The α-jump inversion theorem for spectra of structures

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Degree spectra

Definition. Let $\mathfrak{A}$ be a countable structure. The *spectrum* of $\mathfrak{A}$ is the set of Turing degrees

$Sp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a}$ computes the diagram of an isomorphic copy on $\mathbb{N}$ of $\mathfrak{A}\}$.

For $\alpha < \omega_1^{CK}$ the α-th jump spectrum of $\mathfrak{A}$ is the set $Sp_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in Sp(\mathfrak{A})\}$.

An example

Consider a non-trivial group $G \subseteq Q$.
For every $a \neq 0$ element of G and every prime number p set

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \dots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

If a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$.
Denote by $\mathbf{d}_G = d_e(S_a(G))$, for some non-zero element a of G.

Proposition. *$Sp(G) = \{\mathbf{b} \mid \mathbf{b}$ is total & $\mathbf{d}_G \leq_e \mathbf{b}\}$.*
$Sp_1(G) = \{\mathbf{b} \mid \mathbf{d}'_G \leq_e \mathbf{b}\}$.

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $Sp(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ *and* $\mathfrak{A}$ *be a countable structure such that all elements of* $Sp(\mathfrak{A})$ *are above* $\mathbf{0}^{(\alpha)}$*.*

Does there exist a structure $\mathfrak{M}$ *such that* $Sp_\alpha(\mathfrak{M}) = Sp(\mathfrak{A})$*?*

The jump inversion theorem - the positive solutions

2005 ***S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller, R. Solomon****, Enumerations in computable structure theory, Annals of Pure and Applied Logic,* ***136****, 219-246.*

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Jump inversion theorem

Theorem.*[A. Soskova, I. Soskov] Every jump spectrum is a spectrum of a structure, i.e. for every countable structure $\mathfrak{A}$ there is a structure $\mathfrak{B}$ such that $Sp_1(\mathfrak{A}) = Sp(\mathfrak{B})$.*

Theorem.*[A. Soskova, I. Soskov] Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $Sp(\mathfrak{A}) \subseteq Sp_1(\mathfrak{C})$. There exists a structure $\mathfrak{B}$ such that $Sp(\mathfrak{A}) = Sp_1(\mathfrak{B})$ and $Sp(\mathfrak{B}) \subseteq Sp(\mathfrak{C})$.*

The ω case

In one of his last papers I. Soskov provides a negative solution to the ω-jump inversion problem for degree spectra of structures.

The jump inversion theorem - a negative solution

Theorem.*[Soskov] There is a structure* $\mathfrak{A}$ *with* $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq \mathbf{b}\}$ *for which there is no structure* $\mathfrak{M}$ *with* $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.

Turing reducibility

Let $A \subseteq \mathbb{N}$.
Denote by φ_e^B the Turing computable function by a program with code e with oracle A.

Definition. $A \leq_T B$ if $A = \varphi_e^B$.

Definition. $A \equiv_T B \iff A \leq_T B \;\&\; B \leq_T A$.

Definition.

$$d_T(A) = \{B \mid B \equiv_T A\}.$$

Definition. $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

The Turing jump

$D_T = (D_T, \leq, \oplus, \mathbf{0}_T)$ is an upper semi-lattice, where $\mathbf{0}_T = d_T(\emptyset)$.

Definition. The Turing jump of the set A:
$J_T(A) = K_A = \{x \mid x \in \mathrm{dom}(\varphi_x^A)\}$.

$A \leq_T B \Rightarrow J_T(A) \leq_T J_T(A)$.

Definition. $(d_T(A))' = d_T(J_T(A))$.

Since $A <_T K_A$, but $K_A \not\leq_T A$, then $d_T(A) < d_T(A)'$.

Enumeration reducibility

Definition. Given two sets of natural numbers X and Y, say that X is enumeration reducible to Y ($X \leq_e Y$) if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle x, v\rangle \in W_e \wedge D_v \subseteq Y)).$$

Definition. Let $X \equiv_e Y$ if $X \leq_e Y$ and $Y \leq_e X$.
The enumeration degree of X is $d_e(X) = \{Y \subseteq \mathbb{N} \mid X \equiv_e Y\}$.
By D_e we shall denote the set of all enumeration degrees.

