Degree spectra of sequences of structures

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A parallel between classical computability theory and effective definability in abstract structures

A close parallel between notions of classical computability theory and of the theory of effective definability in abstract structures:

1. The notion of "c.e. in" corresponds to the notion of Σ_1 definability;
2. The "Σ^0_{n+1} in" sets correspond to the sets definable by means of computable infinitary Σ_{n+1} formulae.

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

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Given a set A can we find a set M such that $X \leq_e A$ if and only if X is *c.e. in* M?
There are sets A which are not enumeration equivalent to any set of the form $M \oplus \overline{M}$, so the answer is “No”.

Abstract structures

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- $f^{-1}(X) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}$ for any $X \subseteq A^a$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$ computes the positive atomic diagram of an isomorphic copy of $\mathfrak{A}$.

Definition

A set $X \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ (X c.e. in $\mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$ we have that $f^{-1}(X)$ is c.e. in $f^{-1}(\mathfrak{A})$.

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By Ash, Knight, Manasse, Slaman and independantly Chisholm we have that X is c.e. in $\mathfrak{A}$ if and only if X is definable in $\mathfrak{A}$ by means of a computable infinitary Σ_1 formula with parameters.

Relatively intrinsically enumeration reducible

Definition

A set $X \subseteq A$ is (relatively intrinsically) enumeration reducible to $\mathfrak{A}$ ($X \leq_e \mathfrak{A}$) if for every enumeration f of $\mathfrak{A}$, $f^{-1}(X) \leq_e f^{-1}(\mathfrak{A})$.

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Given a structure $\mathfrak{A} = (A; R_1, \dots R_n)$ let $\mathfrak{A}^+ = (A; R_1, \overline{R_1}, \dots R_n, \overline{R_n})$.

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For every $X \subseteq A$, X c.e. in $\mathfrak{A}$ if and only if $X \leq_e \mathfrak{A}^+$.

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Proposition

For every $X \subseteq A$, X c.e. in $\mathfrak{A}$ if and only if $X \leq_e \mathfrak{A}^+$.

Question

Given a structure $\mathfrak{A}$, does there exist a structure $\mathfrak{M}$, such that for all $R \subseteq |\mathfrak{A}|$, $R \leq_e \mathfrak{A}$ if and only if R is relatively intrinsically Σ_1 in $\mathfrak{M}$?

From sets to sequences of sets

Definition

A sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $A \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.

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Theorem (Selman)

$X \leq_e A$ if an only if for every B, if A is c.e. in B then X is c.e. in B.

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$X \leq_e A$ if an only if for every B, if A is c.e. in B then X is c.e. in B.

Definition

(i) Given a set X of natural numbers and a sequence $\mathcal{Y}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets B, $\mathcal{Y}$ is c.e. in B implies X is Σ^0_{n+1} in B;

(ii) Given sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers, say that $\mathcal{X}$ is ω-enumeration reducible to $\mathcal{Y}$ ($\mathcal{X} \leq_\omega \mathcal{Y}$) if for all sets B, $\mathcal{Y}$ is c.e. in B implies $\mathcal{X}$ is c.e. in B.

Sequences of sets

Ash presents a characterization of "$\leq_n$" and "$\leq_\omega$" using computable infinitary propositional sentences. Soskov and Kovachev give another characterizations in terms of enumeration computability.

Definition

The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

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Theorem (Soskov)

1. *$X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.*
2. *$\mathcal{X} \leq_\omega \mathcal{Y}$ if and only if for every n, $X_n \leq_e \mathcal{P}_n(\mathcal{Y})$ uniformly in n.*

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \dots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.

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An enumeration f of $\vec{\mathfrak{A}}$ is a bijection from $\mathbb{N} \to A$.
$f^{-1}(\vec{\mathfrak{A}})$ is the sequence $\{f^{-1}(A_n) \oplus f^{-1}(R_1^n) \cdots \oplus f^{-1}(R_{m_n}^n)\}_{n<\omega}$.

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For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$.

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Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$.

