfrak{A}$, $DS(\mathfrak{A}^{(n)}) = DS_n(\mathfrak{A})$.

The Jump Inversion Theorem

Theorem. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{C})$. There exists a structure $\mathfrak{B} = \mathfrak{A}^{\exists\forall} \oplus \mathfrak{C}$ such that $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$ and $DS(\mathfrak{B}) \subseteq DS(\mathfrak{C})$.*

Remark. *Similar results by:*

- *A. Montalban (2009) by different approach with complete set of Π^c_n formulas.*
- *A. Stukachev (2009) for Σ reducibility with Marker's extentions.*

Stukachev proves an analogue of this theorem for the semilattices of Σ-degrees of structures with arbitrary cardinalities.

Theorem. *(Stukachev) Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leq_{\mathbf{\Sigma}} \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that $\mathfrak{A} \equiv_{\Sigma} \mathfrak{B}'$.*

We can prove a similar to Stukachev's result.

The Jump Inversion Theorem

Proposition. *If* $\mathcal{O}_A \leq_0^1 \mathfrak{A}$*, then* $\mathfrak{A} \equiv_1^0 \mathfrak{A}^{\exists\forall}$.

Theorem. *Let* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *for some* $k \in \mathbb{N}$*. There exists a structure* $\mathfrak{B} = \mathfrak{A}^{\exists\forall}$ *such that* $\mathfrak{A} \equiv_0^0 \mathfrak{B}^{(k)}$.

Remark. *Note that* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *iff the elements of* $DS(\mathfrak{A})$ *are above* $\mathbf{0}^{(k)}$.

The Jump Inversion Theorem

Proposition. *Let* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *for some* $k \in \mathbb{N}$. *There exists a structure* $\mathfrak{B}$ *such that for every* $n \in \mathbb{N}$, $\mathfrak{A} \equiv_k^n \mathfrak{B}^{(n)}$.

Corollary. *Let* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *for some* $k \in \mathbb{N}$. *There exists a countable structure* $\mathfrak{B}$ *such that*

$$(\forall n \in \mathbb{N})(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{k+1}(\mathfrak{B}^{(n)})].$$

Corollary. *If* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *for some* $k \in \mathbb{N}$ *then for each* $n \in \mathbb{N}$, *there is a structure* $\mathfrak{B}$ *such that*

$$(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{k+1}(\mathfrak{B})].$$

Some problems

- *The definition of* $\mathfrak{A} \equiv_n^k \mathfrak{B}$ *is not symmetric since we suppose that* $A \subseteq B$. *How to define the similar relation more symmetric and for arbitrary* $\mathfrak{A}$ *and* $\mathfrak{B}$*?*
- *How to relativize the Jump Inversion Theorem for structures?*
- *The Jump inversion Theorem for structures for arbitrary constructive ordinal* α.

Thank you!

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