The enumeration reducibility

Definition. Given a set $X \subseteq \mathbb{N}$, denote by $X^+ = X \oplus (\mathbb{N} \setminus X)$.
A set X is called *total* iff $X \equiv_e X^+$.

Theorem. *For any sets X and Y:*

(i) *X is c.e. in Y iff $X \leq_e Y^+$.*

(ii) *$X \leq_T Y$ iff $X^+ \leq_e Y^+$.*

Theorem.*[Selman] $X \leq_e Y$ iff for all total Z*

$$(Y \leq_e Z \Rightarrow X \leq_e Z).$$

The enumeration jump

Definition. For any $X \subseteq \mathbb{N}$ set $J_e(X) = \{\langle e, x \rangle \mid x \in W_e(X)\}$.
The *enumeration jump* X' *of* X is the set $J_e(X)^+$.

- $J_T(X)^+ \equiv_e (X^+)'$.
- $X' \leq_T (X^+)' \leq_T J_T(X)$.
- *for total* X, $X' \equiv_T J_T(X)$.
- *The enumeration jump of an e-degree is always a total degree and agrees with the Turing jump under the standard embedding* $\iota : D_T \to D_e$ *by* $\iota(d_T(X)) = d_e(X^+)$.

Enumeration reducibility of sequences of sets

Definition. Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{X}$ *is enumeration reducible to* $\mathcal{Y}$ ($\mathcal{X} \leq_e \mathcal{Y}$) if for all n, $X_n \leq_e Y_n$ uniformly in n. In other words, if there exists a computable function μ such that for all n, $X_n = W_{\mu(n)}(Y_n)$.

Definition. Let $\mathcal{X} = \{X_n\}_{n<\omega}$ be a sequence of sets of natural numbers. The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

Enumeration reducibility of sequences of sets

By $\mathcal{P}_\omega(\mathcal{X})$ we shall denote the set $\bigoplus_n \mathcal{P}_n(\mathcal{X})$.
Clearly $\mathcal{X} \leq_e \mathcal{P}(\mathcal{X})$ and hence $\bigoplus_n X_n \leq_e \mathcal{P}_\omega(\mathcal{X})$.

Proposition. *For all sequences $\mathcal{X}$ of sets of natural numbers the set $\mathcal{P}_\omega(\mathcal{X})$ is total.*

Proposition. *Let $\mathcal{X} = \{X_n\}_{n<\omega}$ be a sequence of sets of natural numbers, $M \subseteq \mathbb{N}$ and $\mathcal{X} \leq_e \{M^{(n)}\}_{n<\omega}$. Then $\mathcal{P}(\mathcal{X}) \leq_e \{M^{(n)}\}_{n<\omega}$.*

Co-spectra of structures

Definition. Let $\mathfrak{M}$ be a countable structure and $\alpha < \omega_1^{CK}$. The α-th co-spectrum of $\mathfrak{M}$ is the set

$$CoSp_\alpha(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{a} \in D_e \wedge (\forall \mathbf{b} \in Sp_\alpha(\mathfrak{M}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

Computable Σ^c_α formulas

Let L be the language of the structure $\mathfrak{M}$ and $\alpha < \omega_1^{CK}$. The computable Σ^c_α formulas in L are defined inductively:

- A computable Σ^c_0 (Π^c_0) formula is a finitary quantifier-free formula in L;
- A computable Σ^c_α formula $\Phi(\overline{X})$ is a disjunction of c.e. set of formulas of the form
$$(\exists \overline{Y})\Psi(\overline{X}, \overline{Y})$$
Ψ is a finite conjunction of Σ^c_β and Π^c_β formulas for $\beta < \alpha$.
- Π^c_α formulas are the negations of the Σ^c_α formulas.

Σ^c_α definable sets on $\mathfrak{M}$

Definition. Let $\alpha < \omega_1^{CK}$. A subset R of $\mathbb{N}$ is Σ^c_α definable in $\mathfrak{M}$ if there exist a computable function γ taking as values codes of computable Σ^c_α infinitary formulas $F_{\gamma(x)}$ and finitely many parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m).$$

Theorem._[Ash,Knight,Mannase,Slaman][Soskov] Let_ $\alpha < \omega_1^{CK}$. _Then_

1. _If_ $\alpha < \omega$ _then_ **a** $\in CoSp_\alpha(\mathfrak{M})$ _if and only if all elements of_ **a** _are_ $\Sigma^c_{\alpha+1}$ _definable in_ $\mathfrak{M}$.
2. _If_ $\omega \leq \alpha$ _then_ **a** $\in CoSp_\alpha(\mathfrak{M})$ _if and only if all elements of_ **a** _are_ Σ^c_α _definable in_ $\mathfrak{M}$.