Questions

Question

Given a sequence of structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$, such that the Σ_{n+1} definable in $\mathfrak{M}$ sets coincide with sets $R \leq_n \vec{\mathfrak{A}}$?

$\mathcal{X}$ is (r.i.) c.e. in $\mathfrak{M}$ if for each enumeration f of $\mathfrak{M}$, $f^{-1}(X_n)$ is c.e. in $f^{-1}(\mathfrak{M})^{(n)}$ uniformly in n.

Question

Given a sequence of structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$, such that for every sequence $\mathcal{X}$ of subsets of $A = \bigcup_n A_n$, $\mathcal{X} \leq_\omega \vec{\mathfrak{A}}$ if and only if $\mathcal{X}$ is (r.i.) c.e. in $\mathfrak{M}$?

Spectra of sequences of structures

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Definition

The Joint spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists \{f_n\}_{n<\omega} \text{ enumerations of } \vec{\mathfrak{A}})\\ (\forall n)(f_n^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

If $\vec{\mathfrak{A}}$ and $\vec{\mathfrak{A}}^*$ are such that for every n $\mathfrak{A}_n \cong \mathfrak{A}_n^*$ then $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{JSp}(\vec{\mathfrak{A}}^*)$.

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Definition

The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A = \bigcup_n A_n)\\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

Omega enumeration co-spectra

Definition

The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{\mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x})\right\}.$$

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For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition

For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:

1. $d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff*
2. $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$, *for every enumeration f of A iff*
3. *each X_n is definable by a computable sequence of Σ^+_{n+1} formulae with parameters uniformly in n.*

The Question

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Given a sequence of structures $\vec{\mathfrak{A}}$,

1. does there exist a structure $\mathfrak{M}$, such that $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?
2. does there exist a structure $\mathfrak{M}$, such that $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

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The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$.

Fix bijections: $h_0 : R \to X_0$

$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$

$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

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$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(\bar{a}, x_0, \dots x_n)]$.

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2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.

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1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
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The positive answers of the questions [Soskov]

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$, $A = \bigcup_n |\mathfrak{A}_n|$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$.

Theorem

A sequence $\mathcal{Y}$ of subsets of A is (r.i.) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if and only if $\mathcal{Y}$ is (r.i) c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Theorem

For every structure $\mathfrak{A}$, there is a structure $\mathfrak{M}$, s.t. $R \subseteq |\mathfrak{A}|$, $R \leq_e \mathfrak{A}$ if and only if R is relatively intrinsically Σ_1 in $\mathfrak{M}$.

Theorem

For every $R \subseteq A$, $R \leq_n \vec{\mathfrak{A}} \iff R$ is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$.

Theorem

1. *There is a structure $\mathfrak{M}_1$ with $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_1)$.*
2. *There is a structure $\mathfrak{M}_2$ with $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_2)$.*

An example

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.
$\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

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Example

Let $\mathcal{R} = \{R_n\}_{n<\omega}$ be a seq. of sets. Define $\vec{\mathfrak{A}}$ the seq. of structures:

- $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$;
- $\mathfrak{A}_n = (\mathbb{N}; R_n)$ for $n \geq 1$.

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$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

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- There are sets X which are not enumeration equivalent to any set of the form Y^+.
- The enumeration degrees are embedded in to the ω-enumeration degrees by: $\kappa(d_e(X)) = d_\omega(\{X^{(n)}\}_{n<\omega})$.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
    - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
    - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

  Sequences with this property are called *almost zero*.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

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$\mathcal{R} \leq_\omega \mathcal{Q} \iff$
$\{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\} \supseteq \{d_T(B) \mid \mathcal{Q} \text{ is c.e. in } B\} \iff$
$\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) \supseteq \mathrm{Sp}(\mathfrak{M}_\mathcal{Q})$.
Let $\mu(d_\omega(\mathcal{R})) = \mathrm{Sp}(\mathfrak{M}_\mathcal{R})$.

Spectrum with all non low_n degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

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Theorem (Soskov)

There is a structure $\mathfrak{M}$ *with* $\mathrm{Sp}(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} > \mathbf{0}^{(n)})\}$.

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