A property of the ω co-spectra

Theorem. *Let $\mathfrak{M}$ be a countable structure and* $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$. *Then there exists a total enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq_e \mathbf{b}$ *and* $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$,

A property of the ω co-spectra

Proof.

Fix an element R of $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$.
R is Σ^c_ω definable in $\mathfrak{M}$ and hence there exists a computable function γ and parameters $t_1, \dots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \dots, t_m).$$

$F_{\gamma(x)}$ is a c.e. disjunction of computable Σ^c_{n+1} infinitary formulae. Hence there exists a computable function $\delta(n, x)$ such that for all n and x, $\delta(n, x)$ yields a code of some computable Σ^c_{n+1} infinitary formula $F_{\delta(n,x)}$ and

$$x \in R \iff (\exists n)(\mathfrak{M} \models F_{\delta(n,x)}(t_1, \dots, t_m)).$$

□

A property of the ω co-spectra

Proof.

For each $n \in \mathbb{N}$ denote by

$$R_n = \{x \mid x \in \mathbb{N} \wedge \mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_m)\}.$$

Let B be the diagram of some isomorphic copy $\mathfrak{B}$ of $\mathfrak{M}$ on the natural numbers and let κ be an isomorphism from $\mathfrak{M}$ to $\mathfrak{B}$ and $x_1 = \kappa(t_1), \ldots, x_m = \kappa(t_m)$. Then

$$x \in R_n \iff \mathfrak{B} \models F_{\delta(n,x)}(x_1, \ldots, x_m).$$

Hence

$$\mathcal{P}(\{R_n\}_{n<\omega}) \leq_e \{B^{(n)}\}_{n<\omega} \text{ uniformly in } n.$$

Thus

$$\mathcal{P}_\omega(\{R_n\}_{n<\omega}) \leq_e B^{(\omega)}.$$

□

A property of the ω co-spectra

Proof.

Set $\mathbf{b} = d_e(\mathcal{P}_\omega(\{R_n\}_{n<\omega}))$.

- $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$ since $\mathcal{P}_\omega(\{R_n\}_{n<\omega}) \leq_e B^{(\omega)}$ for any isomorphic copy $\mathfrak{B}$ of $\mathfrak{M}$;
- $\mathbf{b}$ is a total degree since $\mathbf{b} = d_e(\mathcal{P}_\omega(\{R_n\}_{n<\omega}))$;
- $\mathbf{a} \leq_e \mathbf{b}$ since $R = \bigoplus_n R_n \leq_e \mathcal{P}_\omega(\{R_n\}_{n<\omega})$.

□

A negative solution for the ω-jump inversion problem

- *Let Y be a set which is quasi-minimal above $\emptyset^{(\omega)}$, i.e. $\emptyset^{(\omega)} <_e Y$ and if X is a total set and $X \leq_e Y$ then $X \leq_e \emptyset^{(\omega)}$, e.g. $Y = \emptyset^{(\omega)} \oplus G$, where G is one-generic relatively $\emptyset^{(\omega)}$.*
- *$d_e(Y)$ does not contain any total set.*

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- *Let Y be a set which is quasi-minimal above $\emptyset^{(\omega)}$, i.e. $\emptyset^{(\omega)} <_e Y$ and if X is a total set and $X \leq_e Y$ then $X \leq_e \emptyset^{(\omega)}$, e.g. $Y = \emptyset^{(\omega)} \oplus G$, where G is one-generic relatively $\emptyset^{(\omega)}$.*
- *$d_e(Y)$ does not contain any total set.*
- *Let $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$. Then $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq_T \mathbf{b}\}$.*
- *Assume that there exists a countable structure $\mathfrak{M}$ such that $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$. Then $CoSp_\omega(\mathfrak{M}) = CoSp(\mathfrak{A})$.*
- *Hence there exists a total degree $\mathbf{b}$ in $CoSp(\mathfrak{A})$ such that $d_e(Y) \leq \mathbf{b} \leq d_e(Y)$.*
  A contradiction.

Theorem. *If Y is quasi-minimal above $\emptyset^{(\omega)}$ and $\mathfrak{A}$ is a structure with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$ then there is no structure $\mathfrak{M}$ with $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.*

A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

Consider a non-trivial group $G \subseteq Q$.
For every $a \neq 0$ element of G and every prime number p set

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \ldots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

If a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$.
Denote by $\mathbf{d}_G = d_e(S_a(G))$, for some non-zero element a of G.

A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

Proposition.*[Coles, Downey, Slaman, Soskov]*
$Sp(G) = \{\mathbf{b} \mid \mathbf{b}$ *is total* & $\mathbf{d}_G \leq_e \mathbf{b}\}$.

Corollary. $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e \mathbf{d}_G\}$.

Proof.

Clearly $\mathbf{a} \in CoSp(G)$ if and only if for all total $\mathbf{b}$, $\mathbf{d}_G \leq_e \mathbf{b} \Rightarrow \mathbf{a} \leq_e \mathbf{b}$.
According Selman's Theorem the last is equivalent to $\mathbf{a} \leq_e \mathbf{d}_G$. □

A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

Consider the set

$$S = \{\langle i, j\rangle : (j = 0) \vee (j = 1 \,\&\, i \in Y)\}.$$

Clearly $S \equiv_e Y$.
Let G be the least subgroup of Q containing the set

$$\{1/p_i^j : \langle i, j\rangle \in S\}.$$

Then $1 \in G$ and $S_1(G) = S$. So, $\mathbf{d}_G = d_e(Y)$.

Theorem. $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$.

Coding a set by a sequence of structures

Let S be a set of natural numbers, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be structures in the same language. We say that the sequence of structures $\{\mathfrak{C}_n\}_n$ codes the set S if

$$\mathfrak{C}_n = \begin{cases} \mathfrak{B}_1, & n \in S; \\ \mathfrak{B}_2, & n \notin S \end{cases}$$

The sequence $\{\mathfrak{C}_n\}_n$ is **uniformly computable**, if it consists of computable copies of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and for each n we can effectively find a computable index for $\mathfrak{C}_n$, although we do not know whether this index corresponds to $\mathfrak{B}_1$ and $\mathfrak{B}_2$. If $\{\mathfrak{C}_n\}_n$ is a uniformly computable sequence, then we say that $\{\mathfrak{C}_n\}_n$ **strongly codes** the set S.

Coding by a sequence of structures

Consider the sequence of structures:

$$\mathfrak{C}_n = \begin{cases} \omega, & n \in S; \\ \omega^*, & n \notin S \end{cases}$$

The following are equivalent:

- the sequence $\{\mathfrak{C}_n\}_n$ strongly codes the set S;
- the set S is Δ^0_2.

The question what sets we can strongly coded by what kind of structures was studied by Ash and Knight (1990).

Theorem. *[Ash & Knight] If α is a computable successor ordinal and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in $\mathcal{L}$ are computable and α friendly and such that and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ satisfied the same Σ_β sentences of $\mathcal{L}$ for each $\beta < \alpha$ then for each Δ^0_α set S there is a sequence consisting of copies of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ which strongly codes S.*

Jump inversion for a successor ordinal

Theorem.*[Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, 2006]*
Let α be a computable successor ordinal and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in $\mathcal{L}$ are computable and α-friendly structures and such that

- *$\mathfrak{B}_1$ and $\mathfrak{B}_2$ satisfy the same Σ_β sentences of $\mathcal{L}$ for each $\beta < \alpha$,*
- *each $\mathfrak{B}_i$ satisfies some Σ^c_α sentence that is not true in the other.*

Then there is a graph $\mathfrak{N}$ built from the sequences which strongly encodes the initial predicates of $\mathfrak{A}$ and
$\mathfrak{N}$ has an X computable copy iff $\mathfrak{A}$ has a $\Delta^0_\alpha(X)$ computable copy.

Jump inversion for a successor ordinal

S. Vatev considers a weak condition for the sequence $\{\mathfrak{C}_n\}_n$ to code the set S - it is not necessarily α-friendly, but

$$\Delta^0_\alpha(\bigoplus_n \mathfrak{C}_n) \leq_T S.$$

Theorem.*[S. Vatev,2013] For every computable successor ordinal $\alpha \geq 2$ and a countable structure $\mathfrak{A}$ such that $Sp(\mathfrak{A}) \subseteq \{\mathbf{a} \mid \mathbf{0}^{(\alpha)} \leq_T \mathbf{a}\}$ there is a structure $\mathfrak{N}$ such that:*

- $Sp_\alpha(\mathfrak{N}) = Sp(\mathfrak{A})$;
- $(\forall X \subseteq A)[X \in \Sigma^c_{\alpha+1}(\mathfrak{N}) \iff X \in \Sigma^c_1(\mathfrak{A})]$.

Spectra of sequences of structures

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Definition. The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A = \bigcup_n A_n) \\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e in } B^{(n)} \text{ uniformly in } n)\}.\end{array}$$

Marker's extensions

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The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$.

Fix bijections: $h_0 : R \to X_0$

$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$

$h_n : (A^m \times X_0 \times X_1 \cdots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \cdots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

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If n is even then:

$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

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$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(\bar{a}, x_0, x_1) \notin G_{h_1}] \iff$

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

Marker's extensions

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$.

Fix bijections: $h_0 : R \to X_0$

$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$

$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

If n is even then:

$\bar{a} \in R \iff \exists x_0 \in X_0[(\bar{a}, x_0) \in G_{h_0}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1[(\bar{a}, x_0, x_1) \notin G_{h_1}] \iff$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(\bar{a}, x_0, \dots x_n)]$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \ldots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \ldots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \ldots R_m, P_1, P_2, \ldots P_k)$.

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Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$.

1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.

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1. For every n construct the n-th Markers's extensions of A_n, R_1^n, $\dots R_{m_n}^n$ with disjoint companions.
2. For every n let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$.

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The positive answers [Soskov]

Theorem. *Let $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ be the Marker's extension of the sequence of structures $\vec{\mathfrak{A}}$. Then for every n:*

$$\mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n).$$

Moreover

$$(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{M}) \iff X \in \Sigma^+_{n+1}(\mathfrak{A}_0, \dots, \mathfrak{A}_n)]$$

Theorem. *For every sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$, there is a structure $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$, such that $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$.*

Theorem. *Let $\mathfrak{M}(\vec{\mathfrak{A}})$ be the Marker's extension of the sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{B}, \mathfrak{A}, \mathfrak{B}, \dots\}$, where $\mathfrak{B} = \{A, =\}$. Then $\mathrm{Sp}_1(\mathfrak{M}(\vec{\mathfrak{A}})) = \mathrm{Sp}(\mathfrak{A})$.*

Omega enumeration reducibility

Definition. Given sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers, say that $\mathcal{X}$ is ω-enumeration reducible to $\mathcal{Y}$ ($\mathcal{X} \leq_\omega \mathcal{Y}$) if for all sets *B*, $\mathcal{Y}$ is c.e. in *B* implies $\mathcal{X}$ is c.e. in *B*.

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Theorem.*[Soskov] $\mathcal{X} \leq_\omega \mathcal{Y}$ if and only if for every n, $X_n \leq_e \mathcal{P}_n(\mathcal{Y})$ uniformly in n.*

Omega enumeration co-spectra

Definition. The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x}) \right\}.$$

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For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition. *For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:*

1. $d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff*
2. $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$, *for every enumeration f of A iff*
3. *each X_n is definable by a computable sequence of Σ^+_{n+1} formulae with parameters uniformly in n.*

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R} = \{R_n\}$: $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$, where G_S is the graph of the successor function $\lambda x.x + 1$.

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- For every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.

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- For every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- $\mathrm{Sp}(\mathfrak{M}_{\mathcal{R}}) = \{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\}$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

Consider the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R} = \{R_n\}$: $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$, where G_S is the graph of the successor function $\lambda x.x + 1$.

- For every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \{d_T(B) \mid \mathcal{R}$ is c.e. in $B\}$.

$\mathcal{R} \leq_\omega \mathcal{Q} \iff$
$\{d_T(B) \mid \mathcal{R}$ is c.e. in $B\} \supseteq \{d_T(B) \mid \mathcal{Q}$ is c.e. in $B\} \iff$
$\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) \supseteq \mathrm{Sp}(\mathfrak{M}_\mathcal{Q})$.
Let $\mu(d_\omega(\mathcal{R})) = \mathrm{Sp}(\mathfrak{M}_\mathcal{R})$.

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Thank you!